1.4 Categorical properties (continued)

<u>Prop 2:</u> $f: V \to W$ G-hom.

Then

(1) $\ker f \overset{\iota}{\hookrightarrow} V$ is the (categorical) kernel of f;

(2) $W \overset{\pi}{\twoheadrightarrow} \operatorname{coker} f := W/\operatorname{im} f$ is the (categorical) cokernel of f;

(3) $\operatorname{im} f = \ker(W \overset{\pi}{\to} \operatorname{coker} f)$ as subrepr. of W.

<u>proof:</u> <u>(1):</u> Consider $g: U \to V$ s.t. $f\circ g = 0 : U \xrightarrow{g} V \xrightarrow{f} W$.

Then $f(g(u)) = 0$ for all $u \in U \Rightarrow g(u) \in \ker f$.

$\Rightarrow \operatorname{im} g \subset \ker f$, i.e. g factors into

$$\begin{tikzcd} \ker f \arrow[r, hook, "\iota"] & V \\ U \arrow[u, "\tilde{g}"] \arrow[ur, "g"'] & \end{tikzcd}$$

Since ι is injective, this factorization is unique (i.e. $\exists ! \tilde{g}$ s.t. $g = \iota \circ \tilde{g}$). $\checkmark$

(2) Given $g: W \to U$ with $g \circ f = 0 : V \xrightarrow{f} W \xrightarrow{g} U$,

we have $g(f(v)) = 0$ for all $v \in V$.

Thus g factors into

$$\begin{tikzcd} W \arrow[rr, "g"] \arrow[dr, two heads, "\pi"'] & & U \\ & W/\operatorname{im} f = \operatorname{coker} f \arrow[ur, "\bar{g}"'] & \end{tikzcd}$$

Since π is surj., $\bar{g}$ is unique.

(3) $\ker(W \overset{\pi}{\twoheadrightarrow} \operatorname{coker} f) = \pi^{-1}(0) = \operatorname{im} f$.

$\square$

Exercise: $f: V \to W$ G-hom.

Then (1) f is a monomorphism iff. f is injective.

(2) f is an epimorphism iff. f is surjective.

(3) f is an isomorphism iff. f is bijective.

Lemma 3: $f: V \to W$ G-hom

Then the induced G-hom. $\bar{f}: V/\ker f \to \operatorname{im} f$

is an isomorphism.

proof: $\bar{f}$ is bijective and thus an isomorphism. $\square$

Rem: (1) $\operatorname{Hom}_G(V, W)$ is an abelian group w.r.t. $f+g: v \mapsto f(v)+g(v)$

(in fact a K-v.s. w.r.t. $a.f: v \mapsto a.f(v)$);

the composition of G-hom's is bilinear:

- $(f_1+f_2)\circ g = f_1\circ g + f_2 \circ g$
- $f\circ(g_1+g_2) = f\circ g_1 + f\circ g_2$

$\Rightarrow$ $\mathrm{Rep}_K(G)$ is pre-additive.

(2) $\mathrm{Rep}_K(G)$ has finite biproducts.

$\Rightarrow$ $\mathrm{Rep}_K(G)$ is additive.

(3) Every morphism has a kernel and a cokernel.

$\Rightarrow$ $\mathrm{Rep}_K(G)$ is pre-abelian.

(4) For every $f: V \to W$, $V/\ker f \simeq \operatorname{im} f$.

$\Rightarrow$ $\mathrm{Rep}_K(G)$ is abelian.

Thm 4: Assume that char $K \nmid \# G$.

Then $\mathrm{Rep}_K(G)$ is semisimple, i.e. every G-repr. V is isomorphic to $\bigoplus_{i=1}^{r} V_i$ for uniquely determined irreducible repr. $V_1 - V_r$, up to isom. and a permutation of indices.

proof: Existence of $V \simeq \bigoplus_{i=1}^{r} V_i$:

By induction on $d = \dim_K V$:

$d=0$: Then $V = \{0\} = \bigoplus_{i=1}^{0} V_i$.

$d>0$: • If V is irreducible, then $V \simeq \bigoplus_{i=1}^{r} V_i$ for $r=1$ and $V_1 = V$.

• If V is not irreducible, then it has an irreducible subrepr. W. By Maschke's thm (Thm. 1.2.4), $\exists\, \pi: V \to W$ s.t. $\pi|_W = \mathrm{id}_W$, and thus $V \simeq W \oplus \ker \pi$.

Since $\dim_K(\ker \pi) < \dim_K(W \oplus \ker \pi) = d$, $\ker \pi \simeq \bigoplus_{i=1}^{r-1} V_i$ with V_i irred. by IH. Thus

$$V \simeq W \oplus \ker \pi \simeq \bigoplus_{i=1}^{r} V_i$$

for $V_r = W$.

Uniqueness of $V_1 \ldots V_r$:

Assume that
$$\bigoplus_{i=1}^{r} V_i \simeq \bigoplus_{i=1}^{s} W_i$$

for irred. $V_1 \ldots V_r$, $W_1 \ldots W_s$.

We prove that $r=s$ and $V_i \simeq W_{\sigma(i)}$ for $i=1 \ldots r$ and some $\sigma \in S_r$ by induction on r.

$r=0$: $\{0\} \simeq \bigoplus_{i=1}^{s} W_i \implies s=0$ ✓

$r>0$: $V_r \xhookrightarrow{\iota_r} \bigoplus_{i=1}^{r} V_i \xrightarrow{\sim} \bigoplus_{i=1}^{s} W_i$ is not 0

(since injective).

$\implies$ for some $k \in \{1 \ldots s\}$,

$$f_{r,k}: V_r \xhookrightarrow{\iota_r} \bigoplus V_i \xrightarrow{\sim} \bigoplus W_i \xrightarrow{\pi_k} W_k$$

is nonzero. Thus by Schur's Lemma (Lemma 1.2.3), $f_{r,k}$ is an isom. (wlog $k=s$).

$$\implies \bigoplus_{i=1}^{r-1} V_i \simeq \left(\bigoplus_{i=1}^{r} V_i\right)/V_r \simeq \left(\bigoplus_{i=1}^{s} W_i\right)/W_s \simeq \bigoplus_{i=1}^{s-1} W_i$$

By IH, $r-1=s-1$ and $V_i \simeq W_{\sigma(i)}$ for $i=1 \ldots r-1$ and $\sigma \in S_{r-1}$.

$\implies r=s$ and $V_i \simeq W_{\tilde\sigma(i)}$ for $i=1 \ldots r$ and $\tilde\sigma \in S_r$ with
- $\tilde\sigma|_{\{1 \ldots r-1\}} = \sigma$
- $\tilde\sigma(r) = r$.

$\square$