Exercises for Algebraic Number Theory Oliver Lorscheid
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To hand in until 29.1.2015 with Emilio Peixoto Assemany or Roberto Alvarenga Jr.

Exercise 1.
Let ζ_3 be a primitive third root of unity and $K = \mathbb{Q}[\zeta_3]$. Show that the integral closure of $\mathbb{Z}$ in K is $\mathbb{Z}[\zeta_3]$ and calculate the canonical volume of $\mathbb{Z}[\zeta_3]$ in the Minkowski space $K_\mathbb{R}$ of K.

Exercise 2.
Let K be a quadratic number field and $\mathcal{O}_K$ its integers. Let $j : K \to K_\mathbb{R}$ be the embedding into the Minkowski space of K, and vol the canonical measure and vol_M the Minkowski measure. Calculate $\mathrm{vol}(\Gamma)$ and $\mathrm{vol}_M(\Gamma)$ for the lattice $\Gamma = j(\mathcal{O}_K)$.

Exercise 3.
Let V be a real vector space and Γ a subgroup of V.

1. Show that Γ is a discrete subgroup if and only if $\Gamma = \langle v_1, \dots, v_m \rangle$ for vectors $v_1, \dots, v_m \in V$ that are linearly independent over $\mathbb{R}$.
2. Show that Γ spans V over $\mathbb{R}$ if and only if there is a bounded subset $M \subset V$ such that $V = \bigcup_{\gamma \in \Gamma} \gamma + M$. Show that if Γ is a lattice in V, then any fundamental mesh Φ satisfies $V = \coprod_{\gamma \in \Gamma} \gamma + \Phi$.

Exercise 4.
Let K be an algebraic number field of degree n with integers $\mathcal{O}_K$ and discriminant d_K.

1. Show that there exists a primitive element $a \in \mathcal{O}_K$, i.e. $K = \mathbb{Q}[a]$.
2. Show that
$$d(1, a, \dots, a^{n-1}) \; = \; (\mathcal{O}_K : \mathbb{Z}[a])^2 \cdot d_K.$$
3. Conclude that $\mathcal{O}_K = \mathbb{Z}[a]$ if $d(1, a, \dots, a^{n-1})$ is squarefree.

Exercise 5.
Let K be a number field. Show that there is a unique $\mathbb{C}$-linear map $\varphi : K \otimes_\mathbb{Q} \mathbb{C} \to \prod_\tau \mathbb{C}$ with $\varphi(a \otimes 1) = (\tau(a))_\tau$ where $\tau : K \to \mathbb{C}$ ranges through all field embeddings, and that this map is an isomorphism of $\mathbb{C}$-vector spaces.

Hint: The linear independence of characters is a general fact (e.g. see [Lang: Algebra, VIII.4 Thm. 7]), which implies that the embeddings $\tau : K \to \mathbb{C}$ are linearly independent over $\mathbb{C}$.

***Exercise 6.** What is the canonical volume of the ideal $(1+i)$ of $\mathbb{Z}[i]$ inside the Minkowski space of $\mathbb{Q}[i]$? Is it equal to the Minkowski volume?

***Exercise 7.** Let A be a ring. Show the following.

1. An element $a \in A$ is prime if and only if the principal ideal (a) is a nonzero prime ideal.

2. An ideal $\mathfrak{p}$ of A is a prime ideal if and only if for all ideals $I_1, \ldots, I_n$ of A, the relation $\mathfrak{p} \mid I_1 \cdots I_n$ implies that $\mathfrak{p} \mid I_k$ for some $k \in \{1, \ldots, n\}$.

***Exercise 8.**
Let V be an n-dimensional Euclidean space with scalar product $\langle -, - \rangle$. Let $\Phi \subset V$ be the parallelepiped spanned by the vectors $v_1, \ldots, v_n \in V$. Show that

$$\operatorname{vol}(\Phi)^2 \;=\; \bigl|\det\bigl((\langle v_i, v_j \rangle)_{i,j}\bigr)\bigr|.$$

This number is also called the *Gram determinant of* $v_1, \ldots, v_n$.

***Exercise 9** (Elementary divisor theorem)**.** Let A be a PID, M be a free module over A and $M' \subset M$ a finitely generated submodule. Then there exists a basis $\mathcal{B}$ of M, $b_1, \ldots, b_n \in \mathcal{B}$ and non-zero elements $(a_1, \ldots, a_n \in A$ such that

1. $(a_1 e_1, \ldots, a_n e_n$ is a basis for M', and

2. $a_i | a_{i+1}$ for $i = 1, \ldots, n-1$.

The sequence of ideals $(a_n) \subset \ldots \subset (a_1)$ is uniquely determined by the previous conditions.

Hint: A proof can be found in [Lang, Algebra, III.7].

The starred exercises are not to hand in.