Exercises for Algebra 1 Instituto Nacional de Matemática Pura e Aplicada
List 6 To be sent to xia.xiao@impa.br until 24.4.2020

Exercise 1.
Let K be a field and $f \in K[T]$ a polynomial.

1. Show for $\deg f = 2$ and $\deg f = 3$ that f is irreducible in $K[T]$ if and only if f does not have a root in K.

2. Find a field K and a polynomial $f \in K[T]$ of degree 4 that is not irreducible and does not have a root in K.

3. Show that there exists a field extension L/K such that f factorizes in $L[T]$ as

$$f = u \prod_{i=1}^{n} (T - a_i)$$

with $u, a_1, \ldots, a_n \in L$.

Exercise 2.
Let A be a ring and let $n\mathbb{Z}$ be the kernel of the unique ring homomorphism $\mathbb{Z} \to A$ where $n \geq 0$. The number $char A = n$ is called the *characteristic of* A.

1. Show that if n is positive, then n is the smallest positive integer such that

$$n \cdot 1 \ = \ \underbrace{1 + \cdots + 1}_{n-\text{times}} \ = \ 0.$$

If $n = 0$, then $k \cdot 1 \neq 0$ for any $k \geq 0$.

2. Show that n is zero or a prime number if A is an integral domain.

3. Let L/K be a field extension. Show that K and L have the same characteristic.

4. Let K be a field of characteristic 0. Show that there is a unique ring homomorphism $\mathbb{Q} \to K$.

5. Let p be a prime number and $\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$ the field with p elements. Let K be a field of characteristic p. Show that there is a unique ring homomorphism $\mathbb{F}_p \to K$.

6. Give an example of a ring homomorphism $A \to B$ where A and B have different characteristics.

Remark: The image of the unique homomorphism $\mathbb{Q} \to K$ (if $char K = 0$) or $\mathbb{F}_p \to K$ (if $char K = p > 0$) is called the *prime field of* K.

Exercise 3.
Let $\mathbb{F}_2 = \mathbb{Z}/2\mathbb{Z}$ be the field with two elements 0 and 1.

1. Show that $f = T^2 + T + 1$ is an irreducible polynomial in $\mathbb{F}_2[T]$.
2. Show that $\mathbb{F}_4 = \mathbb{F}_2[T]/(f)$ is a field with four elements.
3. Show that $\mathbb{F}_4^\times$ is a cyclic group with 3 elements.
4. Show that $T^4 - T = \prod_{a\in\mathbb{F}_4}(T - a)$ (as a polynomial in $\mathbb{F}_4[T]$).
5. Find a factorization of $T^4 - T$ in $\mathbb{F}_2[T]$.

Exercise 4.
Let G be an abelian group with n elements. We define the *exponent of* G as the smallest positive integer m such that $g^m = e$ for all $g \in G$.

1. Show that G is cyclic if and only if its exponent is n.
2. Let K be a field and U a finite subgroup of order n of the multiplicative group $K^\times$ of K. Show that U is cyclic.
   Hint: If m is the exponent of U, then every element of U is a zero of $T^m - 1$.

Exercise 5 (Bonus exercise)**.**

1. Show that all irreducible polynomials in $\mathbb{R}[T]$ are of degree 1 or 2.
2. Define two complex numbers z and z' as equivalent if $z' = z$ or $z' = \overline{z}$, the complex conjugate of z. Denote the corresponding equivalence relation by $\sim$ and the class of z in the quotient set $\mathbb{C}/\sim$ by $[z]$. Show that the map
$$\begin{array}{ccc} \mathbb{C}/\sim & \longrightarrow & \{\text{maximal ideals of } \mathbb{R}[T]\} \\ [z] & \longmapsto & \big(\prod_{z'\in[z]} (T - z')\big) \end{array}$$
   is a bijection.
3. Describe $\operatorname{Spec}\mathbb{C}[T]$, assuming the fundamental theorem of algebra (Exercise 6).
4. Make a drawing of $\operatorname{Spec}\mathbb{R}[T]$ and of the map $f^* : \operatorname{Spec}\mathbb{C}[T] \to \operatorname{Spec}\mathbb{R}[T]$ that is induced by the inclusion $f : \mathbb{R}[T] \to \mathbb{C}[T]$.

***Exercise 6** (Bonus exercise)**.** [1]
Prove the *fundamental theorem of algebra*: given a polynomial $f \in \mathbb{C}[T]$ of positive degree, then there exists a $z \in \mathbb{C}$ such that $f(z) = 0$.

[1] Starred exercises are hard problem for those of you that search for a challenge. To balance the amount of work required to solve these exercises, starred exercises they are worth twice as many points as normal exercises.