Exercises for Algebra II
Series 3
To hand in until 24.8.

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Exercise 1.
Which of the field extensions $\mathbb{Q}(\zeta_3)$, $\mathbb{Q}(\sqrt[3]{2})$ and $\mathbb{Q}(\zeta_3, \sqrt[3]{2})$ of $\mathbb{Q}$ are normal?

Exercise 2.
Let $f = X^6 + X^3 + 1 \in \mathbb{Q}[T]$ and $L = \mathbb{Q}[T]/(f)$. Show that f is irreducible and find all field homomorphisms $L \to \mathbb{C}$. Is $L/\mathbb{Q}$ normal?

Hint: f divides $X^9 - 1$.

Exercise 3.
Proof Fermat's small theorem: If K is a field of characteristic p, then $(a+b)^p = a^p + b^p$. Conclude that $\mathrm{Frob}_p : K \to K$ with $\mathrm{Frob}_p(a) = a^p$ is a field automorphism of K.

Remark: Frob_p is called the *Frobenius homomorphism in characteristic p.*

Exercise 4.
Let $\mathbb{F}_p(x)$ be the quotient field of the polynomial ring $\mathbb{F}_p[x]$ in the indeterminant x, i.e. $\mathbb{F}_p(x) = \{f/g | f, g \in F_p[x] \text{ and } g \neq 0\}$.

1. Show that f is not separable over $\mathbb{F}_p(x)$.

   Hint: Use Fermat's little theorem.

2. Show that $f = T^p - x$ is irreducible over $\mathbb{F}_p(x)$.

   Hint: For a direct calculation, use the factorization of f over $\mathbb{F}_p(\sqrt[p]{x})$; or you can apply the Eisenstein criterium to show that f is irreducibel in $\mathbb{F}_p[x, T]$ and conclude with the help of Gauss' lemma that f is irreducible in $\mathbb{F}_p(x)[T]$.

3. Conclude that $\mathbb{F}_p(\sqrt[p]{x})/\mathbb{F}_p(x)$ is not separable. Is $\mathbb{F}_p(\sqrt[p]{x})/\mathbb{F}_p(x)$ normal?

***Exercise 5.** [1]
Recall the proofs of the Eisenstein criterium and Gauss' lemma, i.e. the content of fg equals the product of the contents of f and g for polynomials f, g over a unique factorization domain.

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[1] The starred exercises are not to hand in. But it is advised to work on these exercises, and possibly to discuss them in the exercise class.