3.3 The character of the induced representation

K any field
G finite group
$H < G$ subgroup

<u>Thm 1:</u> $T \subset G$ set of representatives for G/H

V H-representation with character χ

$g \in G$

Then

$$\chi_{\mathrm{Ind}_H^G V}(g) = \sum_{\substack{t \in T \text{ s.t.} \\ t^{-1} g t \in H}} \chi(t^{-1} g t)$$

$$= \frac{1}{\# H} \sum_{\substack{s \in G \text{ s.t.} \\ s^{-1} g s \in H}} \chi(s^{-1} g s).$$

[if char $K \nmid \# H$]

<u>proof:</u> • $\mathrm{Ind}\, V = \bigoplus_{\sigma \in G/H} V_\sigma$

$\left[= \{ t \otimes v \mid v \in V \} = \text{"} t \otimes V \text{"} \right.$
$\left. \text{if } t \in \sigma \right]$

• Fix a basis $\{v_1, \ldots, v_d\}$ of V;

then $\{ t \otimes v_i \}_{\substack{t \in T \\ i = 1 \ldots d}}$ is a basis of $\mathrm{Ind}\, V$.

• Let $A(g)$ be the repr. matrix w.r.t. $\{ t \otimes v_i \}$:

$$A(g) = \begin{array}{c} \left[\begin{array}{ccc} A_{\sigma_1}(g) & & * \\ & \ddots & \\ * & & A_{\sigma_r}(g) \end{array}\right] \begin{array}{l} \}\, V_{\sigma_1} \\ \\ \}\, V_{\sigma_r} \end{array} \\ \underbrace{\quad}_{V_{\sigma_1}} \quad \cdots \quad \underbrace{\quad}_{V_{\sigma_r}} \end{array}$$

where $r = \# T$. Define $\chi_\sigma(g) = \operatorname{tr} A_\sigma(g)$.

Then $\displaystyle \chi_{\operatorname{Ind} V}(g) = \sum_{\sigma \in G/H} \chi_\sigma(g)$

- Let $t \in T$. Then $gt = sh$ for some $s \in T$ and $h \in H$, and

$$g \cdot V_{tH} = g \cdot (t \otimes V) = gt \otimes V$$
$$= sh \otimes V = s \otimes hV = s \otimes V = V_{sH}.$$

<u>Thus:</u> • if $s \neq t$ $\left(\Leftrightarrow t^{-1} g t = \underbrace{t^{-1} s}_{\notin H} h \notin H\right)$,

then $\chi_{tH}(g) = 0$

• if $s = t$ $\left(\Leftrightarrow t^{-1} g t = t^{-1} t h = h \in H\right)$,

then

$$g \cdot \underbrace{(t \otimes V)}_{= V_{tH}} = t \underbrace{t^{-1} g t}_{\in H} \otimes V = t \otimes (t^{-1} g t) . V$$

and $\chi_{tH}(g) = \chi(t^{-1} g t)$. (i.e. g acts on V_{tH} as $t^{-1} g t$ acts on V)

- Thus

$$\chi_{\operatorname{Ind} V}(g) = \sum_{t \in T} \chi_{tH}(g) = \sum_{\substack{t \in T \\ \text{s.t. } t^{-1} g t \in H}} \chi(t^{-1} g t) \qquad \text{(first formula).}$$

• If $sH = tH$, i.e. $s = th$ for some $h \in H$,

then: • $s^{-1}gs = h^{-1}t^{-1}gth \in H \iff t^{-1}gt \in hHh^{-1} = H$;

• $\chi(s^{-1}gs) = \chi(h^{-1}t^{-1}gth) = \chi(t^{-1}gt)$

[χ is a class fct.]

Thus the second formula. $\square$

<u>Ex:</u> $D_4 = \langle r, s \mid r^4 = s^2 = (rs)^2 = e \rangle$

$\cup$

$H = \langle r \rangle$

conj. cl. of H	e	r	r^2	r^3
χ	1	$\chi(r)$	$\chi(r)^2$	$\chi(r)^3 = \overline{\chi(r)}$

