5.2 Transcendental bases

Def: L/K

A subset $S \subset L$ is algebraically independent over K if the K-linear homomorphism

$$ev_S : K[T_a \mid a \in S] \longrightarrow L$$
$$T_a \longmapsto a$$

is injective. Otherwise, S is called algebraically dependent over K.

A transcendence basis for L/K is a maximal subset $S \subset L$ that is algebraically independent over K.

Rem: L/K

$S \subset L$ alg. indep. over K

Then

$$K(S) = \bigcap_{\substack{K \subset E \subset L \\ \text{s.t. } S \subset E}} E \simeq \mathrm{Frac}\left(K[T_a \mid a \in S]\right).$$

Lemma 1: L/K

$S \subset L$ alg. indep.

Then S is a transcendence basis for L/K if and only if $L/K(S)$ is algebraic.

proof: "$\Rightarrow$": S trans. basis for L/K

- Then there is for every $b \in L$ a non-zero polynomial

$f \in K[T_a \mid a \in S][T_b]$ s.t. $f((a)_{a\in S}, b) = 0$

since $S \cup \{b\}$ is alg. dependent over K.

We can write

$$f = \sum_{i=0}^{n} f_i((T_a)_{a\in S}) \cdot T_b^i.$$

- Since S is alg. indep. over K,

$f_i((a)_{a\in S}) \neq 0$ if $f_i((T_a)_{a\in S}) \neq 0$.

Thus $f((a)_{a\in S}, T) \in K(S)[T]$ is

a non-zero polynomial with root b.

Thus b is algebraic over $K(S)$, which shows

that L/K is algebraic.

"$\Leftarrow$": Assume that $L/K(S)$ is algebraic.

Then there is for every $b \in L$ a nonzero

polynomial $f \in K(S)[T]$ s.t. $f(b) = 0$.

Since $K(S) = \mathrm{Frac}(K[T_a \mid a \in S])$,

$$f = \sum_{i=0}^{n} \frac{g_i((a)_{a\in S})}{h_i((a)_{a\in S})} \cdot T^i$$

for some $g_i, h_i \in K[T_a \mid a \in S]$ with $h_i \neq 0$.

$$\Rightarrow \tilde{f} = f \cdot \prod_{i=0}^{n} h_i = \sum_{i=0}^{n} \left(\prod_{j \neq i} h_j\right) \cdot g_i \cdot T^i,$$

a nonzero polynomial in $K[T_a \mid a \in S][T]$

with root $((a)_{a \in S}, b)$. Thus $S \cup \{b\}$

is alg. dependent over K for every $b \in L$

$\Rightarrow S$ maximally alg. indep. over K. $\square$

<u>Thm 2:</u> L/K

$T_0 \subset T_1 \subset L$ subsets such that

- T_0 is alg. indep. over K
- $L/K(T_1)$ algebraic

Then L/K has a transcendence basis $S \subset L$

with $T_0 \subset S \subset T_1$.

proof: Let $\mathcal{S}$ be the poset of algebraically independent sets $T \subset T_1$ with $T_0 \subset T$, ordered by inclusion. Since every chain

$$T_0' \subset T_1' \subset \ldots$$

in $\mathcal{S}$ has $T' = \bigcup_{i \geq 0} T_i'$ as an upper bound, Zorn's Lemma implies that $\mathcal{S}$ contains a maximal element S.

- By the maximality of S, every $a \in T_1 - S$ is algebraic over $K(S)$. Thus $T_1 \subset \overline{K(S)}$, i.e. $K(T_1)/K(S)$ is algebraic.
- Since $L/K(T_1)$ is algebraic, $L/K(S)$ is algebraic. Thus S is trans. basis for L/K by Lemma 1. $\square$

<u>Lemma 3:</u> L/K
S trans. basis for L/K
$a \in L$ transcendental over K
Then there is a $b \in S$ such that $(S - \{b\}) \cup \{a\}$ is a transcendence basis for L/K.

<u>proof</u>: • Since a is algebraic over $K(S)$, it is the root of a nonzero polynomial $f \in K(S)[T]$. After clearing denominators, we can assume that $f \in K[S][T]$.

Moreover, we can assume that is irreducible in $K[T][S]$.

- Since a is transcendental over K, $\deg_{T_b} f \geq 1$ for some $b \in S$, i.e.

$$f = f((T_{b'})_{b' \in S-\{b\}}, T_a, T_b) = \sum_{j=0}^{n_b} g_j((T_{b'})_{b' \in S-\{b\}}, T_a) \cdot T_b^{\,j}$$

has positive degree $n_b \geq 1$ and $g_{n_b} \neq 0$.

