3 Induced representations

3.1 Definition

K any field

G finite group

$H < G$ subgroup

Def: • $\rho : G \to GL(V)$ G-repr.

The <u>restriction of (ρ, V) to H</u> is

$$\rho|_H : H \hookrightarrow G \xrightarrow{\rho} GL(V).$$

We also write $\mathrm{Res}_H^G V$ for this H-repr.

• $\sigma : H \to GL(V)$ H-repr.

The <u>induced representation of (σ, V) on G</u> is

$$\mathrm{Ind}_H^G V := \text{"} K[G] \otimes_{K[H]} V \text{"} := \overbrace{K[G] \otimes_K V}^{(\text{as } K\text{-v.s.})} \Big/ \left\langle \begin{array}{l} gh \otimes v - g \otimes h.v \\ \text{for } g \in G, h \in H, \\ v \in V \end{array} \right\rangle$$

together with the G-action

$$g \cdot \Big(\sum_i g_i \otimes v_i \Big) = \sum_i g g_i \otimes v_i .$$

Rem: If the context is clear, we sometimes write

$$\mathrm{Res}\, V = \mathrm{Res}_H^G V \quad \text{and} \quad \mathrm{Ind}\, V = \mathrm{Ind}_H^G V.$$

Lemma 1: $\mathrm{Ind}_H^G V$ is a well-defined G-repr.

proof: well-def: $g'.(gh\otimes v) = g'gh\otimes v = g'g\otimes h.v = g'.(g\otimes h.v)$

(1) $e.(\sum g_i\otimes v_i) = \sum eg_i\otimes v_i = \sum g_i\otimes v_i$

(2) $(gh).(\sum g_i\otimes v_i) = \sum ghg_i\otimes v_i = g.(\sum hg_i\otimes v_i)$

$$= g.(h.(\sum g_i\otimes v_i))$$

(3) & (4) $G \curvearrowright \mathrm{Ind}\, V$ is linear by definition. $\square$

Rem: (0) If $H=G$, then $\mathrm{Ind}\, V = V$.

(1) If $H=\{e\}$, then

$$\mathrm{Ind}\, V = K[G]\otimes_K V = V_{reg}\otimes_K V \underset{\uparrow}{\simeq} V_{reg}^{\dim V}$$

$(V\simeq K^{\dim V}$ trivial$)$

(2) Let $\{g_1 - g_r\}$ be a set of representatives for G/H. Then every element of $\mathrm{Ind}\, V$ can be written as $\sum_{i=1}^{r} g_i\otimes v_i$ for unique $v_1 - v_r \in V$.

In particular,

$$\dim_K(\mathrm{Ind}\, V) = r\cdot\dim_K V = [G:H]\cdot\dim_K V.$$

Ex: • $H = \langle r \rangle \subset D_4 = \langle r, s \mid r^4 = s^2 = (rs)^2 = e \rangle$

then: $[D_4 : H] = 2$ and $D_4 / H = \{[e], [s]\}$

• V 1-dim. complex H-repr.

$\chi : H \to \mathbb{C}^\times$ its character

$\Rightarrow \mathrm{Ind}_H^{D_4} V = \{ e \otimes v + s \otimes w \mid v, w \in \mathbb{C} \} \cong \mathbb{C}^2$ (as $\mathbb{C}$-v.s.)

$\mathbb{C}$-basis: $\{ e \otimes 1, s \otimes 1 \}$

$\rightsquigarrow$ have: • $r.(e \otimes v + s \otimes w) = r \otimes v + rs \otimes w = e \otimes r.v + s \otimes r^{-1}.w$ $(rs = sr^{-1})$

$= e \otimes \chi(r) v + s \otimes \chi(r^{-1}) w$

$$\Rightarrow A(r) = \begin{pmatrix} \chi(r) & 0 \\ 0 & \chi(r^{-1}) \end{pmatrix}$$

• $s.(e \otimes v + s \otimes w) = s \otimes v + s^2 \otimes w = e \otimes w + s \otimes v$

$$\Rightarrow A(s) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

conj. cl.	e	r^2	r, r^3	$s, r^2 s$	$rs, r^3 s$
χ_1	1	1	1	1	1
χ_2	1	1	1	-1	-1
χ_3	1	1	-1	1	-1
χ_4	1	1	-1	-1	1
χ_5	2	-2	0	0	0
$\chi_{\mathrm{Ind} V}$	2	$2\chi(r^2)$	$\chi(r) + \chi(r^{-1})$	0	0

$\rightsquigarrow$

$\chi(r)$	$\chi_{\mathrm{Ind} V} =$
1	$\chi_1 + \chi_2$
-1	$\chi_3 + \chi_4$
$\pm i$	χ_5

