Exercises for Algebra 1 — Instituto Nacional de Matemática Pura e Aplicada
List 8 — Oliver Lorscheid
To hand in at 18.5. in the exercise class — Esteban Arreaga (monitor)

Exercise 1.
Let M and N be A-modules. Show that $\operatorname{Hom}_A(M,N)$ is an A-module with respect to the operations $f+g : m \mapsto f(m)+g(m)$ and $a.f : m \mapsto a.f(m)$ for $a \in A$ and $f, g \in \operatorname{Hom}_A(M,N)$. Show that

$$\begin{array}{ccc} \operatorname{Hom}_A(M,N) \times \operatorname{Hom}_A(N,P) & \longrightarrow & \operatorname{Hom}_A(M,P) \\ (f,g) & \longmapsto & g \circ f \end{array}$$

is an A-bilinear homomorphism.

Exercise 2.
Let k be a field, $A = k[T]$ and $M = N = k^2$, as an additive group. Define a map $A \times M \to M$ by

$$\Big(\sum a_i T^i\Big).(m,n) \quad = \quad \Big(\sum (a_i m)\,,\, \sum a_i n\Big)$$

and a map $A \times N \to N$ by

$$\Big(\sum a_i T^i\Big).(m,n) \quad = \quad \Big(\sum (a_i m + i a_i n)\,,\, \sum a_i n\Big)$$

where $a_i, m, n \in k$. Show that M and N are A-modules with respect to these maps. Show that neither M nor N is simple, but that N is indecomposable while M is not.
Hint: T acts on M as the matrix $\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$, and it acts on N as the matrix $\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$.

Exercise 3.
Verify the following assertions.

1. $\mathbb{R}^n \otimes_{\mathbb{R}} \mathbb{R}^m \simeq \mathbb{R}^{m \cdot n}$.
2. $A[T_1] \otimes_A A[T_2] \simeq A[T_1, T_2]$ for any ring A.
3. $\mathbb{Z}/n\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/m\mathbb{Z} \simeq \mathbb{Z}/d\mathbb{Z}$ where d is a greatest common divisor of the natural numbers m and n.
4. $K \otimes_{\mathbb{Z}} L = \{0\}$ if K and L are fields of different characteristics.
5. $\mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Q} \simeq \mathbb{Q}$.

Exercise 4.
Let $\{M_i\}_{i \in I}$ be a family of A-modules.

1. Show that $\prod_{i \in I} M_i$ together with the projections $\pi_j : \prod M_i \to M_j$ is the categorical product in $A - \mathrm{Mod}$.
2. Show that $\bigoplus_{i \in I} M_i$ together with the inclusions $\iota_j : M_j \to \bigoplus M_i$ is the categorical coproduct in $A - \mathrm{Mod}$.

Exercise 5.
Prove the universal properties of the kernel and the cokernel of an A-linear homomorphism.

Exercise 6.
Let $f : M \to N$ be an A-linear homomorphism.

1. Show that it is equivalent
   a) that f is a monomorphism in $A - \mathrm{Mod}$,
   b) that f is injective and
   c) that f is the kernel of a homomorphism $g : N \to Q$.
2. Show that it is equivalent
   a) that f is an epimorphism in $A - \mathrm{Mod}$,
   b) that f is surjective and
   c) that f is the cokernel of a homomorphism $h : P \to M$.
3. Show that f is an isomorphism in $A - \mathrm{Mod}$ if and only if f is bijective.

Remark. A monomorphism is *normal* if it is a kernel. An epimorphism is *normal* if it is a cokernel. A category with a zero object, kernels, cokernels, finite products and finite coproducts such that all monomorphisms and epimorphisms are normal is called an *abelian category.* Thus it follwos from the previous exercises that $A - \mathrm{Mod}$ is an abelian category.

Exercise 7 (Bonus)**.**
Let M, N, N' and P be A-modules and $f : N \to N'$ an A-linear homomorphism. Define the maps

$$f_M : \begin{array}{ccc} M \otimes_A N & \longrightarrow & M \otimes_A N', \\ m \otimes n & \longmapsto & m \otimes f(n) \end{array} \qquad f^* : \begin{array}{ccc} \mathrm{Hom}(N', P) & \longrightarrow & \mathrm{Hom}(N, P) \\ g & \longmapsto & g \circ f \end{array}$$

$$\text{and} \quad f_* : \begin{array}{ccc} \mathrm{Hom}(M, N) & \to & \mathrm{Hom}(M, N'). \\ h & \longmapsto & f \circ h \end{array}$$

1. Show that these maps are homomorphisms of A-modules.
2. Conclude that $M \otimes_A (-)$, $\mathrm{Hom}(-, P)$ and $\mathrm{Hom}(M, -)$ are functors from $A - \mathrm{Mod}$ to $A - \mathrm{Mod}$. Which of them are covariant, which of them are contravariant?
3. Show that the two functors $\mathrm{Hom}(M \otimes_A (-), P) = \mathrm{Hom}(-, P) \circ (M \otimes_A -)$ and $\mathrm{Hom}(M, \mathrm{Hom}(-, P)) = \mathrm{Hom}(M, -) \circ \mathrm{Hom}(-, P)$ are isomorphic.

Remark: One says that $M \otimes_A -$ is *left adjoint* to $\mathrm{Hom}(-, P)$, or that $\mathrm{Hom}(-, P)$ is *right adjoint* to $M \otimes_A -$.