Exercises for Algebra 1 Instituto Nacional de Matemática Pura e Aplicada
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Solutions are not to hand in Esteban Arreaga (monitor)

Let G be a group with neutral element e.

Exercise 1 (Uniqueness of the neutral element and inverses)**.**

1. Let $e' \in G$ be an element such that $e'a = a$ for all $a \in G$. Then $e' = e$.
2. Let $a, b \in G$ be elements such that $ab = e$. Then $b = a^{-1}$.
3. Show that $(ab)^{-1} = b^{-1}a^{-1}$ for all $a, b \in G$.

Exercise 2 (Properties of group homomorphisms)**.**
Let $f : G \to H$ be a group homomorphism. Show that $f(e) = e$ and that $f(a)^{-1} = f(a^{-1})$. Show that f is an isomorphism if and only if there exists a group homomorphism $g : H \to G$ such that $f \circ g$ is the identity map on H and such that $g \circ f$ is the identity map on G.

Exercise 3 (Properties of subgroups)**.**
Let H be a subset of G. Show that H is a subgroup of G if and only if $e \in H$, $m(H \times H) \subset H$ and $i(H) \subset H$. In other words, H is a subgroup if and only if it is a group with respect to the restrictions of m and i to H.

Exercise 4 (The center)**.**
Show that the *center of G*

$$Z(G) \;=\; \{\, a \in G \,|\, ab = ba \text{ for all } b \in G \,\}$$

is a subgroup of G. Show that $Z(G)$ is commutative. Show that $Z(G)$ is normal. Is every commutative subgroup normal?

Exercise 5 (The subgroup generated by a subset)**.**

1. Let $\{H_i\}_{i \in I}$ be a family of subgroups of G. Show that the intersection $\bigcap_{i \in I} H_i$ is a subgroup of G.
2. Let $S \subset G$ be a subset. Show that

$$\bigcap_{H < G \text{ with } S \subset H} H \quad = \quad \{\, a_1 a_2^{-1} \cdots a_{2n-1} a_{2n}^{-1} \,|\, n \geq 1 \text{ and } a_1, \ldots, a_n \in S \cup \{e\} \,\}$$

   and conclude that there is a unique smallest subgroup $\langle S \rangle$ of G that contains S.

Exercise 6 (Orders of elements in abelian groups)**.**
Let G be an abelian group and $a, b \in G$. Then $\operatorname{ord} ab$ is a divisor $\operatorname{ord} a \cdot \operatorname{ord} b$. Is this also true if G is not abelian?

Exercise 7 (Cyclic groups and the Klein four-group)**.**

1. Classify all cyclic groups up to isomorphism. Which of them are abelian?
2. Show that a cyclic group of order n has a unique subgroup of order d for each divisor d of n.
3. Is the *Klein four-group* $V = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ cyclic? Is it abelian?

Exercise 8 (Dihedral groups)**.**
Let D_n be the group of symmetries of a regular polygon with n sides. Show that $D_n = \langle r, s \rangle$ where r is a rotation around the center of the polygon by an angle of $2\pi/n$ and s is the reflection at a line passing through the center of the polygon and one of its vertices. What is the number of elements of D_n? Show that $D_3 \simeq S_3$, and that for $n \geq 4$, the dihedral group D_n is not isomorphic to a symmetric group.

Exercise 9 (Symmetric groups)**.**
The *symmetric group* S_n is the group of permutations of the numbers $1, \dots, n$, together with composition as multiplication, i.e. $\sigma \cdot \tau = \sigma \circ \tau$. An element σ of S_n is called a *cycle (of length l)* if $\operatorname{ord} \sigma = l$ and if there is an $i \in \{1, \dots, n\}$ such that $\sigma(j) = j$ if $j \notin \{i, \sigma(i), \dots, \sigma^{l-1}(i)\}$; we write $\sigma = \big(i, \sigma(i), \dots, \sigma^{l-1}(i)\big)$ in this case.

1. Show that $\big(i, \dots, \sigma^{l-1}(i)\big) = \big(j, \dots, \sigma^{l-1}(j)\big)$ if $j = \sigma^n(i)$ for some $n \geq 0$.
2. Two cycles $\sigma = \big(i, \dots, \sigma^{l-1}(i)\big)$ and $\tau = \big(j, \dots, \tau^{k-1}(j)\big)$ are called *disjoint* if the sets $\{i, \dots, \sigma^{l-1}(i)\}$ and $\{j, \dots, \tau^{k-1}(j)\}$ are disjoint. Show that σ and τ are disjoint if and only if $\sigma\tau = \tau\sigma$.
3. Show that every element of S_n can be written as a product of disjoint cycles.
4. A *transposition* is a cycle (i, j) of length 2. Show that every element of S_n can be written as a product of transpositions.

