
Exercise 1.
Show that the additive group of $\mathbb{Q}$ is a torsionfree $\mathbb{Z}$-module. Show that every free submodule of $\mathbb{Q}$ is cyclic, and show that the same is true for finitely generated submodules of $\mathbb{Q}$. Give an example of a proper submodule $N \subsetneq \mathbb{Q}$ that is not cyclic.

Exercise 2.
Let A be an integral domain and M and N A-modules.

1. Show that $T(M \times N) \simeq T(M) \times T(N)$. Conclude that for $r \geq 0$, nonzero ideals $I_1, \ldots, I_s$ of A and $M = A^r \times \prod_{i=1}^s A/I_i$, we have $T(M) \simeq \prod_{i=1}^s A/I_i$ and $M/T(M) \simeq A^r$.
2. Show that a homomorphism $f : M \to N$ of A-modules restricts to a homomorphism $T(M) \to T(N)$ between their respective torsion modules. Show that this defines a left exact functor $T : \mathrm{Mod}_A \to \mathrm{Mod}_A$.

Exercise 3.

1. Let $M = \mathbb{Z}^3$ and N the submodule generated by $(1, 1, 6)$ and $(1, -1, 6)$. Determine a basis $\{v_1, v_2, v_3\}$ of M, an integer $s \geq 0$ and $d_1, \ldots, d_s \in \mathbb{Z}$ with $\langle d_1 \rangle \subset \ldots \subset \langle d_s \rangle$ such that $\{d_1.v_1, \ldots, d_s.v_s\}$ is a basis of N. Determine the invariants and the elementary divisors of M/N. What is the rank of $(M/N)/T(M/N)$?
2. Let $f : \mathbb{Z}^3 \to \mathbb{Z}^3$ be the $\mathbb{Z}$-linear map given by multiplication of row vectors in $\mathbb{Z}^3$ with the matrix
$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}.$$
Determine the Smith normal form and the invariants of f. Determine the invariants and the elementary divisors of $\mathbb{Z}^3/\mathrm{im} f$. What is the rank of $\mathbb{Z}^3/\mathrm{im} f$ divided by its torsion submodule?

Exercise 4.
An A-module M is *flat* if $- \otimes_A M$ is exact.

1. Show that every free A-module is flat. Conclude that every projective A-module is flat.
2. Let I be an ideal of A. Show that $I \otimes_A M \simeq IM$ if M is flat.

Hint: For (1), Exercise 2 from List 10 is useful. For (2), the proof of Proposition 3.7.9 is helpful.

*__Exercise 5__ (Bonus). [1]
Let A be a principal ideal domain and $r, s \geq 0$ integers.

1. Show that every A-linear map $f : A^r \to A^s$ is of the form $f(a_i) = U \cdot (a_i)$ for some $r \times s$-matrix U with coefficients in A where $U \cdot (a_i)$ denotes the usual multiplication of a matrix with a vector.

2. Let $\mathcal{B} = \{v_1, \dots, v_r\}$ be a subset of A^r and $v_{i,j}$ the j-th coordinate of v_i for $i, j = 1, \dots, r$. Show that $\mathcal{B}$ is a basis of A^r if and only if the $r \times r$-matrix U with coefficients $U_{i,j} = v_{i,j}$ is invertible.

3. Show that there are for every $r \times s$-matrix U an $r \times r$-matrix V and an $s \times s$-matrix W such that $D = WUV$ is in *Smith normal form*, i.e. there is an integer t with $0 \leq t \leq \min\{r, s\}$ and elements $d_1, \dots, d_t \in A$ with $0 \neq \langle d_1 \rangle \subset \dots \subset \langle d_t \rangle$ such that $D_{i,i} = d_i$ for $i = 1, \dots, t$ and $D_{i,j} = 0$ if $i \neq j$ or $i = j > t$.

4. Exhibit invertible matrices V such that multiplying U with V from the right (from the left) results in *(a)* multiplying a column (row) by a unit of A; *(b)* an exchange of columns (rows); *(c)* adding a multiple of a column (row) to another. Such matrices V are called *elementary matrices*.

5. Let A be a Euclidean domain with Euclidean norm $N : A \to \mathbb{N}$. Develop an algorithm using elementary column and row operations to bring U into Smith normal form.

   Hint: One can refine the Gaussian algorithm appropriately using the Euclidean norm for the pivot search. If a pivot does not divide all coefficients of a given column and row, then one can produce a new pivot of smaller norm with the help of the Euclidean algorithm.

Remark: An algorithm as in part (5) does not exist for principal ideal domains in general since in this generality, there are examples of invertible matrics that are not products of elementary matrices.

[1] Starred exercises are hard problem for those of you that search for a challenge. To balance the amount of work required to solve these exercises, starred exercises they are worth twice as many points as normal exercises.