
Exercise 1.
Show that every finite integral domain is a field.

Exercise 2 (Gaussian integers)**.**
Let $i \in \mathbb{C}$ be a square root of -1. Show that the subset $\mathbb{Z}[i] = \{a + bi \in \mathbb{C} | a, b \in \mathbb{Z}\}$ is a subring of $\mathbb{C}$. Is $\mathbb{Z}[i]$ an integral domain? Is it a field? Show that $\mathbb{Q}[i] = \{a + bi | a, b \in \mathbb{Q}\}$ is a subring of $\mathbb{C}$ that is a field.

Remark: $\mathbb{Z}[i]$ is called the *ring of Gaussian integers.*

Exercise 3.
Let A, B and C be rings. Show the following.

1. The identity map $\mathrm{id}_A : A \to A$ is a ring homomorphism.
2. The composition $g \circ f : A \to C$ of two ring homomorphisms $f : A \to B$ and $g : B \to C$ is a ring homomorphism.
3. Given a ring homomorphism $f : A \to B$ and $a \in A$, we have $f(0) = 0$, $f(-a) = -f(a)$ and $f(a^{-1}) = f(a)^{-1}$, provided a has a multiplicative inverse a^{-1}.
4. A ring homomorphism $f : A \to B$ is injective if and only if $f^{-1}(0) = \{0\}$.
5. The image $\mathrm{im} f = \{f(a) | a \in A\}$ of a ring homomorphism $f : A \to B$ is a subring of B.
6. A ring homomorphism $f : A \to B$ is an isomorphism if and only if there is a ring homomorphism $g : B \to A$ such that $g \circ f = \mathrm{id}_Z$ and $f \circ g = \mathrm{id}_B$.

Exercise 4.

1. Show that the set $C^\infty(\mathbb{R})$ of all smooth functions $f : \mathbb{R} \to \mathbb{R}$ is a ring w.r.t. value-wise addition and multiplication, i.e. $(f + g)(x) := f(x) + g(x)$ and $(f \cdot g)(x) := f(x) \cdot g(x)$. Which of the following maps are ring homomorphisms?
   a) $\mathrm{ev}_a : \mathbb{C}^\infty(\mathbb{R}) \to \mathbb{R}$ with $\mathrm{ev}_a(f) := f(a)$ where $a \in \mathbb{R}$;
   b) $d : \mathbb{C}^\infty(\mathbb{R}) \to \mathbb{C}^\infty(\mathbb{R})$ with $d(f) := \frac{df}{dt}$.

2. Show that the set $\mathbb{F}_p[T] = \{\sum_{i=0}^n a_i T^i | n \geq 0, a_i \in \mathbb{F}_p\}$ of polynomials with coefficients in $\mathbb{F}_p$ forms a ring w.r.t. to the addition $\sum a_i T^i + \sum b_i T^i = \sum (a_i + b_i) T^i$ and the multiplication

$$\Big(\sum_{i=0}^{n} a_i T^i\Big) \cdot \Big(\sum_{j=0}^{m} b_j T^j\Big) := \sum_{k=0}^{n+m} \Big(\sum_{i+j=k} a_i b_j\Big) T^k.$$

Which of the following maps are ring homomorphisms?

a) $\mathrm{ev}_c : \mathbb{F}_p[T] \to \mathbb{F}_p$ with $\mathrm{ev}_c(\sum a_i T^i) = \sum a_i c^i$ where $c \in \mathbb{F}_p$;

b) $\mathrm{Frob} : \mathbb{F}_p[T] \to \mathbb{F}_p[T]$ with $\sum a_i T^i \to \sum a_i^p T^i$.

Exercise 5 (Bonus exercise[1]).
Show that the embedding $i : \mathbb{R} \to \mathbb{R}[T]$ of real numbers as constant polynomials is a ring homomorphism. Show that $\mathbb{R}[T]$ together with $i : \mathbb{R} \to \mathbb{R}[T]$ satisfies the following universal property: for every ring homomorphism $f : \mathbb{R} \to B$ and for every map $\tilde{f} : \{T\} \to B$, there is a unique ring homomorphism $F : \mathbb{R}[T] \to B$ such that $f = F \circ i$ and $F(T) = \tilde{f}(T)$.

Use the universal property to describe all ring homomorphisms $\mathbb{R}[T] \to \mathbb{R}[T]$. Which of them are isomorphisms?

[1] Points for solutions of bonus exercises count as a bonus at the end of the course.