2.2 Orthogonality (continued)

G finite group of order n

$K = \mathbb{C}$

Thm 7: The simple characters of G form an orthonormal basis for the space $\mathbb{C}^G = \{\alpha : C(G) \to \mathbb{C}\}$ of class functions on G.

In particular, $\#\{\text{simple characters of } G\} = c(G)$.

proof: Let $\chi_1 - \chi_s$ be the simple characters of G.

By Cor. 2, $\chi_1 - \chi_s$ are orthogonal and thus linearly independent. We are left with showing that $\chi_1 - \chi_s$ span $\mathbb{C}^{C(G)}$.

- Consider any class fct. $\alpha : C(G) \to \mathbb{C}$.

Then

$$\beta = \alpha - \sum_{i=1}^{s} \langle \alpha, \chi_i \rangle \cdot \chi_i$$

is also a class function and satisfies for all $j = 1 - s$,

$$\begin{aligned} \langle \beta, \chi_j \rangle &= \langle \alpha, \chi_j \rangle - \sum_{i=1}^{s} \langle \alpha, \chi_i \rangle \cdot \underbrace{\langle \chi_i, \chi_j \rangle}_{= \delta_{i,j}} \\ &= \langle \alpha, \chi_j \rangle - \langle \alpha, \chi_j \rangle \\ &= 0. \end{aligned}$$

Claim: $\beta = 0$. Then: $\alpha = \sum_{i=1}^{s} \langle \alpha, \chi_i \rangle \chi_i$ is in the span of $\chi_1 - \chi_s$ $\Rightarrow$ $\mathbb{C}^{C(G)} = \langle \chi_1 - \chi_s \rangle_{\mathbb{C}}$.

Proof:

For a G-repr. $\rho: G \to GL(V)$, define

$$\rho_\beta = \sum_{g \in G} \beta(g) \rho(g) : V \longrightarrow V,$$
$$v \longmapsto \sum_{g \in G} \beta(g) \cdot (g.v)$$

which is G-equivariant since

$$\rho_\beta(g.v) = \sum_{h \in G} \beta(h) \cdot (h.(g.v))$$
$$= g.\Big(\sum_{h \in G} \underbrace{\beta(g^{-1}hg)}_{\beta \text{ is a class fct.}} \cdot ((g^{-1}hg).v) \Big)$$
$$\underset{(h' = g^{-1}hg)}{=} g.\Big(\sum_{h' \in G} \beta(h') \cdot (h'.v) \Big)$$
$$= g.\rho_\beta(v).$$

Thus $\rho_\beta \in \mathrm{Hom}_G(V, V)$.

- Let $\rho_1 - \rho_s$ be irred. G-repr. with $\chi_{\rho_i} = \chi_i$.

For $\rho = \rho_i$, we have $\rho_{i,\beta} = \lambda_i \cdot \mathrm{id}$ for some $\lambda_i \in \mathbb{C}$ by Schur's Lemma.

Thus

$$\underbrace{(\dim \rho_i)}_{= d_i > 0} \cdot \lambda_i = tr(\rho_{i,\beta})$$

$$= \sum_{g \in G} \beta(g) \cdot tr(\rho_i(g))$$

$$= \sum_{g \in G} \beta(g) \cdot \chi_i(g)$$

$$= n \cdot \langle \beta, \chi_i^* \rangle$$

$$= 0.$$

↑ character of the contragredient representation ρ_i^* since $\chi_i^*(g) = \sum \mu_i^{-1} = \overline{\sum \mu_i} = \overline{\chi_i(g)}$,

$\mu_1 - \mu_{d_i} \in \mu_n(\mathbb{C})$ eigenvalues of g.

Thus $\lambda_i = 0$ and $\rho_{i,\beta} = 0$.

- For the regular representation $\rho = \rho_{reg} \simeq \bigoplus \rho_i^{d_i} \curvearrowright \mathbb{C}^G$, (Cor. 6) we have

canonical basis:

$$\left\{ \delta_g : \begin{array}{l} G \longrightarrow \mathbb{C} \\ h \mapsto \delta_{gh} \end{array} \right\}_{g \in G}$$

$$\sum_{g \in G} \beta(g) \cdot \delta_g = \sum_{g \in G} \beta(g) \cdot (\rho(g) . \delta_e)$$

$$= \rho_\beta(\delta_e)$$

$$= \sum_{i=1}^{s} d_i \cdot \rho_{i,\beta}(\delta_e) = 0.$$

Since $(\delta_g)_{g \in G}$ forms a basis for $\mathbb{C}^G$,

$\beta(g) = 0$ for all $g \in G$. Thus $\beta = 0$. $\square$

Def: The <u>character table of G</u> is the table

	c_1	$\cdots$	c_s
χ_1	$\chi_1(c_1)$	$\cdots$	$\chi_1(c_s)$
$\vdots$	$\vdots$	$\ddots$	$\vdots$
χ_s	$\chi_s(c_1)$	$\cdots$	$\chi_s(c_s)$

where

- $s = c(G)$
- $C(G) = \{c_1, \dots, c_s\}$
- $\chi_1, \dots, \chi_s$ are the simple characters of G.

Rem: Typically we choose:

- $c_1 = \{e\}$
- $\chi_1 = \chi_{triv}$, the trivial character
- $\chi_1, \dots, \chi_s$ in increasing order of dim's $\chi_1(e) \le \dots \le \chi_s(e)$.

Prop 8: $C(G) = \{c_1, \dots, c_s\}$ where $s = c(G)$

$\chi_1, \dots, \chi_s$ simple characters of G

Then we have for all $i, j \in \{1, \dots, s\}$:

(1) <u>Row orthogonality</u>:

$$\sum_{k=1}^{s} \frac{\# c_k}{n} \cdot \chi_i(c_k) \cdot \overline{\chi_j(c_k)} = \delta_{ij};$$

(2) <u>Column orthogonality</u>:

$$\sum_{k=1}^{s} \frac{\# c_j}{n} \cdot \chi_k(c_i) \cdot \overline{\chi_k(c_j)} = \delta_{ij}.$$

proof: (1): $\sum_{k=1}^{s} \frac{\# c_k}{n} \cdot \chi_i(c_k) \cdot \overline{\chi_j(c_k)} = \frac{1}{n} \cdot \sum_{g \in G} \chi_i(g) \cdot \overline{\chi_j(g)}$

$= \langle \chi_i, \chi_j \rangle = \delta_{ij}$.

(2): Let $\delta_{c_j} : G \to \mathbb{C}$ be the characteristic function of c_j.

$g \mapsto \begin{cases} 1 & : \text{if } g \in c_j \\ 0 & : \text{if } g \notin c_j \end{cases}$

Then $\delta_{c_j} = \sum_{k=1}^{s} a_k \chi_k$ for

$a_k = \langle \delta_{c_j}, \chi_k \rangle = \frac{1}{n} \sum_{g \in G} \delta_{c_j}(g) \overline{\chi_k(g)} = \frac{\# c_j}{n} \cdot \overline{\chi_k(c_j)}$.

$\Rightarrow \sum_{k=1}^{s} \frac{\# c_j}{n} \cdot \chi_k(c_i) \cdot \overline{\chi_k(c_j)} = \sum_{k=1}^{s} a_k \chi_k(c_i) = \delta_{c_j}(c_i) = \delta_{ij}$. □