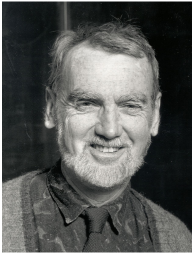
From quadratic reciprocity to the Langlands program

(Oliver Lorscheid, April 2018)



In honor of the 2018 Abel prize winner Robert Langlands

What is the Langlands program?

The Langlands program consists of a web of conjectures, partially relying on conjectural objects, that connect L-functions for Galois representations and automorphic representations.

A simple, but yet unproven and tantalizing instance is the following.

Conjecture (Langlands 1970)

For every positive integer n, there is a bijection

$$\left\{ \begin{array}{c} \textit{irreducible representations} \\ \rho : L_{\mathbb{Q}} \to \mathrm{GL}_n(\mathbb{C}) \end{array} \right\} \xrightarrow{\Phi} \left\{ \begin{array}{c} \textit{irreducible automorphic} \\ \textit{cuspidal representations} \\ \pi \textit{ of } \mathrm{GL}_n \textit{ over } \mathbb{Q} \end{array} \right\}$$

such that $L(\rho, s) = L(\pi, s)$ if $\pi = \Phi(\rho)$ where $L_{\mathbb{Q}}$ is the (conjectural) Langlands group of $\mathbb{Q}$, which is an extension of absolute Galois group $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ of $\mathbb{Q}$.

Part 1: Reciprocity laws and class field theory

Quadratic reciprocity

Let p be an odd prime number and a an integer that is not divisible by p. The **Legendre symbol of a mod p** is

$$\left(\frac{a}{p}\right) = \begin{cases} 1 & \text{if } a \text{ is a square modulo } p, \\ -1 & \text{if not.} \end{cases}$$

Adrien-Marie Legendre

Conjecture (Euler 1783, Legendre 1785)

Let p and q be distinct odd primes. Then

$$\left(\frac{q}{p}\right)\left(\frac{p}{q}\right) = (-1)^{\frac{p-1}{2}\frac{q-1}{2}}.$$

- First complete proof by Gauß in 1796.
- Up to today, there are more than 240 different proofs.

Carl Friedrich Gauß

The p-adic numbers

Let p be a prime number. The **p-adic absolute value** $|\cdot|_p : \mathbb{Q} \to \mathbb{R}_{\geq 0}$ is defined as $\left|p^i \frac{a}{b}\right|_p = p^{-i}$ whenever ab is not divisible by p.

The **p-adic numbers** are the completion $\mathbb{Q}_p$ of $\mathbb{Q}$ with respect to the norm $|\cdot|_p$.

The **p-adic integers** are $\mathbb{Z}_p = \left\{a \in \mathbb{Q}_p \middle| \, |a|_p \leq 1\right\}$.

Since $\mathbb{R}$ is the completion of $\mathbb{Q}$ at the **archimedean absolute value** $|a|_\infty = \operatorname{sign}(a) \cdot a$, we often write $\mathbb{Q}_\infty = \mathbb{R}$ and $p \leq \infty$ to express that p is a prime number or the symbol ∞.

Hilbert reciprocity

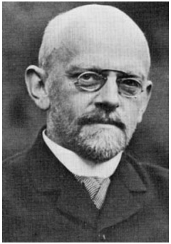
Let a and b be rational numbers and $p \leq \infty$. The **Hilbert symbol** is

$$(a,b)_p = \begin{cases} 1 & \text{if } ax^2 + by^2 = z^2 \text{ has a solution } (x,y,z) \in \mathbb{Q}_p^3, \\ -1 & \text{if not.} \end{cases}$$

Theorem (Hilbert 1897)

For all $a, b \in \mathbb{Q}$, we have $(a,b)_p = 1$ for almost all p and

$$\prod_{p \leq \infty} (a,b)_p \;=\; 1.$$



David Hilbert

Remark: Note that this implies quadratic reciprocity since for distinct odd prime numbers p and q, we have

$$\prod_{\ell \leq \infty} (p,q)_\ell \;=\; (p,q)_2 (p,q)_p (p,q)_q \;=\; (-1)^{\frac{p-1}{2}\frac{q-1}{2}} \left(\frac{q}{p}\right)\left(\frac{p}{q}\right).$$

Class field theory and Artin reciprocity

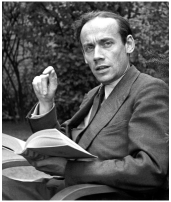
Teiji Takagi

Theorem (Takagi 1920, Artin 1927)

Let $L/\mathbb{Q}_p$ be a finite Galois extension with abelian Galois group $G = \mathrm{Gal}(L/\mathbb{Q}_p)$ and norm $N : L \to \mathbb{Q}_p$, defined by $N(a) = \prod_{\sigma \in G} \sigma(a)$. Then there is a group isomorphism

$$(-, L/\mathbb{Q}_p) : \quad \mathbb{Q}_p^\times / N(L^\times) \xrightarrow{\sim} \mathrm{Gal}(L/\mathbb{Q}_p),$$

which is called the local Artin symbol.



