2.2 Orthogonality

Standing assumption for this section:

G finite group of order n

$K = \mathbb{C}$

Def: For two functions $\alpha, \beta : G \to \mathbb{C}$, we define

complex conjugate

$$\langle \alpha, \beta \rangle = \frac{1}{n} \cdot \sum_{g \in G} \alpha(g)\, \overline{\beta(g)}.$$

Rem: $\langle -,- \rangle$ is the standard Hermitian form on

$$\mathbb{C}^G = \{ \alpha : G \to \mathbb{C} \}$$

w.r.t. the canonical basis $\{ \delta_g : G \to \mathbb{C},\ h \mapsto \delta_{g,h} \}_{g \in G}$.

In particular, $\langle -,- \rangle$ is:

- sesquilinear: $\langle \alpha + \alpha', \beta \rangle = \langle \alpha, \beta \rangle + \langle \alpha', \beta \rangle$

  $\langle \alpha, \beta + \beta' \rangle = \langle \alpha, \beta \rangle + \langle \alpha, \beta' \rangle$

  $\langle \lambda \alpha, \beta \rangle = \lambda \langle \alpha, \beta \rangle = \langle \alpha, \overline{\lambda} \beta \rangle$

- conjugate symmetric: $\langle \alpha, \beta \rangle = \overline{\langle \beta, \alpha \rangle}$
- positive definite: $\langle \alpha, \alpha \rangle \in \mathbb{R}_{>0} \quad \forall \alpha \neq 0$

Thm 1: V, W G-repr. with respective characters χ_V and χ_W.

Then $$\langle \chi_V, \chi_W \rangle = \underbrace{\dim \operatorname{Hom}_G(W, V)}_{\in \mathbb{N} \subset \mathbb{C}}.$$

proof: • Let $g \in G$. Choose bases of eigenvectors for g $\{e_1 - e_d\}$ of V and $\{f_1 - f_c\}$ of W with respective eigenvalues $\lambda_1 - \lambda_d$ and $\mu_1 - \mu_c$.

Then
$$\left\{ \begin{array}{l} \alpha_{ij} : W \longrightarrow V \\ \qquad f_k \longmapsto \delta_{jk} e_i \end{array} \right\}_{\substack{i=1-d \\ j=1-c}}$$
is a basis for $\mathrm{Hom}_{\mathbb{C}}(W, V)$.

• Since
$$(g.\alpha_{ij})(\Sigma \underbrace{a_k}_{\in \mathbb{C}} f_k) = g.\left(\alpha_{ij}\left(g^{-1}.(\Sigma a_k f_k)\right)\right)$$
$$= g.\left(\alpha_{ij}(\Sigma \mu_k^{-1} a_k f_k)\right)$$
$$= g.(\mu_j^{-1} a_j e_i)$$
$$= \lambda_i \mu_j^{-1} a_j e_i$$
$$= \lambda_i \mu_j^{-1} \alpha_{ij}(\Sigma a_k f_k),$$

we conclude that $g.\alpha_{ij} = \lambda_i \mu_j^{-1} \cdot \alpha_{ij}$.

Thus $\{\alpha_{ij}\}_{ij}$ is a basis of eigenvectors for $\mathrm{Hom}_{\mathbb{C}}(W, V)$, with respective eigenvalues $\lambda_i \mu_j^{-1}$.

$\Rightarrow$ (for $\rho : G \to \mathrm{Hom}_{\mathbb{C}}(W, V)$)

$$tr(\rho(g)) = \sum_{i,j} \lambda_i \mu_j^{-1} = (\Sigma_i \lambda_i) \cdot \underbrace{(\Sigma_j \mu_j^{-1})}_{= \Sigma \overline{\mu_j} = \overline{\Sigma \mu_j}}$$

$$= \chi_V(g) \cdot \overline{\chi_W(g)}.$$

• Consider

$$\pi = \frac{1}{n} \sum_{g \in G} \rho(g) : \mathrm{Hom}_{\mathbb{C}}(W,V) \longrightarrow \mathrm{Hom}_{\mathbb{C}}(W,V)$$
$$\varphi \longmapsto \frac{1}{n} \sum g.\varphi$$

Then
$$\mathrm{tr}\, \pi = \frac{1}{n} \sum_{g \in G} \mathrm{tr}(\rho(g))$$
$$= \frac{1}{n} \sum \chi_V(g) \cdot \overline{\chi_W(g)}$$
$$= \langle \chi_V, \chi_W \rangle .$$

