
***Exercise 1.**
Let ζ_n be a primitive n-th root of unity.

1. Determine its minimal polynomial over $\mathbb{Q}$ and the Galois group $\mathrm{Gal}(\mathbb{Q}(\zeta_n)/\mathbb{Q})$ for $n = 1, \dots, 20$.
2. Calculate $\mathrm{N}_{\mathbb{Q}(\zeta_n)/\mathbb{Q}}(\zeta_n)$ and $\mathrm{Tr}_{\mathbb{Q}(\zeta_n)/\mathbb{Q}}(\zeta_n)$.
3. Find all $n \geq 0$ such that $\mathbb{Q}(\zeta_n)/\mathbb{Q}$ is quadratic.
4. Determine all subfields of $\mathbb{Q}(\zeta_n)$ for your 5 favorite values of n.

***Exercise 2.**
Let K be $\mathbb{Q}$, $\mathbb{F}_3$ or $\mathbb{F}_5$, $n = 3$ or 4 and $a = 1$, 2 or 3. Consider the polynomial $f = T^n - a$ in $K[T]$ and its splitting field L over K.

1. Is L/K separable? If so, calculate $\mathrm{Gal}(L/K)$.

   (*Remark:* Notice the different outcomes for $\mathrm{Gal}(L/K)$ if K or a varies.)
2. Determine all intermediate fields E of L/K and find primitive elements for E/K.
3. Which of the subextensions F/E (with $K \subset E \subset F \subset L$) are separable, normal, cyclic, cyclotomic, abelian, solvable, Kummer, Artin-Schreier or radical?

***Exercise 3.**
Which of the following elements are constructible over $\mathbb{Q}$?

1. $\sqrt{3}$, $\sqrt{-3}$, $\sqrt{6}$, $\sqrt{2} + \sqrt{3}$, $\sqrt[3]{3}$, $\sqrt[4]{3}$.
2. ζ_n for $n = 1, \dots, 20$.
3. $1+\zeta_4$, $\zeta_3+\zeta_6$, $\zeta_3+\zeta_9$, $\zeta_6+\zeta_6^{-1}$, $\zeta_9+\zeta_9^{-1}$, $\zeta_9+\zeta_9^4+\zeta_9^7$, $\zeta_7+\zeta_7^{-1}$, $\zeta_7+\zeta_7^2+\zeta_7^4$.

Let a be any of the above elements and L the normal closure of $\mathbb{Q}(a)/\mathbb{Q}$. Calculate $N_{L/\mathbb{Q}}(a)$ and $\mathrm{Tr}_{L/\mathbb{Q}}(a)$.

***Exercise 4.** Give three examples and three non-examples for the following types of extensions: algebraic, transzendental, separable, purely inseparable, normal, Galois, cyclic, cyclotomic, abelian, solvable, Kummer, Artin-Schreier and radical.

***Exercise 5.**
How many primitive elements has $\mathbb{F}_8$ over $\mathbb{F}_2$?

***Exercise 6.**
Find normal bases for the following extensions: $\mathbb{Q}(\zeta_3)/\mathbb{Q}$, $\mathbb{Q}(\sqrt[3]{2})/\mathbb{Q}$, $\mathbb{F}_4/\mathbb{F}_2$ and $\mathbb{F}_8/\mathbb{F}_2$.

***Exercise 7.**
Consider the purely transcendental extension $K = \mathbb{F}_3(x)/\mathbb{F}_3$ of transcendence degree 1, and let $\overline{K}$ be an algebraic closure of K. Let $a \in \overline{K}$ be a root of $f = T^3 - x$ and $b \in \overline{K}$ a root of $g = T^2 - 2$. Find the separable closure E of K in $K(a, b)$. What are the degrees $[K(a, b) : E]$ and $[E : K]$? What are the corresponding separable degrees and inseparable degrees?

***Exercise 8.**
Solve all exercises of Chapters V and VI of Lang's "Algebra".