Exercise 1.
Let M be a free A-module of rank $r > 0$ with basis $\{v_1, \ldots, v_r\}$. Show that the map

$$\begin{array}{ccc} A[T_1, \ldots, T_r] & \longrightarrow & \mathrm{Sym}(M) \\ T_i & \longmapsto & v_i \end{array}$$

is a graded isomorphism of graded A-algebras. Show that $T^i(M)$ is free of rank r^i and that $\mathrm{Sym}^i(M)$ is free of rank $\binom{r+i-1}{i}$.

Exercise 2.
Let A be a $\mathbb{Q}$-algebra and M a finitely generated A-module. The *exponential map* $\exp : \Lambda(M) \to \Lambda(M)$ is defined by the formula

$$\exp(x) \;=\; 1 + \sum_{k \geq 1} \frac{1}{k!} \underbrace{(x \wedge \ldots \wedge x)}_{k-\text{times}}.$$

Show that $\exp(x)$ is equal to a finite sum, and therefore well-defined as an element of $\Lambda(M)$. Calculate the expressions $\exp(m)$ and $\exp(m \wedge n + o \wedge p)$ where $m, n, o, p \in M$. Does the formula $\exp(x + y) = \exp(x) \wedge \exp(y)$ hold for any $x, y \in \Lambda(M)$?

Exercise 3.
Let M be an A-module. Consider the ideals

$$I \;=\; \langle\, m \otimes m \,|\, m \in M \,\rangle \quad \text{and} \quad J \;=\; \langle\, m \otimes n + n \otimes m \,|\, m, n \in M \,\rangle$$

of $T(M)$. Show that $I = J$ if 2 is invertible in A. Give an example for A and M where $I \neq J$.

Exercise 4.
Let $l \leq r$ be positive integers and M a free A-module with basis $\{v_1, \ldots, v_r\}$. For $i = 1, \ldots, r$ and $j = 1, \ldots, l$, let $a_{i,j} \in A$ and define the elements

$$m_j \;=\; \sum_{i=1}^{r} a_{i,j} v_i$$

of M.

1. Show that there is a unique element $\delta_\sigma \in A$ for every strictly order preserving maps $\sigma : \{1, \ldots, l\} \to \{1, \ldots, n\}$ such that

$$m_1 \wedge \ldots \wedge m_l \;=\; \sum \delta_\sigma . \big(v_{\sigma(1)} \wedge \ldots \wedge v_{\sigma(l)}\big)$$

   as elements of $\Lambda^l(M)$ where σ ranges through all stictly order preserving maps $\sigma : \{1, \ldots, l\} \to \{1, \ldots, n\}$. Show that

$$\delta_\sigma \;=\; \det(a_{i,j})_{\substack{i \in \operatorname{im}\sigma \\ j=1,\ldots,l}} .$$

2. Let $f : M \to M$ be an endomorphism and $\Lambda^r(f) : \Lambda^r(M) \to \Lambda^r(M)$ the induced linear map. Let $a_{i,j} \in A$ such that $f(v_i) = \sum_{j=1}^{r} a_{i,j} . v_j$ for $i = 1, \ldots, r$. Conclude that

$$\delta \;=\; \det(a_{i,j})_{i,j=1,\ldots,r}$$

   is the unique element of A such that

$$\big(\Lambda^r(f)\big)\big(v_1 \wedge \ldots \wedge v_r\big) \;=\; \delta . \big(v_1 \wedge \ldots \wedge v_r\big).$$

Exercise 5 (Bonus)**.**
This is a continuation of Exercise 4. However, we assume that $A = k$ is field for this exercise. Consequently, M is a k-vector space.

1. Show that $m_1 \wedge \ldots \wedge m_l \neq 0$ if and only if $\{m_1, \ldots, m_l\}$ is linearly independent.

2. Assume that $\{m_1, \ldots, m_l\}$ and $\{m'_1, \ldots, m'_l\}$ are linearly independent subsets of M. Show that there is a $\lambda \in k^\times$ such that

$$m'_1 \wedge \ldots \wedge m'_l \;=\; \lambda \cdot m_1 \wedge \ldots \wedge m_l .$$

   if and only if $\{m_1, \ldots, m_l\}$ and $\{m'_1, \ldots, m'_l\}$ span the same l-dimensional subvector space N of M.
   Hint: If they span the same subvector space N, then one can find a $l \times l$-base change matrix. What is the effect of this matrix on the coefficients δ_σ from Exercise 4?

3. Define $\mathbb{P}\big(\Lambda^l(M)\big) = \big(\Lambda^l(M) - \{0\}\big)/k^\times$ as the set of equivalent classes of nonzero elements of $\Lambda^l(M)$ modulo scalar multiplication by nonzero $\lambda \in k^\times$. Conclude from the previous part of the exercise that there is a well-defined inclusion

$$\big\{l\text{-dimensional subvector spaces of } M\big\} \;\longrightarrow\; \mathbb{P}\big(\Lambda^l(M)\big).$$

   Remark: The set $\mathbb{P}\big(\Lambda^l(M)\big)$ is called the *projective space of* $\Lambda^l(M)$, the above inclusion is called the *Plücker embedding* and its image is called the *Grassmann variety* $\operatorname{Gr}(l, n)$ *of l-subspaces in n-space.*