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**Exercise 1.**
Show that the following polynomials are irreducible in $\mathbb{Q}[T]$.

- $T^2 + 1$;
- $2T^4 - 18T - 12$;
- $T^3 + T^2 + 1$;
- $T^3 - 13T + 5$;
- $T^5 + 3T^3 + 6T^2 + 1$;
- $T^{12} - 2$.

Which of them are irreducible in $\mathbb{Z}[T]$? Find one that is irreducible in $\mathbb{Z}[i][T]$.

**Exercise 2.**
Let $p$ be a prime number. Show that $T^{p-1} + \cdots + T + 1$ is irreducible in $\mathbb{Z}[T]$.
**Hint:** Show that $f(T+1)$ is irreducible and conclude that $f = f(T)$ is irreducible.

**Exercise 3** (Bonus exercise)**.**
Let $\mathcal{C}$ and $\mathcal{D}$ be categories.

1. Let $\mathcal{F} : \mathcal{C} \to \mathcal{D}$ be a functor and $\alpha : A \to B$ an isomorphism in $\mathcal{C}$. Show that $\mathcal{F}(\alpha)$ is an isomorphism in $\mathcal{D}$.

2. Give an example of a functor $\mathcal{F} : \mathcal{C} \to \mathcal{D}$ and an epimorphism $\alpha$ in $\mathcal{C}$ such that $\mathcal{F}(\alpha)$ is not an epimorphism.

3. Let $\{A_i\}_{i\in I}$ a family of objects in $\mathcal{C}$. Assume that $\{A_i\}_{i\in I}$ has a product $\prod_{i\in I} A_i$ and a coproduct $\coprod_{i\in I} A_i$ in $\mathcal{C}$. Let $B$ be another object of $\mathcal{C}$. Show that there are bijections
$$\mathrm{Hom}_{\mathcal{C}}\Big(B, \prod_{i\in I} A_i\Big) \longrightarrow \prod_{i\in I} \mathrm{Hom}_{\mathcal{C}}(B, A_i)$$
and
$$\mathrm{Hom}_{\mathcal{C}}\Big(\coprod_{i\in I} A_i, B\Big) \longrightarrow \prod_{i\in I} \mathrm{Hom}_{\mathcal{C}}(A_i, B).$$

**Exercise 4** (Bonus exercise).
Let $\mathcal{C}$ be a category. A *zero object of* $\mathcal{C}$ is an object $\mathbf{0}$ that is both initial and terminal. If $\mathcal{C}$ has a zero object $\mathbf{0}$, then we call for any two objects $A$ and $B$ of $\mathcal{C}$, the unique morphism $0 : A \to \mathbf{0} \to B$ from $A$ to $B$ the *zero morphism*.

1. Show that the categories Ab and $\mathrm{Vect}_K$ have a zero object. Show that in both categories a morphism $\alpha : A \to B$ is a zero morphism if and only if $\alpha(a) = 0$ for all $a \in A$ (where 0 stays for the zero element of $B$).

2. Show that the categories Sets and Rings do not have a zero object.

Assume that $\mathcal{C}$ has a zero object $\mathbf{0}$.

3. Show that the composition of a morphism with a zero morphism (in any order) is a zero morphism.

A *(categorical) kernel of a morphism* $\alpha : A \to B$ is an object $\ker \alpha$ together with a morphism $\iota : \ker \alpha \to A$ such that $\alpha \circ \iota = 0$ that satisfies the following universal property: for every object $C$ and every morphism $\iota' : C \to A$ such that $\alpha \circ \iota' = 0$, there is a unique morphism $\beta : C \to \ker \alpha$ such that $\iota' = \iota \circ \beta$.

4. Draw a diagram taking all the above objects and morphisms into consideration.

5. Let $\alpha : A \to B$ be a morphism of abelian groups. Show that $\ker \alpha = \{a \in A | \alpha(a) = 0\}$ together with the inclusion $\ker \alpha \to A$ as subgroup is a categorical kernel of $\alpha$.

6. What is the problem with categorical kernels in Rings?