Exercise 10 (The signum)**.**
Let σ be an element of S_n and $\sigma = \tau_n \circ \cdots \circ \tau_1$ and $\sigma = \tau'_m \circ \cdots \circ \tau'_1$ two representations of σ as a product of transpositions $\tau_1, \dots, \tau_n$ and $\tau'_1, \dots, \tau'_m$.

1. Show that $n - m$ is even. Conclude that the map $\operatorname{sign} : S_n \to \{\pm 1\}$ that sends σ to $(-1)^n$ is well-defined.
2. Show that sign is a group homomorphism.

Exercise 11 (Theorem of Cayley)**.**
Let $G = \{a_1, \cdots, a_n\}$ be of finite order n. Define the map $f : G \to S_n$ that sends a_l to the permutation σ_l with $\sigma_l(i) = j$ such that $a_l a_i = a_j$. Show that f is an injective group homomorphism. Conclude that every finite group is isomorphic to a subgroup of a symmetric group.

Exercise 12 (The alternating group).
The *alternating group* A_n is defined as the kernel of sign : $S_n \to \{\pm 1\}$. A group G is called *simple* if $G \neq \{e\}$ and if the only normal subgroups of G are $\{e\}$ and G.

1. Show that a cyclic group G of order n is simple if and only if n is a prime number.
2. Show that A_3 is simple. Show that A_4 is not simple. What about A_1 and A_2?
3. Show that A_n is simple for $n \geq 5$.[1]

Exercise 13 (Quaternion group).
The quaternion group Q consists of the elements $\{\pm 1, \pm i, \pm j, \pm k\}$, and the multiplication is determined by the following rules: 1 is the neutral element, $(-1)^2 = 1$ and

$$i^2 = j^2 = k^2 = -1, \quad (-1)i = -i, \quad (-1)j = -j, \quad (-1)k = -k, \quad ij = k = -ji.$$

1. Is Q abelian?
2. Describe all subgroups of Q.
3. Which subgroups are normal? What are the respective quotient groups?

Exercise 14.
Classify all groups with 6 elements and all groups with 8 elements up to isomorphism.

Exercise 15 (Transitivity of index).
Let H be a subgoup of G and K a subgroup of H. Show that $(G : K) = (G : H)(H : K)$.

Exercise 16 (Quotients by non-normal subgroups).
Let H be subgroup of G. Show that the association $([a], [b]) \mapsto [ab]$ is not well-defined on cosets $[a], [b] \in G/H$ if H is not normal in G.

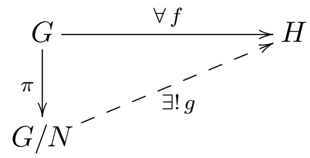
Exercise 17 (Alternative characterization of normal subgroups).
A subgroup H of G is normal if and only if $gHg^{-1} \subset H$ for every $g \in G$.

Exercise 18 (Exercises on normal subgroups).
Show the following statements.

1. Every subgroup of index 2 is normal.
2. Every subgroup of an abelian group is normal. Is there a nonabelian group G such that every subgroup H of G is normal?
3. The intersection of two normal subroups is a normal subgroup. If both normal subgroups have finite index, then their intersection has also finite index.

[1]This exercise is more difficult than others, but solutions can be found in the literature.

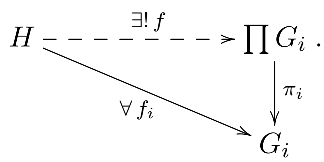
Exercise 19 (Universal property of the quotient)**.**
Let N be a normal subgroup of G. Show that the quotient map $\pi : G \to G/N$ satisfies the following universal property: for every group homomorphism $f : G \to H$ with $f(a) = e$ for $a \in N$ there exists a unique group homomorphism $g : G/N \to H$ such that $f = g \circ \pi$:



Exercise 20 (Universal property of the product)**.**
Let $\{G_i\}_{i\in I}$ be a family of groups and $G = \prod G_i$ their product.

1. Show that the map $\pi_i : G \to G_i$ that sends $(g_i)_{i\in I}$ to g_i is a surjecive group homomorphism for every $i \in I$. These maps are called the *canonical projections.*

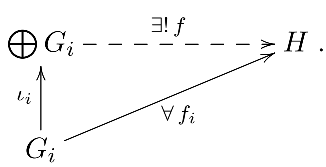
2. Show that the product together with the canonical projections satisfies the following universal property: for every family of group homomorphisms $\{f_i : H \to G_i\}_{i\in I}$, there is a unique group homomorphism $f : H \to \prod G_i$ such that $f_i = \pi_i \circ f$ for every $i \in I$:



Exercise 21 (Universal property of the direct sum)**.**
Let $\{G_i\}_{i\in I}$ be a family of abelian groups and $G = \bigoplus G_i$ their direct sum.

