2.2 Algebraic closure

Def: L/K

$f = \sum c_i T^i \in K[T]$

$a \in L$

Then a is a <u>root of f</u> :iff $f(a)=0$.

Lemma 1: $f \in K[T]$ irreducible

$L = K[T]/(f)$

$a = [T] \in L$

Then a is a root of f.

proof: • Note that (f) is a maximal ideal since f is irreducible; thus L is a field.

• $ev_a : K[T] \to L$ sends f to 0 by the definition of $a=[T]$ and $L = K[T]/(f)$. Thus $f(a) = ev_a(f) = 0$. □

Cor. 2: Every $f \in K[T]$ of degree ≥ 1 has a root in some field extension L/K.

proof: Since $\deg f \geq 1$, $f = gh$ for some $g, h \in K[T]$ where g is irreducible. By Lemma 1 g has a root $a = [T]$ in $L = K[T]/(g)$.

Thus $f(a) = ev_a(f) = \underbrace{ev_a(g)}_{=0} \cdot ev_a(h) = 0$. □

Def: A field K is algebraically closed if every polynomial $f \in K[T]$ of degree ≥ 1 has a root $a \in K$.

Lemma 3: K algebraically closed
$f \in K[T]$ of degree $n \geq 0$.
Then $f = u \cdot \prod_{i=1}^{n} (T - a_i)$ for some $u, a_1, \ldots, a_n \in K$.

proof: By induction on $n = \deg f$:

$n=0$: Then $f = u$ for some $u \in K$.

$n>0$: • Since K is alg. cl., f has a root $a \in K$, i.e. $f \in \ker(ev_a : K[T] \to K)$

• Since $ev_a(T-a) = a - a = 0$ and $T-a$ irred., $f = (T-a) \cdot g$ for some $g \in K[T]$, and $\deg g = \deg f - 1 = n-1$.

• By (IH) $g = u \prod_{i=1}^{n-1} (T_i - a_i)$, thus

$$f = u \prod_{i=1}^{n} (T_i - a_i) \quad \text{where } a_n = a. \qquad \square$$

Cor. 4: K alg. cl.
L/K algebraic
Then $L = K$.

proof: Let $a \in L$. Then a has a minimal polynomial $f \in K[T]$ (L/K is alg). By Lemma 3, $f = 1 \cdot \prod_{i=1}^{n} (T - a_i)$ for some $a_1, \ldots, a_n \in K$. Since f is irreducible, $n = 1$ & $f = T - a_1 \Rightarrow a = a_1 \in K$. $\square$

Cor 5: A field K is alg. cl. iff. every irreducible polynomial $f \in K[T]$ has degree 1.

proof: "$\Rightarrow$": K alg. cl., $f \in K[T]$

$\underset{(\text{Lemma 3})}{\Rightarrow}$ $f = u \prod (T - a_i)$ in $K[T]$

Thus f is irred. iff. deg $f = 1$.

"$\Leftarrow$": Let $f \in K[T]$ be of deg $f \geq 1$ with factorization $f = \prod g_i$ into irreducible $g_i \in K[T]$. Then deg $g_i = u_i(T_i - a_i)$ for some $u_i, a_i \in K$

$\Rightarrow$ a_i is a root of g_i and thus of f.

$\Rightarrow$ K alg. cl. $\square$

Thm. 6: Every field K is contained in an algebraically closed field.

proof: Set $L_0 = K$. We will define recursively an infinite "tower"

$L_0 \subset L_1 \subset \_ \subset L_n \subset \_$

of field extensions.

$\rightarrow$ Given L_i, we construct L_{i+1} as follows.

Let $S_i = \{ X_f \mid f \in L_i[T] \text{ of degree} \geq 1 \}$.

Then for $g = \sum c_i T^i \in L_i[T]$,

$$g(X_g) = \sum c_i X_g^i \in L_i[S_i] = L_i\left[X_f \;\middle|\; \begin{array}{l} f \in L_i[T] \text{ of} \\ \text{degree} \geq 1 \end{array}\right].$$

claim 1: $I = \left(g(X_g) \;\middle|\; \begin{array}{l} g \in L_i[T] \\ \deg g \geq 1 \end{array} \right)$ is a proper ideal of $L_i[S_i]$.

proof: Assume that $I = L_i[S_i]$. Then

$$1 = h_1 \cdot g_1(X_{g_1}) + \dots + h_n g_n(X_{g_n})$$

for some $g_1 - g_n \in L_i[T]$ of degree ≥ 1

and some $h_1 - h_n \in L_i[S_i]$.

- By Cor. 2, $\exists$ finite E/L_i s.t. every g_j has a root a_j in E. Define

$$\begin{aligned} \chi : L_i[S_i] &\longrightarrow E. \\ X_{g_j} &\longmapsto a_j \\ X_f &\longmapsto 0 \quad \text{for } f \notin \{g_1 - g_n\} \end{aligned}$$

Then

$$1 = \chi(1) = \sum_{j=1}^{n} \chi(h_j) \underbrace{\chi(g_j(X_{g_j}))}_{= g_j(a_j) = 0} = 0. \;\lightning$$

$\rightarrow$ By Zorn's Lemma, I is contained in a maximal ideal m. Define $L_{i+1} = L_i[S_i]/m$, which yields

$$L_i \rightarrow L_i[S_i] \rightarrow L_i[S_i]/m = L_{i+1}.$$

Note that for every $g \in L_i[T]$ of degree ≥ 1, $[X_g] \in L_{i+1}$ is a root since $g(X_g) \in I \subset \mathfrak{m}$.

