4.2 Solvable groups

Def: A group G is *simple* if $G \neq \{e\}$ and if the only normal subgroups of G are $\{e\}$ and G.

Ex: $\mathbb{Z}/n\mathbb{Z}$ is simple iff. n is prime.

Thm 1: The alternating group A_n is simple for $n \geq 5$.

proof: claim 1: A_n is generated by 3-cycles.

We have

$$A_n = \langle (ij)(kl) \mid i,j,k,l \in \{1 \ldots n\},\ i \neq j,\ k \neq l \rangle$$

and

$$(ij)(kl) = (ijk)(jkl) \quad \text{if } i,j,k,l \text{ are pw. distinct,}$$
$$(ij)(jl) = (ijl) \quad \text{if } i,j=k,l \text{ are pw. distinct,}$$
$$(ij)(ij) = e.$$

Thus claim 1. $\lrcorner$

claim 2: All 3-cycles are conjugate in A_n.

Let $(ijk), (i'j'k') \in A_n$ and $\gamma \in S_n$ s.t.

$$\gamma(i) = i', \quad \gamma(j) = j', \quad \gamma(k) = k'.$$

Then $\gamma(ijk)\gamma^{-1} = (i'j'k')$, i.e. (ijk) and $(i'j'k')$ are conjugate in S_n.

If $\gamma \notin A_n$, then there are $l, m \in \{1 \ldots n\} - \{i,j,k\}$ with $l \neq m$ $(n \geq 5)$, and thus $\gamma'(ijk)(\gamma')^{-1} = (i'j'k')$ for $\gamma' = \gamma \cdot (lm) \in A_n$. Thus claim 2. $\lrcorner$

claim 3: Every normal subgroup $N \neq \{e\}$ of A_n contains a 3-cycle.

Let $\sigma \neq e$ be an element of N with a maximal number of fixed points, i.e. $i \in \{1 \_ n\}$ s.t. $\sigma(i) = i$. Since $\sigma \neq e$, σ has at least one cycles $(i j \text{--})$ of length ≥ 2.

<u>case 1:</u> All cycles of σ have length ≤ 2.

Then there are at least 2 cycles $(ij), (kl)$ of length 2 ($\sigma \neq e$ & $\mathrm{sign}(\sigma) = 1$).

Let $m \in \{1 \_ n\} - \{i,j,k,l\}$ and $\tau = (klm) \in A_n$.

Then

$$\sigma' = \underbrace{\tau \sigma \tau^{-1}}_{\in N} \underbrace{\sigma^{-1}}_{\in N} \in N$$

and
- $\sigma'(i) = i$,
- $\sigma'(j) = j$,
- $\sigma'(p) = p$ for all $p \neq m$ with $\sigma(p) = p$.

Thus σ' has more fixed points than σ ↯

<u>case 2:</u> σ has a cycle $(ijk \text{--})$ and i,j,k are not the only non-fixed points.

Then there are distinct $l, m \in \{1 \_ n\} - \{i,j,k\}$ s.t. $\sigma(l) \neq l$ and $\sigma(m) \neq m$ ($\mathrm{sign}(\sigma) = 1$).

For $\tau = (klm)$,

$$\sigma' = \underbrace{\tau \sigma \tau^{-1}}_{\in N} \underbrace{\sigma^{-1}}_{\in N} \in N$$

and
- $\sigma'(j) = j$
- $\sigma'(p) = p$ for all fixed points p of σ

Thus σ' has more fixed points than σ ↯

Thus claim 3. ✓

If $N \neq \{e\}$ is a normal subgroup, then it contains a 3-cycle (claim 3), which is conjugate to all other 3-cycles in A_n (claim 2). Since N is normal, it contains all 3-cycles, and thus

$$A_n \underset{\text{(claim 1)}}{=} \langle \text{3-cycles} \rangle = N.$$

$\square$

Def: • A <u>normal series</u> (<u>of length r</u>) of a group G is a sequence

$$\{e\} = G_0 \underset{\neq}{\trianglelefteq} G_1 \underset{\neq}{\trianglelefteq} \cdots \underset{\neq}{\trianglelefteq} G_r = G$$

of normal subgroups $G_{i-1} \trianglelefteq G_i$. Its <u>factors</u> are the quotient groups $Q_i = G_i / G_{i-1}$ for $i = 1, \ldots, r$.

