Exercises for Algebra II
List 9 To hand in at 29.10.2018 in the exercise class

Exercise 1.
Determine all simple characters of A_4, which can be done as follows.

1. Show that A_4 has 4 conjugacy classes and determine representatives for each class.
2. Show that A_4^{ab} is isomorphic to $\mathbb{Z}/3\mathbb{Z}$ and conclude that A_4 has precisely 3 one-dimensional characters χ_1, χ_2 and χ_3. Determine these characters.
3. Show that A_4 has precisely one more simple character χ_4, which is of dimension 3. Determine χ_4. Can you find a representation with character χ_4?

Exercise 2.
Determine all simple characters of the dihedral group D_5 with 10 elements, which can be done as follows.

1. Show that D_5 has 4 conjugacy classes and determine representatives for each class.
2. Show that D_5^{ab} is isomorphic to $\mathbb{Z}/2\mathbb{Z}$ and conclude that D_5 has precisely 2 one-dimensional characters χ_1 and χ_2. Determine these characters.
3. Show that the action of D_5 on the regular pentagon defines a 2-dimensional representation and calculate its character χ_3.
4. Show that $\chi_4 = \chi_3^*$ is the forth character of D_5.

Exercise 3.
Let G be a group acting on a set X and $\mathbb{C}^X$ the corresponding permutation representation. Show that the contragredient representation of $\mathbb{C}^X$ is isomorphic to $\mathbb{C}^X$.

Exercise 4.
Let U, V and W be representations of G over a field K. Find an isomorphism

$$\mathrm{Hom}_K(U \otimes_K V, W) \longrightarrow \mathrm{Hom}_K(U, \mathrm{Hom}_K(V, W))$$

of G-representations. Conclude that $\mathrm{Hom}_K(U, V) \simeq U^* \otimes_K V$ as G-representations.

Bonus: Show that $- \otimes_K V$ and $\mathrm{Hom}_K(V, -)$ are naturally functors from $\mathrm{Rep}_K(G)$ to $\mathrm{Rep}_K(G)$, and that $- \otimes_K V$ is left-adjoint to $\mathrm{Hom}_K(V, -)$.

Exercise 5 (Bonus)**.**
Let G be a finite group. Let X be the set of isomorphism classes $[V]$ of complex representations V of G.

1. Show that $([V_1],[W_1]) \sim ([V_2],[W_2])$ if and only if $V_1 \oplus W_2 \simeq V_2 \oplus W_1$ defines an equivalence relation $\sim$ on $X \times X$. Define the *(0-th) K-group of G* as the quotient set
$$K_0(G) \;=\; X \times X/\sim .$$
We write $V - W$ for the equivalence class of $([V],[W])$ in $K_0(G)$ and call $V - W$ a *virtual representation of* G. We write V for $V - \{0\}$ where $\{0\}$ is the zero-dimensional representation.

2. Show that the addition
$$(V_1 - W_1) + (V_2 - W_2) \;=\; V_1 \oplus V_2 \;-\; W_1 \oplus W_2$$
and the multiplication
$$(V_1 - W_1) \cdot (V_2 - W_2) \;=\; (V_1 \otimes V_2) \oplus (W_1 \otimes W_2) \;-\; (V_1 \otimes W_2) \oplus (V_2 \otimes W_1)$$
turn $K_0(G)$ into a ring whose zero is $\{0\}$ and whose one is the trivial one-dimensional representation $\mathbb{C}$.

3. Show that $K_0(G)$ is freely generated over $\mathbb{Z}$ by the classes $[V_1], \ldots, [V_s]$ of the irreducible representations of G, i.e. $K_0(G) \simeq \mathbb{Z}[T_1, \ldots, T_s]$.

4. Show that the association $V \mapsto \chi_V$ defines an injective ring homomorphism $K_0(G) \to \{C(G) \to \mathbb{C}\}$.