By Exercise 1 of List 11, π is a projection onto
$$\mathrm{Hom}_{\mathbb{C}}(W,V)^G \underset{\substack{\uparrow \\ \text{(cf. section 1.1)}}}{=} \mathrm{Hom}_G(W,V).$$

Thus $\mathrm{tr}\, \pi = \dim(\mathrm{im}\, \pi) = \dim \mathrm{Hom}_G(W,V)$. □

<u>Def</u>: A <u>simple character of G</u> is a character of an irreducible representation of G.

The <u>dimension of a character χ</u> is $\dim \chi = \chi(e)$.

(i.e. $\dim \chi_V = \dim V$)

<u>Cor 2</u>: V, W irred. G-repr.

Then
$$\langle \chi_V, \chi_W \rangle = \begin{cases} 1 & \text{if } V \simeq W, \\ 0 & \text{if not.} \end{cases}$$

<u>proof</u>: By Schur's Lemma, $\mathrm{Hom}_G(W,V) = \begin{cases} \mathbb{C} & \text{if } V \simeq W, \\ 0 & \text{if not.} \end{cases}$

Thus $\langle \chi_V, \chi_W \rangle \underset{\text{(Thm. 1)}}{=} \dim \mathrm{Hom}_G(W,V)$ is as claimed. □

<u>Cor 3</u>: The number of isomorphism classes of irred. G-repr. is at most equal to

$$c(G) = \# C(G) = \#\{\text{conj. classes in } G\}.$$

<u>proof</u>: • Every (simple) character is an element of the $c(G)$-dimensional $\mathbb{C}$-vector space $\mathbb{C}^{C(G)} = \{ f : C(G) \to \mathbb{C} \}$.

• By Cor. 2, the simple characters are pw. orthogonal and thus linearly independent. Thus

$$\#\{\text{isom.cl. of irred. repr.}\} \le \dim \mathbb{C}^{C(G)} = c(G).$$

□

<u>Cor 4</u>: V G-repr. with character χ_V

Then V is irreducible iff. $\langle \chi_V, \chi_V \rangle = 1$.

<u>proof</u>: • For $V = \{0\}$, $\chi_V(g) = 0$ for all $g \in G$ and thus $\langle \chi_V, \chi_V \rangle = 0$.

• If V is irred., then

$$\langle \chi_V, \chi_V \rangle = \dim_{\mathbb{C}} \mathrm{Hom}_G(V,V) = 1$$

• If $V \simeq \bigoplus_{i=1}^{r} W_i$ with W_i irred. and $r \geq 2$, then

$$\langle \chi_V, \chi_V \rangle = \sum_{i,j} \underbrace{\langle \chi_{W_i}, \chi_{W_j} \rangle}_{\geq 0}$$

$$\geq \underbrace{\langle \chi_{W_1}, \chi_{W_1} \rangle}_{=1} + \dots + \underbrace{\langle \chi_{W_r}, \chi_{W_r} \rangle}_{=1} = r \geq 2.$$

□

Cor. 5: V, W G-repr.

Then $V \simeq W$ iff. $\chi_V = \chi_W$.

proof "$\Rightarrow$" If $\exists$ G-isom. $\varphi: V \xrightarrow{\sim} W$, then for all $g \in G$,

$$\chi_V(g) = tr(\rho_V(g)) = tr(\varphi^{-1} \circ \rho_W(g) \circ \varphi) = tr(\rho_W(g)) = \chi_W(g).$$

"$\Leftarrow$" If $\chi_V = \chi_W$, then consider decompositions

$$V \simeq \bigoplus_{i=1}^{s} V_i \quad \text{and} \quad W \simeq \bigoplus_{j=1}^{r} W_j$$

into irred. G-repr. $V_1, \ldots, V_s, W_1, \ldots, W_r$, and

let U be an irred. G-repr. Then

$$\# \{ i \in \{1, \ldots, s\} \mid V_i \simeq U \} = \sum_{i=1}^{s} \langle \chi_U, \chi_{V_i} \rangle$$

(Cor. 2)

$$= \langle \chi_U, \chi_V \rangle$$
$$= \langle \chi_U, \chi_W \rangle$$
$$= \# \{ j \in \{1, \ldots, r\} \mid W_j \simeq U \}$$

Thus $V \simeq \bigoplus V_i \simeq \bigoplus W_j \simeq W$. $\square$

Cor 6: $\mathbb{C}^G$ regular G-repr.