3.2 Frobenius reciprocity

K field

G finite group

$H < G$ subgroup

<u>Lemma 1:</u> Res_H^G and Ind_H^G are functorial:

(1) Given a G-hom. $f: V_1 \to V_2$, then

$$\mathrm{Res}\, f : \mathrm{Res}\, V_1 \longrightarrow \mathrm{Res}\, V_2$$
$$v \longmapsto f(v)$$

is an H-hom., which defines a functor

$$\mathrm{Res}_H^G : \mathrm{Rep}_K(G) \longrightarrow \mathrm{Rep}_K(H).$$

(2) Given an H-hom $f: V_1 \to V_2$, then

$$\mathrm{Ind}\, f : \mathrm{Ind}\, V_1 \longrightarrow \mathrm{Ind}\, V_2$$
$$g \otimes v \longmapsto g \otimes f(v)$$

is a G-hom, which defines a functor

$$\mathrm{Ind}_H^G : \mathrm{Rep}_K(H) \longrightarrow \mathrm{Rep}_K(G).$$

<u>proof:</u> (1) This is tautological.

(2) $\mathrm{Ind}\, f$ is clearly K-linear. It is G-equiv. since $\mathrm{Ind}\, f\,(g.(h \otimes v)) = \mathrm{Ind}\, f\,(gh \otimes v)$

$= gh \otimes f(v) = g.(h \otimes f(v)) = g.(\mathrm{Ind}\, f\,(h \otimes v)).$ $\square$

Thm. 2: Ind_H^G is left adjoint to Res_H^G, i.e. there are bijections

$$\Phi_{V,W} : \mathrm{Hom}_H(V, \mathrm{Res}\, W) \longrightarrow \mathrm{Hom}_G(\mathrm{Ind}\, V, W)$$

for all H-repr's V and all G-repr's W such that

$$\begin{tikzcd}
\mathrm{Hom}_H(V', \mathrm{Res}\, W) \arrow[r, "\Phi_{V',W}"] \arrow[d] & \mathrm{Hom}_G(\mathrm{Ind}\, V', W) \arrow[d] \\
\mathrm{Hom}_H(V, \mathrm{Res}\, W') \arrow[r, "\Phi_{V,W'}"] & \mathrm{Hom}_G(\mathrm{Ind}\, V, W')
\end{tikzcd}$$

$\alpha \mapsto \mathrm{Res}\, \varphi' \circ \alpha \circ \varphi$ $\qquad$ $\beta \mapsto \varphi' \circ \beta \circ \mathrm{Ind}\, \varphi$

commutes for all G-hom's $\varphi : V \to V'$ and all H-hom's $\varphi' : W \to W'$.

proof: • Definition of $\Phi_{V,W}$:

Consider an H-hom $\alpha : V \longrightarrow \mathrm{Res}\, W$

Define $\hat{\alpha} = \Phi_{V,W}(\alpha) : \mathrm{Ind}\, V \longrightarrow W.$

$$g \otimes v \longmapsto g.\alpha(v)$$

Note: $\Phi_{V,W}$ is K-linear.

claim: $\hat{\alpha}$ is a G-hom.

proof: • well-def: $\hat{\alpha}(gh \otimes v) = (gh).\alpha(v) = g.(h.\alpha(v))$
$= g.\alpha(h.v) = \hat{\alpha}(g \otimes h.v)$

• K-linear: clear

• G-equiv.: $\hat{\alpha}(g.(h \otimes v)) = \hat{\alpha}(gh \otimes v) = (gh).\alpha(v)$
$= g.(h.\alpha(v)) = g.\hat{\alpha}(h \otimes v).$ ⚡

- The unit η of the adjunction:

The K-linear map

$$\eta_V : V \longrightarrow \operatorname{Res}(\operatorname{Ind} V) = K[G] \otimes_{K[H]} V$$
$$v \longmapsto 1 \otimes v \qquad \text{("unit" of the adjunction)}$$

is an H-hom since

$$\eta.(h.v) = 1 \otimes h.v = h \otimes v = h.(1 \otimes v) = h.\eta(v).$$

- The inverse bijection $\Psi_{V,W}$:

Consider a G-hom. $\beta : \operatorname{Ind} V \to W$ and define

$$\check{\beta} = \Psi_{V,W}(\beta) : V \xrightarrow{\eta_V} \operatorname{Res} \operatorname{Ind} V \xrightarrow{\operatorname{Res}\beta} \operatorname{Res} W,$$
$$v \mapsto 1 \otimes v \longmapsto \beta(1 \otimes v)$$

which is an H-hom.