<u>Claim 7:</u> $L = \bigcup_{i \geq 0} L_i$ is an algebraically closed field.

<u>proof:</u> • L is a field since for all $x, y \in L$, $\exists\, i \geq 0$ s.t. $x, y \in L_i$. Thus $x+y$, $x-y$, $x \cdot y$, x/y (if $y \neq 0$) are in $L_i \subset L$.

• Let $f = \sum c_j T^j \in L[T]$ of deg $f \geq 1$. Then $c_0 - c_n \in L_i$ for some $i \geq 0$. Thus $f \in L_i[T]$ has a root in $L_{i+1} \subset L$.

This proves Thm. 6. □

<u>Lemma 7:</u> E/K algebraic, L alg. cl. field

$$\begin{tikzcd} E \arrow[r, dashed, "\exists \sigma_E"] & L \\ K \arrow[u, dash] \arrow[ur, "\sigma"'] & \end{tikzcd}$$

Then every homomorphism $\sigma : K \to L$ extends to a homomorphism $\sigma_E : E \to L$.

<u>proof:</u> • Consider the set S of pairs $(F/K, \sigma_F)$ where $K \subset F \subset E$ is an intermediate field and $\sigma_F : F \to L$ extends σ:

$$\begin{tikzcd} E \arrow[rr, dashed, "\exists ?"] & & L \\ & F \arrow[ul, dash] \arrow[ur, "\sigma_F"] & \\ & K \arrow[u, dash] \arrow[uur, "\sigma"'] & \end{tikzcd}$$

• S is a poset w.r.t. the partial order

$(F/K, \sigma_F) \leq (F'/K, \sigma_{F'})$ iff • $F \subset F'$
• $\sigma_{F'}|_F = \sigma_F$

- Then every linearly ordered subset

$$(F_1/K, \sigma_{F_1}) \le (F_2/K, \sigma_{F_2}) \le \dots$$

has the upper bound $(F/K, \sigma_F)$

where
- $F = \bigcup F_i$
- $\sigma_F : F \to L$, $\quad a \mapsto \sigma_{F_i}(a)$ if $a \in F_i$ $(i \gg 0)$

By Zorn's Lemma, $\mathcal{S}$ has a maximal element $(F/K, \sigma_F)$.

→ <u>want</u>: $F = E$

If $F \neq E$, then $\exists a \in E - F$, which is alg. $/F$. Let f be the min. pol. of a, i.e. $(f) = \ker(ev_a : F[T] \to E)$. Then since L is alg. cl. f has a root b in L, i.e. $f \in \ker(ev_b : F[T] \to L)$.

Thus $\ker(ev_a) = (f) \subset \ker(ev_b)$, and thus

$$\begin{tikzcd}
F(a) \arrow[rrr, dashed, "\exists \sigma_{F(a)}"] & & & L, \\
& F[T] \arrow[ul, "ev_a"] \arrow[urr, "ev_b"] & & \\
& & F \arrow[ul] \arrow[uur, "\sigma_F"'] &
\end{tikzcd}$$

which is an extension of σ_F to $\sigma_{F(a)} : F(a) \to L$ and contradicts the maximality of $(F/K, \sigma_F)$. ↯

We conclude that $F = E$. $\square$

Def: K field

An <u>algebraic closure of K</u> is an algebraic field extension L/K where L is algebraically closed. We often write $\bar{K}$ for an algebraic closure of K.

Thm. 8: Every field K has an algebraic closure, and any two algebraic closures of K are isomorphic.

proof: <u>Existence</u>: By Thm. 6, K is contained in some alg. cl. field L. Define

$$\bar{K} = \bigcup_{\substack{K \subset E \subset L \\ E/K \text{ alg.}}} E = \{a \in L \mid a \text{ alg.}/K\},$$

which is an alg. extension of K.

If $f \in \bar{K}[T] \subset L[T]$ is of positive degree, then f has a root $a \in L$, which is alg. over $\bar{K}$ as a root of f. By Cor. 2.1.6, a is alg. $/K$ and thus $a \in \bar{K}$.

Thus $\bar{K}$ is alg. cl.

- <u>Uniqueness</u>: Let $\bar{K}'/K$ be another alg. cl. of K. By Lemma 7, there is a homomorphism $\sigma : \bar{K}' \to \bar{K}$

that extends the inclusion $K \hookrightarrow \bar{K}$:

$$\begin{tikzcd}
\bar{K}' \arrow[rr, "\sigma"] & & \bar{K}. \\
& K \arrow[ul, dash] \arrow[ur, dash] &
\end{tikzcd}$$

Thus $\sigma(\bar{K}') \subset \bar{K}$ is an alg. cl. subfield of $\bar{K}$, and $\bar{K}/\sigma(\bar{K}')$ is algebraic. By Cor. 4, $\bar{K} = \sigma(\bar{K}')$, which shows that $\sigma: \bar{K}' \to \bar{K}$ is an isomorphism. $\square$