$T = \{e, s\}$ represents D_4 / H

<u>note:</u> $e^{-1}ge = g$ and $s^{-1}gs = sgs$

V 1-dim. H-repr. with character $\chi : H \longrightarrow \mathbb{C}^\times$
$r \longmapsto \chi(r)$

repr. g of conj. cl. of D_4	$\chi_{\mathrm{Ind}\,V}(g) =$	
e	$\chi(e) + \chi(e) = 2$	$(e,\ ses \in H)$
r^2	$\chi(r^2) + \chi(sr^2s) = 2\chi(r^2)$	$(r^2,\ sr^2s = r^2 \in H)$
r	$\chi(r) + \chi(srs) = \chi(r) + \overline{\chi(r)}$	$(r,\ srs = r^3 \in H)$
s	0	$(s,\ sss \notin H)$
rs	0	$(rs,\ srss \notin H)$

3.4 Mackey's irreducibility criterion

Context:

$K = \mathbb{C}$

$H, H' < G$ subgroups

$\rho : H \longrightarrow GL(V)$ H-repr.

S set of representatives for $H' \backslash G / H = \{ \overbrace{H'gH}^{\{h'gh \mid h' \in H', h \in H\}} \mid g \in G \}$

with $e \in S$

Fact: $G = \coprod_{g \in S} H'gH$

For $s \in S$, define

- $H_s = sHs^{-1} \cap H' < H'$
- $\rho_s : H_s \longrightarrow GL\,V$ an H_s-repr; also write: V_s
  $g \longmapsto \rho(s^{-1}gs)$

Lemma 1: $\mathrm{Res}^G_{H'} \mathrm{Ind}^G_H V \simeq \bigoplus_{s \in S} \mathrm{Ind}^{H'}_{H_s} V_s$

as H'-repr.

$$\begin{tikzcd}
 & & G \arrow[r, dashed, "\mathrm{Ind}\,\rho"] & GL(\mathrm{Ind} V) & \\
GL(-) & H' \arrow[l, dashed, "\mathrm{Res}\,\mathrm{Ind}\,\rho"'] \arrow[ur, hook] \arrow[dl, dashed, "\mathrm{Ind}\,\rho_s"] & & H \arrow[ul, hook] \arrow[r, dashed, "\rho"] & GL(V) \\
GL(-) & & H_s \arrow[ul, hook] \arrow[r, dashed, "\rho_s"] & GL(V_s) &
\end{tikzcd}$$

proof: $W = \mathrm{Ind}^G_H V = \bigoplus_{t \in T} t \otimes V$

where T is a set of repr. for G/H.

- For $s \in S$, define

$$W(s) = \bigoplus_{\substack{t \in T \\ \text{s.t. } t \in H'sH}} t \otimes V = \langle t \otimes V \mid t \in H'sH \rangle$$

Claim: $\cdot\ \forall g \in H',\quad g.W(s) = W(s)$

$$\cdot\ \mathrm{Stab}_{H'}(s \otimes V) = \{g \in H' \mid g.(s\otimes V) = s \otimes V\}$$
$$= \{g \in H' \mid \underbrace{s^{-1}gs \in H}_{\Leftrightarrow\, g \in sHs^{-1}}\} = H_s$$

$$\Rightarrow W(s) = \bigoplus_{[g] \in H'/H_s} g.(s \otimes V)$$

thus: $W(s) \;{}^{\text{"}}{=}\; \mathbb{C}[H'] \otimes_{\mathbb{C}[H_s]} (s \otimes V)\,{}^{\text{"}} = \mathrm{Ind}_{H_s}^{H'}(s \otimes V)$

Since
$$V_s \longrightarrow s \otimes V \quad \text{is an isom. of } H_s\text{-repr.},$$
$$v \longmapsto s \otimes v$$

we get $\mathrm{Res}^G_{H'} W \simeq \bigoplus_{s \in S} W(s) \simeq \bigoplus_{s \in S} \mathrm{Ind}^{H'}_{H_s} V_s$, as claimed. $\square$

Let $H' = H$ and the rest as before.

Thm 2: $\mathrm{Ind}_H^G V$ is an irred. G-repr. iff.

$$\begin{tikzcd} G \arrow[r, "\mathrm{Ind}\, s"] \arrow[d, dash] & GL(\mathrm{Ind} V) \\ H \arrow[r, "s"] \arrow[d, dash] & GL(V) \\ H_s \arrow[r, "\mathrm{Res}\, s"] \arrow[dr, bend left, "s_s"'] & GL(V) \\ & GL(V_s) \end{tikzcd}$$

(1) V is an irred. H-repr., and

(2) $\forall s \in S - \{e\}$, $\langle \chi_{V_s}, \chi_{\mathrm{Res}^H_{H_s} V} \rangle_{H_s} = 0$.