- $\Psi_{V,W} \circ \Phi_{V,W} = id$: Consider $\alpha : V \to \operatorname{Res} W$, $v \in V$,

$$(\hat{\alpha})^{\vee}(v) = \hat{\alpha}(1 \otimes v) = 1.\alpha(v) = \alpha(v)$$

$$\Rightarrow (\hat{\alpha})^{\vee} = \alpha$$

- $\Phi_{V,W} \circ \Psi_{V,W} = id$: Consider $\beta : \operatorname{Ind} V \to W$, $g \otimes v \in \operatorname{Ind} V$,

$$(\check{\beta})^{\wedge}(g \otimes v) = g.\check{\beta}(v) = g.\beta(1 \otimes v) = \beta(g \otimes v)$$

$$\Rightarrow (\check{\beta})^{\wedge} = \beta$$

Thus

$$\mathrm{Hom}_H(V, \mathrm{Res}\, W) \underset{\Psi_{V,W}}{\overset{\Phi_{V,W}}{\rightleftarrows}} \mathrm{Hom}_G(\mathrm{Ind}\, V, W)$$

are mutually inverse bijections.

- <u>Functoriality:</u>

Consider $\varphi : V \to V'$, $\varphi' : W' \to W$,

$\alpha : V' \to \mathrm{Res}\, W$ and $g \otimes v \in \mathrm{Ind}\, V$. Then

$$\begin{aligned}
\Phi_{V,W'}(\mathrm{Res}\, \varphi' \circ \alpha \circ \varphi)(g \otimes v) &= g.(\mathrm{Res}\, \varphi' \circ \alpha \circ \varphi(v)) \\
&= g.(\varphi' \circ \alpha(\varphi(v))) \\
&= \varphi'(g.\alpha(\varphi(v))) \\
&= \varphi' \circ \hat{\alpha}(g.(\mathrm{Ind}\, \varphi(1 \otimes v))) \\
&= (\varphi' \circ \Phi_{V',W}(\alpha) \circ \mathrm{Ind}\, \varphi)(g \otimes v)
\end{aligned}$$

Thus

$$\Phi_{V,W'}(\mathrm{Res}\, \varphi' \circ \alpha \circ \varphi) = \varphi' \circ \Phi_{V',W}(\alpha) \circ \mathrm{Ind}\, \varphi,$$

as desired. $\square$

<u>Rem:</u> $\Phi_{V,W}$ is in fact a bijective K-linear map, and thus an isomorphism of K-vector spaces.

Thm 3: (Frobenius reciprocity)

$K = \mathbb{C}$

V H-repr. with character χ_V

W G-repr. with character χ_W

Then $\langle \chi_V, \chi_{\mathrm{Res}\, W} \rangle_H = \langle \chi_{\mathrm{Ind}\, V}, \chi_W \rangle_G$.

proof:

$$\begin{aligned}
\langle \chi_V, \chi_{\mathrm{Res}\, W} \rangle_H &\underset{(\text{Thm. } 2.2.1)}{=} \dim_{\mathbb{C}} \mathrm{Hom}_H(V, \mathrm{Res}\, W) \\
&\underset{(\text{Thm. } 2)}{=} \dim_{\mathbb{C}} \mathrm{Hom}_G(\mathrm{Ind}\, V, W) \\
&\underset{(\text{Thm. } 2.2.1)}{=} \langle \chi_{\mathrm{Ind}\, V}, \chi_W \rangle_G .
\end{aligned}$$

$\square$

Cor 4: $K = \mathbb{C}$

$V_1 - V_s$ irred. G-repr. up to isom.

W H-repr.

Then
$$\chi_{\mathrm{Ind}\, W} = \sum_{i=1}^{s} \langle \chi_V, \chi_{\mathrm{Res}\, V_i} \rangle_H \cdot \chi_{V_i} .$$

proof: Exercise !

Ex: $H = \langle r \rangle \subset D_4$

V 1-dim. H-repr.

$\chi : H \longrightarrow \mathbb{C}^\times$ its character
$r \longmapsto \chi(r)$

Then

$$\chi_{\mathrm{Ind}\, V} = \sum_{i=1}^{5} \left(\frac{1}{4} \sum_{j=1}^{4} \chi(r^j) \cdot \chi_i(r^j) \right) \cdot \chi_i$$

conj. cl. of H	e	r	r^2	r^3
χ_1	1	1	1	1
χ_2	1	1	1	1
χ_3	1	-1	1	-1
χ_4	1	-1	1	-1
χ_5	2	0	-2	0