2 Character theory

2.1 Definitions and first properties

G finite group of order n

K field (often required: • $\#\mu_n(K) = n$
• K algebraically closed)

<u>Def</u>: $\rho : G \longrightarrow GL(V)$ repr.

• The <u>character of ρ</u> is the function

$$\chi = \chi_V : \begin{array}{ccc} G & \longrightarrow & \mathbb{C} \\ g & \longmapsto & tr(\rho(g)) \end{array}$$

• A function $f : G \to \mathbb{C}$ is a <u>class function</u> if it is constant on conjugacy classes, i.e.

$$f(ghg^{-1}) = f(h)$$

for all $g, h \in G$.

<u>Ex</u>: If $\dim V = 1$, then $\chi = \rho : G \longrightarrow K^\times \subset K$ since $tr(a) = a$.
(1×1)-matrix

<u>Rem</u>: $C(h) := \{ghg^{-1} \mid g \in G\}$ <u>conjugacy class of h</u>

$\mathcal{C}(G) := \{C(h) \mid h \in G\}$ set of conjugacy classes in G

Then a function $f : G \to \mathbb{C}$ is a class function iff. it factors into

$$\begin{tikzcd} G \arrow[r, "f"] \arrow[d, "h \mapsto C(h)"'] & \mathbb{C} \\ \mathcal{C}(G) \arrow[ur, dashed, "\exists \bar{f}"'] & \end{tikzcd}$$

(necessarily uniquely!).

Prop 1: $\rho: G \to GL(V)$ repr. of dim. d

$\chi: G \to K$ its character

$\varepsilon = \exp(G) = \mathrm{lcm}\{\mathrm{ord}(g) \mid g \in G\}$ exponent of G

Then $(= \min\{i \geq 1 \mid g^i = e \ \forall g \in G\})$

(1) $\chi(e) = d$.

(2) χ is a class function.

(3) If $\#\mu_\varepsilon(K) = \varepsilon$, then the eigenvalues of $\rho(g)$ are contained in $\mu_\varepsilon(K)$ and $\rho(g)$ is diagonalizable for every $g \in G$, i.e. $\rho(g)$ is conjugated to a diagonal matrix ($\iff V$ has a basis of eigenvectors).

(4) If $K \subset \mathbb{C}$, then $\chi(g^{-1}) = \overline{\chi(g)}$.

$\uparrow$ complex conjugate

proof: (1) $\chi(e) = \mathrm{tr}\begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix} = \dim V = d$.

(2) $\chi(ghg^{-1}) = \mathrm{tr}(\rho(g)\rho(h)\rho(g)^{-1}) = \mathrm{tr}(\rho(h)) = \chi(h)$.

(3) For all $g \in G$, we have $g^\varepsilon = 1$.

prim. ε-th root of 1

Thus the min. pol. f of $\rho(g)$ divides $T^\varepsilon - 1 = \prod_{i=1}^{\varepsilon}(T - \zeta_\varepsilon^i)$,

and therefore the eigenvalues of g, which are the roots of f, are contained in $\mu_\varepsilon(K)$, and f decomposes into pairwise distinct linear factors (f is separable). Thus V decomposes into a direct sum of 1-dim. eigenspaces and has a basis of eigenvectors.

Note: $A(g)$ is diagonal w.r.t. to this basis.

(4) Wlog, we can assume that $K = \mathbb{C}$. Then

$$\rho(g) \sim \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_d \end{pmatrix} \Rightarrow \rho(g^{-1}) \sim \begin{pmatrix} \lambda_1^{-1} & & \\ & \ddots & \\ & & \lambda_d^{-1} \end{pmatrix}$$

Since $\lambda_i \in \mu_\varepsilon(\mathbb{C})$, we have $\lambda_i^{-1} = \overline{\lambda_i}$.

Thus $\chi(g^{-1}) = \lambda_1^{-1} + \dots + \lambda_d^{-1} = \overline{\lambda_1} + \dots + \overline{\lambda_d} = \overline{\chi(g)}$. □

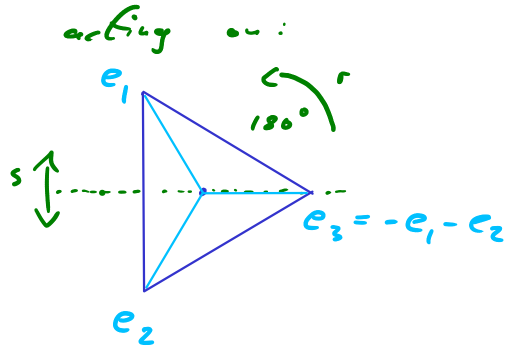
<u>Rem:</u> If $h' = ghg^{-1}$, then $\rho(h')$ and $\rho(h)$ have the same eigenvalues, char. pol. and min. pol.; it suffices thus to study a representative for each conj. cl.

