Exercises for Algebraic Number Theory
List 2 to hand in at 16.1.2017 in the exercise class

Exercise 1.
Every unique factorization domain is integrally closed. Conclude that $\mathbb{Z}$ is integrally closed in $\mathbb{Q}$.

Exercise 2.
Let L/K be a separable field extension of degree n with basis $(1, a, \ldots, a^{n-1})$ for some $a \in L$. Then $d(1, a, \ldots, a^{n-1}) = \prod_{i<j}(a_i - a_j)^2$.

Exercise 3. Let L be a number field and $\mathcal{O}_L$ its integers.

1. Let M be a non-trivial finitely generated $\mathcal{O}_L$-submodule of L. Show that the discriminant $d(M) = d(b_1, \ldots, b_n)$ of M does not depend on the choice of a $\mathbb{Z}$-basis $(b_1, \ldots, b_n)$ of M.
2. Let $M \subset M'$ be two non-trivial finitely generated $\mathcal{O}_L$-submodules of L. Show that the index $(M' : M)$ is finite and that $d(M) = (M' : M)^2 d(M')$.

Exercise 4.

1. Show that every quadratic extension of $\mathbb{Q}$ is of the form $\mathbb{Q}[\sqrt{D}]$ with $D \in \mathbb{Z}$.
2. Let $D \in \mathbb{Z}$ be squarefree and different from 0 and 1. Consider $L = \mathbb{Q}[\sqrt{D}]$ and its integers $\mathcal{O}_L$. Show that a basis of $\mathcal{O}_L$ over $\mathbb{Z}$ is given by
$$\begin{array}{ll} \{1, \sqrt{D}\} & \text{if } D \equiv 2 \text{ or } 3 \pmod 4, \\ \{1, (1+\sqrt{D})/2\} & \text{if } D \equiv 1 \pmod 4. \end{array}$$
3. Calculate the discriminant of L in either case.

Hint: Use the norm and trace to calculate the minimal polynomial of an element $a \in L$.

Exercise 5. Consider the ring of integers $\mathbb{Z}[\sqrt{-5}] = \{a + b\sqrt{-5} | a, b \in \mathbb{Z}\}$ of $\mathbb{Q}[\sqrt{-5}]$.

1. Show that $N_{\mathbb{Z}[\sqrt{-5}]/\mathbb{Z}}(a + b\sqrt{-5}) = a^2 + 5b^2$.
2. Calculate all elements of $\mathbb{Z}[\sqrt{-5}]$ with norm in $\{0, \ldots, 9\}$. Conclude that 2, 3, $1 + \sqrt{-5}$ and $1 - \sqrt{-5}$ are irreduzible in $\mathbb{Z}[\sqrt{-5}]$.
3. Find all factorizations of 6 into irreduzible elements in $\mathbb{Z}[\sqrt{-5}]$. Is $\mathbb{Z}[\sqrt{-5}]$ a unique factorization domain? Which of 2, 3, $1 + \sqrt{-5}$ and $1 - \sqrt{-5}$ are prime in $\mathbb{Z}[\sqrt{-5}]$?

***Exercise 6.**
Which of the following numbers are algebraic integers?

$$\sqrt{2}, \quad \frac{1}{2}, \quad \sqrt{\frac{1}{2}}, \quad \frac{1+\sqrt{5}}{2}, \quad \frac{3+2\sqrt{6}}{1-\sqrt{6}}.$$

***Exercise 7.** Verify all properties of norm and trace as claimed in Proposition 2 of the lecture. To keep the proofs simple, you may assume that all field extensions are separable.

***Exercise 8** (Chinese Remainder theorem)**.**
Let A be a ring, $I_1, \dots, I_n$ be ideals of R with $I_i + I_j = A$ for $i \neq j$ and $J = \bigcap_{i=1}^n I_i$. Then there is a canonical isomorphism of rings

$$A/J \xrightarrow{\sim} \bigoplus_{i=1}^{n} A/I_i.$$

***Exercise 9.** Let $f : A \to B$ be a ring homomorphism and I an ideal of B. Show that $f^{-1}(I)$ is an ideal of A, and show that if I is a prime ideal, then so is $f^{-1}(I)$. Is the converse statement true?

***Exercise 10.** Let A be a ring. Show that the A-submodules of A are precisely the ideals of A.

***Exercise 11.** Let A be a ring, $a, b \in A$ and $I, J \subset A$ ideals.

1. Show that $a \mid b$ if and only if $(a) \mid (b)$.
2. Show that $I \cap J \mid I \cdot J$. Is the converse true?
3. Show that $I \cap J$ is the least common multiple of I and J.
4. Show that $I + J$ is the greatest common divisor of I and J.
5. Show that (a) is a nonzero prime ideal if and only if a is a prime element.
6. Show that I is a prime ideal if and only if for all ideal $J_1, \dots, J_n$ of A such that $I \mid J_1 \cdots J_n$, there is an $i \in \{1, \dots, n\}$ such that $I \mid J_i$.

The starred exercises are not to hand in.