Exercises for Algebra II
List 5 To hand in at 24.9.2018 in the exercise class

Exercise 1.
Show that there is an n_i for $i = 1, 2, 3$ such that the following fields E_i are contained in $\mathbb{Q}(\zeta_{n_i})$. What are the smallest values for n_i?

1. $E_1 = \mathbb{Q}(\sqrt{2})$;
2. $E_2 = \mathbb{Q}(\sqrt{3})$;
3. $E_3 = \mathbb{Q}(\sqrt{-3})$;

Hint: Try to realize $\sqrt{2}$ and $\sqrt{3}$ as the side length of certain rectangular triangles. Which angles do occur?

Exercise 2.
Let ζ_{12} be a primitive 12-th root of unity. What is $\mathrm{Gal}(\mathbb{Q}(\zeta_{12}/\mathbb{Q}))$? Find primitive elements for all subfields E of $\mathbb{Q}(\zeta_{12})$.

Exercise 3. Let L be the splitting field of $T^3 - 2$ over $\mathbb{Q}$. Show that $\sqrt[3]{2}$, $\sqrt{-3}$ and ζ_3 are elements of L. Calculate $\mathrm{N}_{L/\mathbb{Q}}(a)$ and $\mathrm{Tr}_{L/\mathbb{Q}}(a)$ for $a = \sqrt[3]{2}$, $a = \sqrt{-3}$ and $a = \zeta_3$. Calculate $\mathrm{N}_{\mathbb{Q}(\zeta_3)/\mathbb{Q}}(\zeta_3)$ and $\mathrm{Tr}_{\mathbb{Q}(\zeta_3)/\mathbb{Q}}(\zeta_3)$.

Exercise 4.
Let L be the splitting field of $f = T^4 - 3$ over $\mathbb{Q}$. What is the Galois group of $L/\mathbb{Q}$? Make a diagram of all subgroups of $\mathrm{Gal}(L/K)$ that illustrates which subgroups are contained in others. Which of the subextensions of $L/\mathbb{Q}$ are elementary radical? Is $L/\mathbb{Q}$ radical?

Hint: Find the four complex roots $a_1, \ldots, a_4$ of f. Which permutations of $a_1, \ldots, a_4$ extend to field automorphisms of L?

*__Exercise 5__ (Bonus). [1]
Let L/K be a Galois extension and let

$$\begin{array}{rccc} M_a: & L & \longrightarrow & L \\ & b & \longmapsto & a \cdot b \end{array}$$

be the K-linear map associated with an element $a \in L$. Show that the trace of M_a equals $\mathrm{Tr}_{L/K}(a)$ and that the determinant of M_a equals $\mathrm{N}_{L/K}(a)$.

Hint: Use Exercise 1 from List 2.

*__Exercise 6__ (Bonus).
Find all composition series and their factors for the dihedral group

$$D_6 = \langle r, s \,|\, r^6 = s^2 = e, \, srs = r^{-1} \rangle.$$

*__Exercise 7__ (not to hand in).
Let p be a prime number and $n \geq 1$ and $\zeta \in \mathbb{F}_{p^n}$ a generator of $\mathbb{F}_{p^n}^\times$. Exhibit an embedding $i : \mathrm{Gal}(\mathbb{F}_{p^n}/\mathbb{F}_p) \to (\mathbb{Z}/(p^n-1)\mathbb{Z})^\times$ and conclude that n divides $\varphi(p^n-1)$. Can you find a proof for $n|\varphi(p^n-1)$ that does not use Galois theory?

[1] It is advised to work on the bonus exercises and to hand in solution. The resulting points count as a bonus.