Emil Artin

Theorem (Artin 1927)

Let $L/\mathbb{Q}$ be a Galois extension with abelian Galois group $G = \mathrm{Gal}(L/\mathbb{Q})$. Then $\prod_{p \leq \infty}(a, L\,\mathbb{Q}_p/\mathbb{Q}_p) = 1$ for all $a \in \mathbb{Q}^\times$.

Remark: This generalizes Hilbert's reciprocity law since $(a, b)_p = \frac{(a, L\mathbb{Q}_p/\mathbb{Q}_p)(\sqrt{b})}{\sqrt{b}}$ where $L = \mathbb{Q}[\sqrt{b}]$.

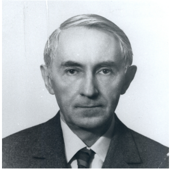
Chevalley's idelic formulation

The **idele group** of $\mathbb{Q}$ is the group

$$\mathbb{I}_{\mathbb{Q}} = \left\{(a_p) \in \prod_{p\leq\infty} \mathbb{Q}_p^{\times} \middle| a_p \in \mathbb{Z}_p^{\times} \text{ for almost all } p < \infty\right\}.$$

Note that $\mathbb{Q}^{\times}$ embeds diagonally into $\mathbb{I}_{\mathbb{Q}}$.

The **idele class group** of $\mathbb{Q}$ is $C_{\mathbb{Q}} = \mathbb{I}_{\mathbb{Q}}/\mathbb{Q}^{\times}$.



Claude Chevalley

Theorem (Chevalley 1936)

Let $\mathbb{Q}^{\mathrm{ab}}$ be the maximal abelian extension of $\mathbb{Q}$. The product over all local Artin symbols defines a surjective group homomorphism

$$r : C_{\mathbb{Q}} \longrightarrow \mathrm{Gal}(\mathbb{Q}^{\mathrm{ab}}/\mathbb{Q})$$

whose kernel is the connected component $C_{\mathbb{Q}}^0$ of the identity component of $C_{\mathbb{Q}}$ (w.r.t. the "idelic topology" on $C_{\mathbb{Q}}$).

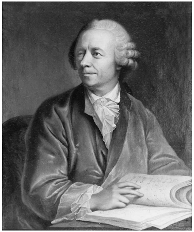
Part 2: L-series

The Euler product

In 1740, Euler describes explicit formulas for

$$\zeta(s) = \sum_{n\geq 1} \frac{1}{n^s}$$

where s is an even positive integer. The case $s = 2$ solved the long standing Basel problem.



Leonard Euler

Moreover, Euler showed that

$$\zeta(s) = \prod_{p<\infty} \frac{1}{1 - p^{-s}}$$

for real numbers $s > 1$, which is known as the *Euler product* nowadays.

Dirichlet series

Let $\chi : \mathbb{Z} \to \mathbb{C}^\times$ be a group homomorphism of finite order, i.e. $\chi(n) = 1$ for some $n \geq 1$. The **Dirichlet series of χ** is

$$L(\chi, s) \;=\; \sum_{n\geq 1} \frac{\chi(n)}{n^s} \;=\; \prod_{p<\infty} \frac{1}{1-\chi(p)p^{-s}}.$$

Peter Gustav Lejeune Dirichlet

Application to primes in arithmetic progression:

Theorem (Dirichlet 1840)

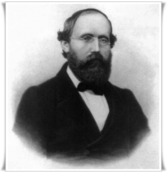
Let a and n be positive integers. Then there are infinitely many prime numbers p congruent to a modulo n.

Riemann's analysis of $\zeta(s)$

Theorem (Riemann 1856)

As a complex function, $\zeta(s)$ converges absolutely in the halfplane $\{s \in \mathbb{C} | \mathrm{Re}(s) > 1\}$, has a meromorphic continuation to $\mathbb{C}$ with a simple pole at 1, and it satisfies a functional equation of the form

$$\zeta(1-s) \;=\; (\textit{well-behaved factor}) \cdot \zeta(s).$$



Bernard Riemann

In the following, we shall refer to such properties as *arithmetic*.