Since f cannot be a divisor of g_{n_b}

in $K(S)[T]$, and f is (a constant multiple of)

the minimal polynomial of a over $K(S)$,

we have $g_{n_b}((s')_{s' \in S-\{s\}}, a) \neq 0$.

- Thus for $S' = (S - \{s\}) \cup \{b\}$, $f((s')_{s' \in S'}, T_b)$

  is a nonzero element of $K(S')[T_b]$ with root b.

  Thus s is algebraic over $K(S')$,

  and L is algebraic over $K(S')$.

- Since $S - \{b\} \subset S'$ is not maximally

  alg. indep. over K, Thm. 2 implies

  that S' is a transcendence basis for L/K. $\square$

Thm 4: L/K

Any two transcendence bases for L/K have

the same cardinality.

proof: • Let S and T be two transcendence bases for L/K.

Let $\mathcal{S}$ be the set of all bijections $\alpha : S' \to T'$

between subsets $S' \subset S$ and $T' \subset T$ such that

$S^{S'}_{T'} = (S - S') \cup T'$ is a transcendence basis for L/K.

- $\mathcal{S}$ becomes a poset w.r.t. the partial order defined by $\alpha_1 \le \alpha_2$ for bijections $\alpha_i : S_i' \longrightarrow T_i'$ in $\mathcal{S}$ $(i=1,2)$ if $S_1' \subset S_2'$, $T_1' \subset T_2'$ and $\alpha_1 = \alpha_2|_{S_1'}$:

$$\begin{tikzcd} S_1' \arrow[r, "\alpha_1"] \arrow[d, hook] \arrow[dr, phantom, "="] & T_1' \arrow[d, hook] \\ S_2' \arrow[r, "\alpha_2"] & T_2' \end{tikzcd}$$

<u>Exercise:</u> Every chain $\alpha_1 \le \alpha_2 \le \dots$ in $\mathcal{S}$ is bounded by $\alpha : S' = \bigcup_{i\ge 0} S_i' \longrightarrow \bigcup_{i \ge 0} T_i' = T'$ where $\alpha|_{S_i'} = \alpha_i$. (In particular: $S^{S'}_{T'}$ is a trans. basis of L/K)

Thus we can apply Zorn's Lemma and find a maximal element $\alpha : S' \longrightarrow T'$ in $\mathcal{S}$.

<u>Claim:</u> $T' = T$.

<u>proof:</u> If this was not the case, then there is a $t \in T - T'$. By Lemma 3, there is an $s \in S^{S'}_{T'}$ s.t. $S^{S' \cup \{s\}}_{T' \cup \{t\}} = (S^{S'}_{T'} - \{s\}) \cup \{t\}$ is a transcendental basis of L/K. Note that since $T' \cup \{t\} \subset T$ is alg. indep. over K, s.t. $T' \subset S^{S'}_{T'}$, but $s \in S - S'$. Thus we can extend $\alpha : S' \to T'$ to a bijection $\alpha' : S' \cup \{s\} \longrightarrow T' \cup \{t\}$ with $\alpha'(s) = t$ that is in $\mathcal{S}$, which contradicts the maximality of α. ↯ ∎

- Since $T'=T$, $S^{\alpha}_{T} = (S-S')\cup T$ is a trans. basis of L/K, which is only possible if $S'=S$ since T is maximally alg. indep. over K.
Thus we find a bijection $S=S' \xrightarrow{\alpha} T'=T$, which shows that S and T have the same cardinality. $\square$

<u>Def:</u> L/K
The <u>transcendence degree of L/K</u> is the cardinality of a transcendence basis of L/K. The extension L/K is <u>purely transcendental</u> if $L=K(S)$ for some transcendence basis S of L/K.

<u>Ex:</u> (1) A rational function field $L=K(a_1 \dots a_n)$ over K is the same as a purely transcendental field extension L/K with trans. basis $S=\{a_1 \dots a_n\}$; e.g.

$$K(T) = \left\{ \frac{f}{g} \;\middle|\; f,g \in K[T],\ g\neq 0 \right\}$$

is of trans. degree 1 over K.

(2) $L = \mathrm{Frac}\left(K[x,y]/(y^2-x^3-x) \right)$ (char $K \neq 2$) is not a rational function field (exercise!)

But $L = K(x)[T]/(T^2-(x^3+x))$ is algebraic over $K(x)$, and thus L/K has trans. degree 1 over K.