Sometimes we write

$$\underset{Q_1}{G_0 \trianglelefteq G_1} \underset{Q_2}{\trianglelefteq} \cdots \underset{Q_r}{\trianglelefteq G_r}$$

• A <u>refinement</u> of $G_0 \trianglelefteq \ldots \trianglelefteq G_r$ is a normal series $H_0 \trianglelefteq \ldots \trianglelefteq H_s$ of G such that $\{G_0, \ldots, G_r\} \subset \{H_0, \ldots, H_s\}$.

• A <u>composition series of G</u> is a normal series $G_0 \trianglelefteq \ldots \trianglelefteq G_r$ whose factors $Q_i = G_i / G_{i-1}$ are simple groups for $i = 1, \ldots, r$.

Rem: A normal series is a composition series iff it has no proper refinement.

Ex: $\{e\} \triangleleft A_4 \triangleleft S_4$ is a normal series, but not a composition series since it has the refinement

$$\underset{\mathbb{Z}/2}{\{e\} \triangleleft} \{e, (12)(34)\} \underset{\mathbb{Z}/2}{\triangleleft} \{e, (12)(34), (13)(24), (13)(24)\} \underset{\mathbb{Z}/3}{\triangleleft} A_4 \underset{\mathbb{Z}/2}{\triangleleft} S_4,$$

which is a composition series. In particular, A_4 is not simple.

Rem: Every finite group has a composition series, but there are infinite groups without composition series; e.g. $G = \mathbb{Z}$.

Def: Two normal series $G_0 \triangleleft \ldots \triangleleft G_r$ and $H_0 \triangleleft \ldots \triangleleft H_s$ of a group $G = G_r = H_s$ are equivalent if $r = s$ and if there factors agree up to permutation.

Ex: $\{\bar{0}\} \underset{\mathbb{Z}/2}{\triangleleft} \{\bar{0}, \bar{3}\} \underset{\mathbb{Z}/3}{\triangleleft} \mathbb{Z}/6$ and $\{\bar{0}\} \underset{\mathbb{Z}/3}{\triangleleft} \{\bar{0}, \bar{2}, \bar{4}\} \underset{\mathbb{Z}/2}{\triangleleft} \mathbb{Z}/6$

are equivalent

Thm 2: (Schreier)

Any two normal series

$$G_0 \triangleleft \ldots \triangleleft G_r \qquad \text{and} \qquad H_0 \triangleleft \ldots \triangleleft H_s$$

of a group $G = G_r = H_s$ have equivalent refinements.

proof: We define

$$G_{i,j} = G_{i-1}(G_i \cap H_j) \qquad \text{for } i = 1 \ldots r,\ j = 0 \ldots s$$

$$H_{i,j} = (G_i \cap H_j)\, H_{j-1} \qquad \text{for } i = 0 \ldots r,\ j = 1 \ldots s$$

and get refinements

$$\{e\} = G_0 = G_{1,0} \lhd G_{1,1} \lhd \cdots \lhd G_{1,s} = G_1 = G_{2,0} \lhd \cdots \lhd G_{r,s} = G_r = G,$$

$$\{e\} = H_0 = H_{0,1} \lhd H_{1,1} \lhd \cdots \lhd H_{r,1} = H_1 = H_{0,2} \lhd \cdots \lhd H_{r,s} = H_s = G$$

where some inclusions might not be proper.