In honor of Riemann's 1856 paper, $\zeta(s)$ is called the *Riemann zeta function* nowadays. Not to forget the *Riemann hypothesis*:

Conjecture (Riemann 1856)

If $\zeta(s) = 0$, then s is an even negative integer or $\mathrm{Re}(s) = 1/2$.

Hecke L-functions

Generalization of Dirichlet series (here only for $\mathbb{Q}$):

A **Hecke character** is a continuous group homomorphism $\chi : C_{\mathbb{Q}} \to \mathbb{C}^\times$. It is **unramified at p** if the composition $\mathbb{Z}_p^\times \to C_K \to \mathbb{C}^\times$ is trivial.
The **Hecke L-function of χ** is

$$L(\chi, s) \;=\; \prod_{\chi \text{ unramified at } p} \frac{1}{1-\chi(p)p^{-s}}.$$

Erich Hecke

Theorem (Hecke 1916)

$L(\chi, s)$ is arithmetic. If χ is nontrivial, then $L(\chi, s)$ is entire, i.e. without pole at 1.

Problem: How to factorize "large" L-functions into "smaller" L-functions?

Artin L-functions

Let $K/\mathbb{Q}$ be a Galois extension with Galois group $G = \mathrm{Gal}(K/\mathbb{Q})$. An ***n*-dimensional Galois representation of *G*** is a continuous group homomorphism

$$\rho : G \longrightarrow \mathrm{GL}_n(\mathbb{C}).$$

If $n = 1$, then ρ is called a **character**.

The **Artin *L*-function of ρ** is defined as the product

$$L(\rho, s) \;=\; \prod_{p<\infty} L_p(\rho, s)$$

of local factors $L_p(\rho, s)$, which are, roughly speaking, the reciprocals of the characteristic polynomials of $\rho(\mathrm{Frob}_p)$ where Frob_p is a lift of the Frobenius automorphism of the residue field.

Artin reciprocity, part 2

As a consequence of Artin's and Chevalley's theorems, the reciprocity map $r : C_{\mathbb{Q}} \to \mathrm{Gal}(\mathbb{Q}^{\mathrm{ab}}/\mathbb{Q})$ induces an isomorphism

$$r^* : \mathrm{Hom}(\mathrm{Gal}(\mathbb{Q}^{\mathrm{ab}}/\mathbb{Q}), \mathbb{C}^\times) \xrightarrow{\sim} \mathrm{Hom}(C_{\mathbb{Q}}/C_{\mathbb{Q}}^0, \mathbb{C}^\times)$$

between the respective character groups via $r^*(\rho) = \rho \circ r$.

Theorem (Artin 1927)

If $\chi = \rho \circ r$, then $L(\chi, s) = L(\rho, s)$.

Applications: (1) Artin L-functions are arithmetic.
(2) Factorization of large L-functions into smaller L-functions.

The abelian Langlands correspondence

Let $G_\mathbb{Q} = \mathrm{Gal}(\overline{\mathbb{Q}}, \mathbb{Q})$ be the absolute Galois group of $\mathbb{Q}$.
Since $\mathrm{Gal}(\mathbb{Q}^{\mathrm{ab}}/\mathbb{Q}) = G_\mathbb{Q}^{\mathrm{ab}}$ and since every group homomorphism $G_\mathbb{Q} \to \mathbb{C}^\times$ factors through $G_\mathbb{Q}^{\mathrm{ab}}$, we gain an isomorphism

$$\mathrm{Hom}(G_\mathbb{Q}, \mathbb{C}^\times) \;=\; \mathrm{Hom}(\mathrm{Gal}(\mathbb{Q}^{\mathrm{ab}}/\mathbb{Q}), \mathbb{C}^\times) \;\xrightarrow{\sim}\; \mathrm{Hom}(C_\mathbb{Q}/C_\mathbb{Q}^0, \mathbb{C}^\times).$$

To extend this to an isomorphism with the whole character group of $C_\mathbb{Q}$, we have to exchange $G_\mathbb{Q}$ by the *Weil group* $W_\mathbb{Q}$ of $\mathbb{Q}$.

Theorem (Langlands correspondence for GL_1)

Artin reciprocity induces an isomorphism

$$\Phi : \mathrm{Hom}(W_\mathbb{Q}, \mathbb{C}^\times) \;\xrightarrow{\sim}\; \mathrm{Hom}(C_\mathbb{Q}, \mathbb{C}^\times)$$

such that $L(\rho, s) = L(\chi, s)$ if $\chi = \Phi(\rho)$.