<u>Ex:</u> $D_3 = \langle r, s \mid r^3 = s^2 = (rs)^2 = e \rangle \cong S_3$

$r \leftrightarrow (1\,2\,3)$

$s \leftrightarrow (1\,2)$

acting on:



- $\exp D_3 = 2 \cdot 3 = 6 = \# D_3$
- <u>conj. cl.</u>: $\{e\}$, $\{r, r^2\}$, $\{s, rs, r^2 s\}$
- In the basis (e_1, e_2) of $\mathbb{C}^2$, we have:

repr. matrix	trace	char. pol.	eigenvalues	min. pol.
$A(e) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	$\chi(e) = 2$	$T^2 - 2T + 1$	1	$T - 1$
$A(s) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$	$\chi(s) = 0$	$T^2 - 1$	$1, -1$	$T^2 - 1$
$A(r) = \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}$	$\chi(r) = -1$	$T^2 - T + 1$	ζ_3, ζ_3^2	$T^2 - T + 1$

<u>Prop 2:</u> V, W G-repr.; $g \in G$; assume that $\# \mu_\varepsilon(K) = \varepsilon := \exp(G)$.

Then (1) $\chi_{V \oplus W}(g) = \chi_V(g) + \chi_W(g)$,

(2) $\chi_{V \otimes W}(g) = \chi_V(g) \cdot \chi_W(g)$,

(3) $\chi_{V^*}(g) = \chi_V(g^{-1})$.

proof: Choose bases of eigenvectors (e_i) of V and (f_j) of W for $\rho_V(g)$ and $\rho_W(g)$, resp.

Then $A_V(g) = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_d \end{pmatrix}$ and $A_W(g) = \begin{pmatrix} \mu_1 & & \\ & \ddots & \\ & & \mu_c \end{pmatrix}$

w.r.t. these bases where $d = \dim V$ and $c = \dim W$.

(1) $(e_1, \dots, e_d, f_1, \dots, f_c)$ is a basis of $V \oplus W$,

and
$$A_{V\oplus W}(g) = \begin{pmatrix} \lambda_1 & & & & \\ & \ddots & & & \\ & & \lambda_d & & \\ & & & \mu_1 & \\ & & & & \ddots \\ & & & & & \mu_c \end{pmatrix}$$

$$\Rightarrow \chi_{V\oplus W}(g) = tr\left(A_{V\oplus W}\, g\right) = \sum \lambda_i + \sum \mu_j = \chi_V(g) + \chi_W(g).$$

(2) $(e_i \otimes f_j)_{i,j}$ is a basis of $V \otimes W$

$$\Rightarrow g.(e_i \otimes f_j) = (g.e_i) \otimes (g.f_j) = \lambda_i \mu_j \cdot (e_i \otimes f_j)$$

$$\Rightarrow \chi_{V\otimes W}(g) = \sum_{i,j} \lambda_i \mu_j = \left(\sum \lambda_i\right) \cdot \left(\sum \mu_j\right) = \chi_V(g) \cdot \chi_W(g).$$

(3) $$\underset{\text{(Lemma 1.4.6)}}{A_{V^*}(g) = A_V(g^{-1})^T} = \begin{pmatrix} \lambda_1^{-1} & & \\ & \ddots & \\ & & \lambda_d^{-1} \end{pmatrix} = A_V(g^{-1})$$

$$\Rightarrow \chi_{V^*}(g) = \chi_V(g^{-1})$$

□

Cor 3: $\# \mu_\varepsilon(k) = \varepsilon = \exp G$
$\dim V = 1$ $\qquad$ ↓ triv. 1-dim. G-repr.
Then $V \otimes V^* \simeq K$.

proof: · $\dim V \otimes V^* = (\dim V) \cdot (\dim V^*) = 1$
· $\chi(g) = \rho(g) = (a_{11}(g))$ for all $g \in G$ if $\dim \rho = 1$.

$$\Rightarrow \rho_{V\otimes V^*}(g) = \chi_{V\otimes V^*}(g) = \chi_V(g) \cdot \chi_V(g^{-1}) = (a_{11}(g) \cdot a_{11}(g)^{-1}) = (1)$$

$\Rightarrow V \otimes V^*$ triv.

□

Lemma 4: S set, $G \curvearrowright S$

K^S permutation repr.

χ its character

$S^g = \{x \in S \mid g.x = x\}$ for $g \in G$

Then $\chi(g) = \# S^g$.

proof: Let $A(g) = (a_{xy}(g))$ be the representation matrix of g w.r.t. the canonical basis $\left\{ \delta_x : \begin{array}{l} K^S \to K \\ y \mapsto \delta_{xy} \end{array} \right\}$ of K^S. Then

$$a_{x,x}(g) = \begin{cases} 1 & \text{if } g.x = x \\ 0 & \text{if } g.x \neq x \end{cases}$$

(cf. Examples in section 1.1)

Thus $\chi(g) = \operatorname{tr}\left(a_{x,y}(g)\right)_{x,y} = \sum_{x \in S} a_{x,x}(g) = \# S^g$. $\square$

Ex: (1) K^G regular repr. of G (given by $g.h = gh$)

χ its character

Then $G^g = \begin{cases} G & \text{if } g = e \\ \emptyset & \text{if } g \neq e \end{cases}$

Thus $\chi(g) = \begin{cases} \# G & \text{if } g = e \\ 0 & \text{if } g \neq e \end{cases}$

(2) $S_3 \curvearrowright \{1,2,3\}$

K^3 permutation repr.

χ its character

$\rightsquigarrow$

g	X^g	$\chi(g)$
e	$\{1,2,3\}$	3
(12)	$\{3\}$	1
(123)	$\emptyset$	0