1.3 The group algebra

G finite group

K field

Def: The group algebra of G (over K) is

$$K[G] = \left\{ \sum_{g \in G} a_g \cdot g \;\middle|\; a_g \in K \right\}$$

with addition

$$\left(\sum a_g g\right) + \left(\sum b_g g\right) = \sum (a_g + b_g) g$$

and multiplication

$$\left(\sum a_g g\right) \cdot \left(\sum b_g g\right) = \sum_{g \in G} \left(\sum_{g = h_1 h_2} a_{h_1} \cdot b_{h_2} \right) g .$$

Rem: • $K[G]$ is a (not necessarily commutative) ring

with $0 = \sum 0 \cdot g$ and $1 = 1 \cdot e + \sum_{g \neq e} 0 \cdot g$

• The map
$$\begin{aligned} K &\longrightarrow K[G] \\ a &\longmapsto a \cdot 1 = a \cdot e + \sum_{g \neq e} 0 \cdot g \end{aligned}$$

is a ring homomorphism and turns $K[G]$ into a K-algebra.

• $$\begin{aligned} G &\longrightarrow K[G] \\ g &\longmapsto 1 \cdot g + \sum_{h \neq g} 0 \cdot h \end{aligned}$$ is a multiplicative map.

• There is a canonical isomorphism
$$\begin{aligned} \varphi : K^G &\longrightarrow K[G] \\ [f : G \to K] &\longmapsto \sum_{g \in G} f(g) \cdot g \end{aligned}$$

of K-vector spaces. However, the obvious product

$$f_1 f_2 : g \longmapsto f_1(g) \cdot f_2(g)$$

for K^G differs from the product in $K[G]$, which corresponds instead to the <u>convolution product</u>

$$f_1 * f_2 : g \longmapsto \sum_{h \in G} f_1(gh) f_2(h^{-1})$$

for K^G.

<u>Def</u>: A <u>$K[G]$-module</u> is an abelian group V together with a map $\mathscr{A} : K[G] \times V \to V$

$$(x, v) \mapsto x.v$$

such that

(1) $1.v = v$, $\quad 0.v = 0$, $\quad x.0 = 0$,

(2) $(xy).v = x.(y.v)$,

(3) $(x+y).v = x.v + y.v$,

(4) $x.(v+w) = x.v + x.w$

for all $x, y \in K[G]$ and $v, w \in V$.

<u>Rem</u>: · Since $K \subset K[G]$, V is in particular a K-vector space.

· We have $\dim_K V < \infty$ iff. V is finitely generated as a $K[G]$-module, i.e. $\exists$ finite subset $S \subset V$ s.t. $\forall v \in V$ $\exists s_1 - s_n \in S$, $x_1 - x_n \in K[G]$ s.t. $v = \sum x_i . s_i$. (since: $\dim_K K[G] = \# G < \infty$).

· If $\dim_K V < \infty$, then $\rho : G \longrightarrow GL(V)$ with $\rho(g).v = g.v = \mathscr{A}(g, v)$ is a repr. of G.

Def: A <u>$K[G]$-linear map</u> between $K[G]$-modules V and W is a group hom. $f: V \to W$ s.t. $f(x.v) = x.f(v)$ for all $x \in K[G]$ and $v \in V$, i.e.

$$\begin{array}{ccc} K[G]\times V & \xrightarrow{\vartheta_V} & V \\ {\scriptstyle (id,f)}\downarrow & & \downarrow {\scriptstyle f} \\ K[G]\times W & \xrightarrow[\vartheta_W]{} & W \end{array}$$

commutes.

- This defines the category $Mod(K[G])$ of $K[G]$-modules. We denote the full subcategory of finitely generated $K[G]$-modules by $Mod^{fg}(K[G])$.

Rem: A $K[G]$-linear map $f: V \to W$ is, in particular,

- K-linear: $f(a.v) = a.f(v)$ (for $a \in K$, $v \in V$);
- G-equivariant: $f(g.v) = g.f(v)$ (for $g \in G$, $v \in V$).

Prop 1: The functor

$$\begin{aligned} \mathcal{F}: Mod^{fg}(K[G]) &\longrightarrow Rep_K(G) \\ [\vartheta: K[G]\times V \to V] &\longmapsto [\rho: G \to GL(V)] \\ [f: V \to W] &\longmapsto [f: V \to W] \end{aligned}$$

is an equivalence of categories.

proof: An inverse of $\mathcal{F}$ is the following functor $\mathcal{G}: Rep_K(G) \to Mod^{fg}(K[G])$:

- for a repr. (ρ, V) of G, we define

$$\vartheta = \mathcal{G}(g) : K[G] \times V \longrightarrow V$$
$$\left(\sum a_g g \, , v\right) \longmapsto \sum a_g (g.v)$$

- for a G-hom. $f : V \to W$, we define $\mathcal{G}(f) = f : V \to W$.

We leave it as an exercise to verify that $\mathcal{G}$ is well-defined and a quasi-inverse of $\mathcal{F}$. □

1.4 Categorical properties

G finite group
K field

Def: $V_1, \ldots, V_r$ repr. of G

The direct sum of $V_1, \ldots, V_r$ is defined as the K-vector space $\bigoplus_{i=1}^{r} V_i = \prod_{i=1}^{r} V_i$ together the G-action
$$g.(v_1, \ldots, v_r) = (g.v_1, \ldots, g.v_r).$$

It comes with the canonical inclusions

$$\iota_j : V_j \longrightarrow \bigoplus V_i$$
$$v \longmapsto (0, \ldots, 0, v_j, 0, \ldots, 0)$$

↑
j-th coordinate

and the canonical projections

$$\pi_j : \bigoplus V_i \longrightarrow V_j$$
$$(v_1, \ldots, v_r) \longmapsto v_j$$

for $j = 1, \ldots, r$.

Rem: $\bigoplus V_i$ together with $G \curvearrowright \bigoplus V_i$ defines indeed a G-repr. $\varrho : G \to GL(\bigoplus V_i)$. The ι_j and π_j are G-homs.

Prop 1: $\bigoplus V_i$ together with $\{\iota_j\}$ and $\{\pi_j\}$ is a biproduct of $V_1 \dots V_r$ in $\mathrm{Rep}_K(G)$,

i.e. $(\bigoplus V_i, \{\iota_j\})$ is a coproduct and

$(\bigoplus V_i, \{\pi_j\})$ is a product.

proof: • Universal property of a coproduct:

Given G-homs $\varphi_j : V_j \longrightarrow W$ for $j = 1 \dots r$ and some G-repr. W, we define $\varphi : \bigoplus V_i \longrightarrow W$,

$$(v_1 \dots v_r) \longmapsto \varphi_1(v_1) + \dots + \varphi_r(v_r)$$

which is the unique K-linear map s.t. $\varphi \circ \iota_j = \varphi_j$,

i.e.
$$\begin{tikzcd} \bigoplus V_i \arrow[r, "\varphi"] & W \\ V_j \arrow[u, "\iota_j"] \arrow[ur, "\varphi_j"'] & \end{tikzcd}$$
commutes, for $j = 1 \dots r$.

It is G-equivariant since

$$\begin{aligned}
\varphi(g.(v_1 \dots v_r)) &= \varphi(g.v_1 \dots g.v_r) \\
&= \varphi_1(g.v_1) + \dots + \varphi_r(g.v_r) \\
&= g.\varphi_1(v_1) + \dots + g.\varphi_r(v_r) \\
&= g.(\varphi_1(v_1) + \dots + \varphi_r(v_r)) \\
&= g.\varphi(v_1 \dots v_r).
\end{aligned}$$

• Universal property of a product:

Given G-homs $\varphi_j : W \to V_j$ for $j = 1 \dots r$ and some G-repr. W,

$$\varphi : W \longrightarrow \bigoplus V_i$$
$$w \longmapsto (\varphi_1(w), \dots, \varphi_r(w))$$

is the unique k-linear map s.t. $\pi_j \circ \varphi = \varphi_j$,

i.e.
$$\begin{tikzcd} W \arrow[r, "\varphi"] \arrow[dr, "\varphi_j"'] & \bigoplus V_i \arrow[d, "\pi_j"] \\ & V_j \end{tikzcd}$$
commutes, for all $j = 1 \dots r$.

It is G-equivariant since

$$\begin{aligned} \varphi(g.w) &= (\varphi_1(g.w) \dots \varphi_r(g.w)) \\ &= (g.\varphi_1(w) \dots g.\varphi_r(w)) \\ &= g.(\varphi_1(w) \dots \varphi_r(w)) \\ &= g.\varphi(w). \end{aligned}$$

$\square$

<u>Rem:</u> · Note that for $r = 0$, $\bigoplus_{i=1}^{0} V_i = \{0\}$. We conclude that $\{0\}$ is a zero object in $\mathrm{Rep}_k(G)$, i.e. for every G-repr. V there are unique G-homs

$$0 : \{0\} \longrightarrow V \qquad \text{and} \qquad 0 : V \longrightarrow \{0\}$$
$$0 \longmapsto 0 \qquad\qquad\qquad v \longmapsto 0$$

(<u>or:</u> $\{0\}$ is initial and terminal in $\mathrm{Rep}_k(G)$)

· The zero G-hom $0 : V \to W$, $v \mapsto v$ is characterized as the composition $V \xrightarrow{0} \{0\} \xrightarrow{0} W$ of the terminal G-hom. from V with the initial G-hom. to W.