[i.e. V_s and $\mathrm{Res}^H_{H_s} V$ do not have a common subrepresentation $\neq 0$]

proof: $\cdot$ If $V = 0$, then $\mathrm{Ind}_H^G V = 0$, and the thm. holds; thus we can assume that $V \neq 0$.

- By Cor. 2.2.6, $W = \mathrm{Ind}_H^G V$ is irred. iff. $\langle \chi_W, \chi_W \rangle_G = 1$.

- By Thm. 3.2.3 (Frobenius reciprocity),

$$\langle \chi_W, \chi_W \rangle_G = \langle \chi_V, \chi_{\mathrm{Res}_H^G W} \rangle_H$$

- By Lemma 1, $\mathrm{Res}_H^G W \simeq \bigoplus_{s \in S} \mathrm{Ind}_{H_s}^H V_s$

$$\Rightarrow \chi_{\mathrm{Res}_H^G W} = \sum_{s \in S} \chi_{\mathrm{Ind}_{H_s}^H V_s}$$

Thus: W is irred. iff

$$1 = \langle \chi_V, \chi_{\mathrm{Res}_H^G W} \rangle_H \underset{(T=1)}{=} \langle \chi_{\mathrm{Res}_H^G W}, \chi_V \rangle_H$$

$$= \sum_{s \in S} \langle \chi_{\mathrm{Ind}_{H_s}^H V_s}, \chi_V \rangle_H$$

$$\underset{\text{(Frobenius reciprocity)}}{=} \sum_{s \in S} \langle \chi_{V_s}, \chi_{\mathrm{Res}_{H_s}^H V} \rangle_{H_s}$$

Since $V \neq 0$, we have for $s = e$ that

$H_e = eHe^{-1} \cap H = H$ and $\chi_{V_e} = \chi_V$; thus

$$\langle \chi_{V_e}, \chi_{\mathrm{Res}_{H_e}^H V} \rangle_{H_e} = \langle \chi_V, \chi_V \rangle_H \geq 1.$$

Therefore: $1 = \langle \chi_W, \chi_W \rangle_G = \sum_{s \in S} \underbrace{\langle \chi_{V_s}, \chi_{\mathrm{Res}_{H_s}^H V} \rangle_{H_s}}_{= \dim_{\mathbb{C}} \mathrm{Hom}_{H_s}(V_s, \mathrm{Res}_{H_s}^H V) \geq 0}$

iff. (1) $\langle \chi_V, \chi_V \rangle_H = 1$ (i.e. V irred.), and

(2) $\langle \chi_{V_s}, \chi_{\mathrm{Res}^H_{H_s} V} \rangle_{H_s} = 0 \quad \forall s \in S - \{e\}$. $\square$

<u>Ex:</u> • $H = H' = \langle r \rangle \subset G = D_4 = \langle r, s \mid r^4 = s^2 = (rs)^2 = e \rangle$

$\Rightarrow H \backslash G / H = \{H, HsH\}$ has representatives $S = \{e, s\}$;

$H_e = H$ and $H_s = sHs^{-1} \cap H = \{ s r^i s^{-1} \in H \mid i = 1\_4 \} = H$

• V 1-dim. H-repr. with char $\chi : H \to \mathbb{C}^\times$
$\qquad r \mapsto \chi(r)$

Then

• $\chi_{V_s}(r^i) = \chi(s^{-1} r^i s) = \chi(r^{-i})$

• $\mathrm{Res}^H_{H_s} V = V$

• $\langle \chi, \chi \rangle_H = 1$ (condition (1))

$\chi(r)$	$\langle \chi_{V_s}, \chi \rangle_H =$	$\mathrm{Ind}_H^{D_4} V$ irred.?
1	$\frac{1}{4}[1^2 + 1^2 + 1^2 + 1^2] = 1$	No
-1	$\frac{1}{4}[1^2 + (-1)^2 + 1^2 + (-1)^2] = 1$	No
$i, -i$	$\frac{1}{4}[1^2 + \underbrace{(-i)\overline{i}}_{=-1} + (-1)^2 + \underbrace{i \cdot \overline{(-i)}}_{=-1}] = 0$	Yes