1. Show that the map $\iota_i : G_i \to G$ that sends g to $(g_j)_{j\in I}$ with $g_i = g$ and $g_j = e_j$ for $j \neq i$ is an injective group homomorphism for every $i \in I$. These maps are called the *canonical injections.*

2. Show that the direct sum together with the canonical injections satisfies the following universal property: for every family of group homomorphisms $\{f_i : G_i \to H\}_{i\in I}$ of abelian groups, there is a unique group homomorphism $f : \bigoplus G_i \to H$ such that $f_i = f \circ \iota_i$ for every $i \in I$:



3. Is the same true if H is a non-abelian group?

Exercise 22 (Some group actions).
Show that the following maps are group actions.

1. $S_n \times \{1, \dots, n\} \to \{1, \dots, n\}$, with $\sigma.i = \sigma(i)$.
2. $\mathrm{GL}_n(\mathbb{R}) \times \mathbb{R}^n \to \mathbb{R}^n$, with $g.v = g \cdot v$ (usual matrix multiplication).
3. $\mathbb{R}^\times \times \mathbb{R}^n \to \mathbb{R}^n$, with $a.v = a \cdot v$ (scalar multiplication).
4. The permutation of the vertices of a regular n-gon by elements of the dihedral group D_n.

Exercise 23 (Center and centralizer).

1. Show that the definitions of the center $Z(G)$ from the lecture and Exercise 4 agree, i.e. that
$$\{\, x \in G \,|\, \mathcal{O}(x) = \{x\} \,\} \;=\; \{\, a \in G \,|\, ab = ba \text{ for all } b \in G \,\}$$
where the orbit $\mathcal{O}(x)$ is taking w.r.t. the action of G on G by conjugation.
2. Show that $C_G(x) = \{a \in G | ax = xa\}$.
3. Show that
$$Z(G) \;=\; \bigcap_{x \in G} C_G(x).$$

Exercise 24 (Normalizer).
Let H be a subgroup of G. Show that its normalizer $\mathrm{Norm}_G(H)$ is the largest subgroup of G containing H such that H is a normal subgroup of $\mathrm{Norm}_G(H)$. Show further that the following properties are equivalent:

1. H is normal in G;
2. $\mathrm{Norm}_G(H) = G$;
3. H is a fixed point for the action of G on the set of all subgroups of G by conjugation.

Exercise 25 (Short exact sequences).
A short exact sequence of groups is a sequence
$$\{e\} \xrightarrow{f_1} N \xrightarrow{f_2} G \xrightarrow{f_3} Q \xrightarrow{f_4} \{e\}$$
of groups and group homomorphism such that $\mathrm{im} f_i = \ker f_{i+1}$ for $i = 1, 2, 3$.

1. Show that $\mathrm{im} f_i = \ker f_{i+1}$ for $i = 1, 2, 3$ holds if and only if f_2 is injective, if $\mathrm{im} f_2 = \ker f_3$ and if f_3 is surjective.
2. Show that N is a normal subgroup of G and that $G/N \simeq Q$ in case of a short exact sequence.

Exercise 26.
Calculate all orbits and stabilizers for the action of D_4 on itself by conjugation.

Exercise 27 (Commutator subgroup)**.**
The commutator of two elements $a, b \in G$ is $[a, b] = aba^{-1}b^{-1}$. The commutator subgroup of G is the subgroup $[G, G]$ generated by the commutators $[a, b]$ of all pairs of elements a and b of G.

1. Show that $[a, b] = e$ if and only if $ab = ba$. Conclude that $[G, G] = \{e\}$ if and only if G is abelian.

2. Show that $c[a, b]c^{-1} = [cac^{-1}, cbc^{-1}]$ and conclude that $[G, G]$ is a normal subgroup of G.

3. Show that the quotient group $G^{\mathrm{ab}} = G/[G, G]$ is abelian.

4. Show that G^{ab} together with the projection $\pi : G \to G^{\mathrm{ab}}$ satisfies the following universal property: for every group homomorphism $f : G \to H$ into an abelian group H, there exists a unique group homomorphism $f^{\mathrm{ab}} : G^{\mathrm{ab}} \to H$ such that $f = f^{\mathrm{ab}} \circ \pi$: