Exercises for Algebra II
List 3 To hand in at 10.9.2018 in the exercise class

Exercise 1.
Let $f = T^6 + T^3 + 1 \in \mathbb{Q}[T]$ and $L = \mathbb{Q}[T]/(f)$. Show that f is irreducible and find all field homomorphisms $L \to \mathbb{C}$. Is $L/\mathbb{Q}$ normal?

Hint: f divides $T^9 - 1$.

Exercise 2.
Proof Fermat's little theorem: If K is a field of characteristic p, then $(a+b)^p = a^p + b^p$. Conclude that $\mathrm{Frob}_{p^n} : K \to K$ with $\mathrm{Frob}_{p^n}(a) = a^{p^n}$ is a field automorphism of K.

Remark: Frob_p is called the *Frobenius homomorphism in characteristic* p.

Exercise 3.
Let $\mathbb{F}_p(x)$ be the quotient field of the polynomial ring $\mathbb{F}_p[x]$ in the indeterminant x, i.e. $\mathbb{F}_p(x) = \{f/g | f, g \in F_p[x] \text{ and } g \neq 0\}$.

1. Show that $f = T^p - x$ is irreducible over $\mathbb{F}_p(x)$.

   Hint: For a direct calculation, use the factorization of f over $\mathbb{F}_p(\sqrt[p]{x})$; or you can apply the Eisenstein criterium to show that f is irreducibel in $\mathbb{F}_p[x, T]$ and conclude with the help of Gauss' lemma that f is irreducible in $\mathbb{F}_p(x)[T]$.

2. Show that f is not separable over $\mathbb{F}_p(x)$.

   Hint: Use Fermat's little theorem.

3. Conclude that $\mathbb{F}_p(\sqrt[p]{x})/\mathbb{F}_p(x)$ is not separable. Is $\mathbb{F}_p(\sqrt[p]{x})/\mathbb{F}_p(x)$ normal?

Exercise 4.
Let $\zeta_3 = e^{2\pi/3} \in \mathbb{C}$ be a primitive third root of unity, i.e. $\zeta_3^3 = 1$, but $\zeta_3 \neq 1$. Which of the field extensions $\mathbb{Q}(\zeta_3)$, $\mathbb{Q}(\sqrt[3]{2})$ and $\mathbb{Q}(\zeta_3, \sqrt[3]{2})$ of $\mathbb{Q}$ are Galois? What are the respective automorphism groups over $\mathbb{Q}$? Find all intermediate extensions of $\mathbb{Q}(\zeta_3, \sqrt[3]{2})/\mathbb{Q}$ and draw a diagram.

***Exercise 5.** [1]
Let L/K be a finite field extension and E the separable closure of K in L. Show that $[E : K]_s = [E : K]$ and $[L : E]_s = 1$. Conclude that the separable degree $[L : K]_s$ is a divisor of $[L : K]$.

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[1] The starred exercises are bonus exercises. It is advised to work on these exercises and to hand in solution. The resulting points count as a bonus.

***Exercise 6.**

1. Find a finite separable (but not normal) field extension L/K that does not satisfy the Galois correspondence.
2. Find a finite normal (but not separable) field extension L/K that does not satisfy the Galois correspondence.
3. Find a normal and separable (but not finite) field extension L/K that does not satisfy the Galois correspondence.