Exercises for Algebra II
List 6 To hand in at 1.10.2018 in the exercise class

Exercise 1.
Let K be a field and L the splitting field of a cubic polynomial f over K. Assume that $\zeta_3 \in L$ and that L/K is separable. Show that there is a subfield E of L such that $K \subset E \subset L$ is a tower of elementary radical extensions (with possibly $L = E$ or $E = K$). In which situations are E/K and L/E cyclotomic, Kummer and Artin-Schreier? What are E and L if $K = \mathbb{Q}$ and $f = T^3 - b \in \mathbb{Q}[T]$?

Exercise 2.
Let $L/\mathbb{Q}$ be a cubic solvable extension. Show that $L/\mathbb{Q}$ is not radical. Show that such an extension exists.

Hint: Show that if $L/\mathbb{Q}$ was radical, it must contain ζ_3. Lead this to a contradiction.

Exercise 3.
Which roots of the following polynomials are constructible over $\mathbb{Q}$?

1. $f_1 = T^4 - 2$
2. $f_2 = T^4 - T$
3. $f_3 = T^4 - 2T$

Exercise 4.
Let K be a subfield of $\mathbb{C}$ and a a root of $T^2 - b \in K[T]$. Show that every element of $K(a)$ is constructible over K. Use this to explain the relationship between the two definitions of constructible numbers from sections 1.1 and 4.6 of the lecture.

***Exercise 5.** [1]
Let ζ_n be a primitive n-th root of unity.

1. Determine its minimal polynomial over $\mathbb{Q}$ and the Galois group $\mathrm{Gal}(\mathbb{Q}(\zeta_n)/\mathbb{Q})$ for $n = 1, \ldots, 20$.
2. Calculate $\mathrm{N}_{\mathbb{Q}(\zeta_n)/\mathbb{Q}}(\zeta_n)$ and $\mathrm{Tr}_{\mathbb{Q}(\zeta_n)/\mathbb{Q}}(\zeta_n)$.
3. Find all $n \geq 0$ such that $\mathbb{Q}(\zeta_n)/\mathbb{Q}$ is quadratic.
4. Determine all subfields of $\mathbb{Q}(\zeta_n)$ for your 5 favorite values of n.

***Exercise 6.**
Let K be a field and L the splitting field of a polynomial f over K of degree 4 or less. Show that L/K is solvable if it is separable.

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[1] The starred exercises are not to hand in. They are meant as suggestions for your preparation for the exam.

***Exercise 7.**
Let K be $\mathbb{Q}$, $\mathbb{F}_3$ or $\mathbb{F}_5$, $n = 3$ or 4 and $a = 1$, 2 or 3. Consider the polynomial $f = T^n - a$ in $K[T]$ and its splitting field L over K.

1. Is L/K separable? If so, calculate $\mathrm{Gal}(L/K)$.

   (*Remark:* Notice the different outcomes for $\mathrm{Gal}(L/K)$ if K or a varies.)

2. Determine all intermediate fields E of L/K and find primitive elements for E/K.

3. Which of the subextensions F/E (with $K \subset E \subset F \subset L$) are separable, normal, cyclic, cyclotomic, abelian, solvable, Kummer, Artin-Schreier or radical?

***Exercise 8.**
Which of the following elements are constructible over $\mathbb{Q}$?

1. $\sqrt{3}$, $\sqrt{-3}$, $\sqrt{6}$, $\sqrt{2}+\sqrt{3}$, $\sqrt[3]{3}$, $\sqrt[4]{3}$.

2. ζ_n for $n = 1, \dots, 20$.

3. $1+\zeta_4$, $\zeta_3+\zeta_6$, $\zeta_3+\zeta_9$, $\zeta_6+\zeta_6^{-1}$, $\zeta_9+\zeta_9^{-1}$, $\zeta_9+\zeta_9^4+\zeta_9^7$, $\zeta_7+\zeta_7^{-1}$, $\zeta_7+\zeta_7^2+\zeta_7^4$.

Let a be any of the above elements and L the normal closure of $\mathbb{Q}(a)/\mathbb{Q}$. Calculate $N_{L/\mathbb{Q}}(a)$ and $\mathrm{Tr}_{L/\mathbb{Q}}(a)$.

***Exercise 9.** Give three examples and three non-examples for the following types of extensions: algebraic, transzendental, separable, purely inseparable, normal, Galois, cyclic, cyclotomic, abelian, solvable, Kummer, Artin-Schreier, simple radical and radical.

***Exercise 10.**
Find normal bases for the following extensions: $\mathbb{Q}(\zeta_3)/\mathbb{Q}$, $\mathbb{Q}(\sqrt[3]{2})/\mathbb{Q}$, $\mathbb{F}_4/\mathbb{F}_2$ and $\mathbb{F}_8/\mathbb{F}_2$.

***Exercise 11.**
Let $\mathbb{F}_p[x, y]$ be the polynomial ring in two variables x and y and $\mathbb{F}_p(x, y)$ its fraction field. Let $\sqrt[p]{x}$ be a root of $T^p - x$ and $\sqrt[p]{y}$ be a root of $T^p - y$.

1. Show that $\mathbb{F}_p(\sqrt[p]{x}, \sqrt[p]{y})$ is a field extension of $\mathbb{F}_p(x, y)$ of degree p^2.

2. Show that $a^p \in \mathbb{F}_p(x, y)$ for every $a \in \mathbb{F}_p(\sqrt[p]{x}, \sqrt[p]{y})$.

3. Conclude that the field extension $\mathbb{F}_p(\sqrt[p]{x}, \sqrt[p]{y}) \,/\, \mathbb{F}_p(x, y)$ has no primitive element and that it has infinitely many intermediate extensions.

***Exercise 12.**
Consider the purely transcendental extension $K = \mathbb{F}_3(x)/\mathbb{F}_3$ of transcendence degree 1, and let $\overline{K}$ be an algebraic closure of K. Let $a \in \overline{K}$ be a root of $f = T^3 - x$ and $b \in \overline{K}$ a root of $g = T^2 - 2$. Find the separable closure E of K in $K(a, b)$. What are the degrees $[K(a, b) : E]$ and $[E : K]$? What are the corresponding separable degrees and inseparable degrees?

***Exercise 13.**
Solve all exercises of Chapters V and VI of Lang's "Algebra".