$\chi_{reg} = \chi_{\mathbb{C}^G}$ its character

Then $\mathbb{C}^G \simeq \bigoplus_{i=1}^{s} V_i^{d_i}$

where $V_1 \ldots V_s$ is a complete system of representatives of the isomorphism classes $[V_1] \ldots [V_s]$ of irred. G-repr. and $d_i = \dim_{\mathbb{C}} V_i$.

In part.,

$$\chi_{reg} = \sum_{\chi \text{ simple}} (\dim \chi) \cdot \chi \quad \text{and} \quad n = \sum_{\chi \text{ simple}} (\dim \chi)^2.$$

proof: • $\langle \chi_{V_i}, \chi_{reg} \rangle = \frac{1}{n} \sum_{g \in G} \chi_{V_i}(g) \cdot \underbrace{\overline{\chi_{reg}(g)}}_{= \begin{cases} n & \text{if } g = e \\ 0 & \text{if } g \neq e \end{cases} \text{ (Ex. in 2.1)}}$

$$= \frac{1}{n} \cdot \chi_{V_i}(e) \cdot n = \dim V_i$$

(Prop. 2.1.1)

• Let $\mathbb{C}^G \simeq \bigoplus_{j=1}^{r} W_j$ be a decomposition into irred. repr. W_j. Then $\chi_{reg} = \sum_{j=1}^{r} \chi_{W_j}$

and $\underbrace{\langle \chi_{V_i}, \chi_{reg} \rangle}^{= \dim V_i = d_i} = \sum_{j=1}^{r} \langle \chi_{V_i}, \chi_{W_j} \rangle = \# \{ j \in \{1 \ldots r\} \mid V_i \simeq W_j \}.$

$$\Rightarrow \mathbb{C}^G \simeq \bigoplus_{j=1}^{r} W_j \simeq \bigoplus_{i=1}^{s} V_i^{d_i}.$$

• By Prop. 2.1.2, $\chi_{reg} = \sum d_i \cdot \chi_{V_i} = \sum_{\chi \text{ simple}} (\dim \chi) \cdot \chi$.

By Prop. 2.1.1, $n = \chi_{reg}(e) = \sum (\dim \chi) \cdot \chi(e) = \sum (\dim \chi)^2$. $\square$

<u>Exercise:</u> $m = \# G^{ab}$

Then G has precisely m isom. cl. of 1-dim. repr.,

and G has an irred. repr. of dim. ≥ 2

iff. G is not abelian.

<u>Ex:</u> Simple characters of S_3:

- By Cor. 6, $6 = \# S_3 = \sum (\dim \chi)^2 = \underbrace{1^2 + 1^2 + 2^2}$

$\Rightarrow \chi_{reg} = \underset{\text{1-dim.}}{\chi_1 + \chi_2} + 2 \underset{\text{2-dim.}}{\chi_3}$ (only possibility as S_3 is not abelian)

- trivial (1-dim.) character $\chi_1 = \chi_{triv} : g \mapsto 1 \quad (\forall g \in G)$

- $S_3^{ab} = S_3 / \langle (123) \rangle \simeq \mathbb{Z}/2\mathbb{Z} \xrightarrow{\chi_2 = \chi_{sign}} \mathbb{C}^\times$

$\qquad i \longmapsto (-1)^i$

- <u>recall:</u> $C(S_3) = \{[e], [(12)], [(123)]\}$

<u>character table of S_3:</u>

simple characters \ conj. classes	e	(12)	(123)
χ_{triv}	1	1	1
χ_{sign}	1	-1	1
χ_3	2	0	-1
(χ_{reg}	6	0	0)

$\left(\chi_3 = \frac{1}{2}(\chi_{reg} - \chi_{triv} - \chi_{sign})\right)$

<u>double check:</u> $\chi_3 = \chi_V$ for $G \curvearrowright V = \mathbb{C}^2$ via:

$\rho(e) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \rho(12) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \rho(123) = \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix};$

<u>traces:</u> $\chi_V(e) = 2, \quad \chi_V(12) = 0, \quad \chi_V((123)) = -1.$