Exercises for Algebra 1 Instituto Nacional de Matemática Pura e Aplicada
List 8 To be sent to xia.xiao@impa.br until 8.5.2020

Exercise 1.
Verify the following assertions.

1. $\mathbb{R}^n \otimes_{\mathbb{R}} \mathbb{R}^m$ and $\mathbb{R}^{m \cdot n}$.
2. $A[T_1] \otimes_A A[T_2] \simeq A[T_1, T_2]$ for any ring A.
3. $\mathbb{Z}/n\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/m\mathbb{Z} \simeq \mathbb{Z}/d\mathbb{Z}$ where d is a greatest common divisor of the natural numbers m and n.
4. $K \otimes_{\mathbb{Z}} L = \{0\}$ if K and L are fields of different characteristics.
5. $\mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Q} \simeq \mathbb{Q}$.

Exercise 2.
Let M, N and P be A-modules. Show the following properties.

1. $M \otimes \{0\} = \{0\}$ and $M \otimes A \simeq M$;
2. $M \otimes N \simeq N \otimes M$;
3. $(M \otimes N) \otimes P \simeq M \otimes (N \otimes P)$;
4. $M \otimes (N \oplus P) \simeq (M \otimes N) \oplus (M \otimes P)$.

Hint: Use the universal property of the tensor product to find appropriate isomorphisms.

Exercise 3.
Let M and N be A-modules. Show that $\mathrm{Hom}_A(M,N)$ is an A-module with respect to the operations $f + g : m \mapsto f(m) + g(m)$ and $a.f : m \mapsto a.f(m)$ for $a \in A$ and $f, g \in \mathrm{Hom}_A(M,N)$. Show that

$$\begin{array}{ccc} \mathrm{Hom}_A(M,N) \times \mathrm{Hom}_A(N,P) & \longrightarrow & \mathrm{Hom}_A(M,P) \\ (f,g) & \longmapsto & g \circ f \end{array}$$

is an A-bilinear homomorphism. Conclude that the association $f \otimes g \mapsto g \circ f$ describes a homomorphism $\mathrm{Hom}_A(M,N) \otimes \mathrm{Hom}_A(N,P) \to \mathrm{Hom}_A(M,P)$ of A-modules.

Exercise 4.
Let M, N, N' and P be A-modules and $f : N \to N'$ an A-linear homomorphism. Consider the associations

$$f_M : \begin{array}{ccc} M \otimes_A N & \longrightarrow & M \otimes_A N', \\ m \otimes n & \longmapsto & m \otimes f(n) \end{array} \qquad f^* : \begin{array}{ccc} \mathrm{Hom}(N', P) & \longrightarrow & \mathrm{Hom}(N, P) \\ g & \longmapsto & g \circ f \end{array}$$

$$\text{and} \quad f_* : \begin{array}{ccc} \mathrm{Hom}(M, N) & \to & \mathrm{Hom}(M, N'). \\ h & \longmapsto & f \circ h \end{array}$$

1. Show that f_M is well-defined as a map and that all three maps are homomorphisms of A-modules.
2. Conclude that $M \otimes_A (-)$, $\mathrm{Hom}(-, P)$ and $\mathrm{Hom}(M, -)$ are functors from Mod_A to Mod_A. Which of them are covariant, which of them are contravariant?

Exercise 5 (Bonus)**.**
Let $f : A \to B$ be a ring homomorphism.

1. Show that sending an A-module M to $B \otimes_A M$ and sending an A-linear map $\alpha : M \to M'$ to the B-linear map $\alpha_B : B \otimes_A M \to B \otimes_A M'$ that is defined by $\alpha_B(b \otimes m) = b \otimes \alpha(m)$ defines a functor $B \otimes_A - : \mathrm{Mod}_A \to \mathrm{Mod}_B$.
2. Show that a B-module N is A-module with respect to the action defined by $a.n = f(a).n$ for $a \in A$ and $n \in N$. Show that a B-linear map $\alpha : N \to N'$ is A-linear with respect to this action. Conclude that this defines a functor $\mathcal{F} : \mathrm{Mod}_B \to \mathrm{Mod}_A$.
3. Show that the association $b \otimes n \mapsto b.n$ defines a B-linear map $\eta_N : B \otimes_A \mathcal{F}(N) \to N$.
4. Let M be an A-module and N a B-module. Show that the association

$$\Psi_{M,N} : \begin{array}{ccc} \mathrm{Hom}_A\big(M, \mathcal{F}(N)\big) & \longrightarrow & \mathrm{Hom}_B(B \otimes_A M, N) \\ \gamma : M \to \mathcal{F}(N) & \longmapsto & \eta_N \circ \gamma_B : B \otimes_A M \to N \end{array}$$

   is a well-defined bijection.
5. Let $\alpha : M \to M'$ be an A-linear map and $\beta : N \to N'$ a B-linear map. Show that the diagram

$$\begin{array}{ccccccc} \gamma & \in & \mathrm{Hom}_A\big(M', \mathcal{F}(N)\big) & \xrightarrow{\Psi_{M',N}} & \mathrm{Hom}_B(B \otimes_A M', N) & \ni & \delta \\ \Big\downarrow & & \Big\downarrow & & \Big\downarrow & & \Big\downarrow \\ \mathcal{F}(\beta) \circ \gamma \circ \alpha & \in & \mathrm{Hom}_A\big(M, \mathcal{F}(N')\big) & \xrightarrow{\Psi_{M,N'}} & \mathrm{Hom}_B(B \otimes_A M, N') & \ni & \beta \circ \delta \circ \alpha_B \end{array}$$

   commutes.

Remark: The functor $B \otimes_A - : \mathrm{Mod}_A \to \mathrm{Mod}_B$ is called the *extension of scalars from A to B* and the functor $\mathcal{F} : \mathrm{Mod}_B \to \mathrm{Mod}_A$ is usually called *the restrcition of scalars from B to A*. The properties (4) and (5) say that $\mathcal{F}$ is *right-adjoint* to $B \otimes_A -$.