By the 3rd isomorphism theorem "$H/(H \cap N) \simeq HN/N$",

$$\begin{aligned}
G_{i,j}/G_{i,j-1} &= G_{i-1}(G_i \cap H_j)/G_{i-1}(G_i \cap H_{j-1}) \quad (= HN/N) \\
&\simeq (G_i \cap H_j)/(G_{i-1} \cap H_j)(G_i \cap H_{j-1}) \quad (= H/H \cap N) \\
&\quad \begin{pmatrix} H = G_i \cap H_j \\ N = G_{i,j-1} \end{pmatrix} \\
&\simeq (G_i \cap H_j)H_{j-1}/(G_{i-1} \cap H_j)H_{j-1} \quad (= HN/N) \\
&\quad \begin{pmatrix} H = G_i \cap H_j \\ N = H_{i-1,j} \end{pmatrix} \\
&= H_{i,j}/H_{i-1,j}
\end{aligned}$$

Thus $G_{1,0} \lhd \cdots \lhd G_{r,s}$ and $H_{0,1} \lhd \cdots \lhd H_{r,s}$ have the same factors and thus are equivalent refinement (after removing the non-proper inclusions). $\square$

Cor 3: If G has a composition series, then any normal series of G has a refinement that is a composition series. $\square$

Cor 4: (Jordan-Hölder theorem)

Any two composition series of G are equivalent. $\square$

Def: A group is <u>solvable</u> if it has a normal series with abelian factors.

Ex. (1) G abelian $\Rightarrow$ G solvable

(2) G finite & solvable $\Rightarrow$ G has a composition series whose factors are cyclic of prime order p.

(3) A_3 and S_3 are solvable:

$$\underset{\mathbb{Z}/3}{\{e\} \triangleleft} \underbrace{\{e, (123), (132)\}}_{=A_3} \underset{\mathbb{Z}/2}{\triangleleft} S_3$$

(4) A_4 and S_4 are solvable:

$$\underset{\mathbb{Z}/2 \times \mathbb{Z}/2}{\{e\} \triangleleft} \{e, (ij)(k\ell) \mid \{i,j,k,\ell\} = \{1 \ldots 4\}\} \underset{\mathbb{Z}/3}{\triangleleft} A_4 \underset{\mathbb{Z}/2}{\triangleleft} S_4$$

(5) A_n and S_n are <u>not</u> solvable for $n \geq 5$.

Thm (Feit – Thompson)

G finite, $\# G$ odd

Then G is solvable

(proof beyond this course...)

Def: p prime number

A <u>p-group</u> is a finite group G of order $\# G = p^n$ for some $n \geq 0$.

<u>Lemma 5</u>: Every p-group is solvable.

<u>proof</u>: This is trivial for $G = \{e\}$. Assume $\# G = p^n > 1$.

<u>claim</u>: the center $Z(G)$ of G is non-trivial.

Consider $G \curvearrowright G$ by $g.h = ghg^{-1}$. Then

- $a \in Z(G) \iff G.a = \{a\}$
- $G.e = \{e\}$
- $G.h \simeq G/\mathrm{Stab}_G(h) \Rightarrow \# G.h \mid p^n$

Thus
$$\underbrace{\# G}_{\text{divisible by } p} = \# Z(G) + \sum_{G.h \neq \{h\}} \underbrace{\# G.h}_{\text{divisible by } p},$$

and $\# Z(G)$ is divisible by p. Thus the claim. ✓

- Define $G_1 = G$ and for $i \geq 1$, $G_{i+1} = G_i / Z(G_i)$,

we get a sequence

$$G = G_1 \xrightarrow{\pi_1} G_2 \xrightarrow{\pi_2} G_3 \twoheadrightarrow \dots \twoheadrightarrow G_r = \{e\}$$

Define
- $H_0 = \{e\}$
- $H_i = \ker(\pi_i \circ \dots \circ \pi_1)$ for $i = 1 \dots r-1$

and get a normal series

$$\{e\} = H_0 \triangleleft \underset{\substack{Z(G_1) \\ = Z(G)}}{H_1} \triangleleft \underset{Z(G_2)}{H_2} \triangleleft \dots \triangleleft \underset{\substack{Z(G_{r-1}) \\ = G_{r-1}}}{H_r} = G$$

with abelian factors. Thus G is solvable.

$\square$