André Weil

Part 3: Going nonabelian—the Taniyama-Shimura-Weil conjecture

Dirichlet series of a modular form

Henri Poincaré

The **Poincaré upper half plane** is $\mathbb{H} = \{z \in \mathbb{C} | \mathrm{Im}(z) > 0\}$.

A **modular cusp form (of weight k and level N)** is an holomorphic function $f : \mathbb{H} \to \mathbb{C}$ of the form

$$f(z) = \sum_{n=1}^{\infty} a_n\, e^{2\pi i n z/N} \quad \text{s.t.} \quad f\left(\frac{az+b}{cz+d}\right) = (cz+d)^k f(z)$$

for all $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$ congruent to $\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ modulo N.

It is a **(Hecke) eigenform** if $a_{pq} = a_p a_q$ for all primes $p \neq q$.

The **Dirichlet series of f** is

$$L(f,s) \;=\; \sum_{n \geq 1} \frac{a_n}{n^s} \;=\; \prod_{p<\infty} \frac{1}{1 - a_p p^{-s}}.$$

Theorem (Hecke 1936)

$L(f,s)$ is arithmetic.

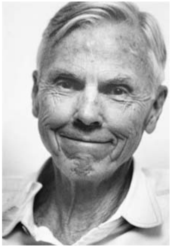
Elliptic curves

An **elliptic curve over $\mathbb{Q}$** is a Lie group $E \simeq \mathbb{S}^1 \times \mathbb{S}^1$ (i.e. a complex torus) that is defined by equations over $\mathbb{Q}$. In particular, it contains a subgroup $E(\overline{\mathbb{Q}}) \subset E$ on which $G_{\mathbb{Q}}$ acts.

For a prime ℓ and $n \geq 1$, let $E(\overline{\mathbb{Q}})[\ell^n]$ be the subgroup of ℓ^n-torsion points of $E(\overline{\mathbb{Q}})$. The **Tate module** of E is

$$T_\ell(E) = \varprojlim_n E(\overline{\mathbb{Q}})[\ell^n],$$

which is isomorphic to $\mathbb{Z}_\ell^2$ and has an action of $G_{\mathbb{Q}}$.


John Tate

For every "good prime" ℓ and embedding $\mathbb{Z}_\ell \to \mathbb{C}$, we get a Galois representation

$$\rho : G_{\mathbb{Q}} \longrightarrow \mathrm{GL}(T_\ell(E) \otimes_{\mathbb{Z}_\ell} \mathbb{C}) \simeq \mathrm{GL}_2(\mathbb{C}).$$

The ***L*-function of *E*** is the Artin L-function $L(E,s) = L(\rho, s)$.

The Taniyama-Shimura-Weil conjecture

Conjecture (Taniyama 1955, Shimura 1957, Weil 1967)

For every elliptic curve E over $\mathbb{Q}$, there is a modular cusp eigenform f of weight 2 such that $L(E, s) = L(f, s)$.

The conjecture was proven, step-by-step, by

- Weil (1967),
- Wiles (1995), with some help of Taylor,
- Diamond (1996),
- Conrad, Diamond, Taylor (1999),
- Brevil, Conrad, Diamond, Taylor (2001).

Nowadays it is called the modularity theorem.

Remark: Wiles contribution gained popularity since it implied "Fermat's last theorem". Wiles received the Abel prize in 2016.

Yukata Taniyama

Goro Shimura

The modularity theorem as a Langlands correspondence

Since every modular cusp eigenform f (of some level N) is an element of an irreducible automorphic cuspidal representation π and $L(\pi, s) = L(f, s)$, the modularity theorem can be rephrased as follows.

There exists a bijection

$$\left\{ \begin{array}{c} \textit{certain} \\ \text{irreducible representations} \\ \rho : G_{\mathbb{Q}} \to \mathrm{GL}_2(\mathbb{C}) \end{array} \right\} \xrightarrow{\Phi} \left\{ \begin{array}{c} \textit{certain} \\ \text{irreducible automorphic} \\ \text{cuspidal representations} \\ \pi \text{ of } \mathrm{GL}_2 \text{ over } \mathbb{Q} \end{array} \right\}$$

such that $L(\rho, s) = L(\pi, s)$ if $\pi = \Phi(\rho)$.

Part 4: The Langlands program

Adele groups

The **adele ring** of $\mathbb{Q}$ is

$$\mathbb{A} \;=\; \Big\{ (a_p) \in \prod_{p \leq \infty} \mathbb{Q}_p \,\Big|\, a_p \in \mathbb{Z}_p \text{ for almost all } p < \infty \Big\}.$$

Note that $\mathbb{I}_{\mathbb{Q}} = \mathbb{A}^\times = \mathrm{GL}_1(\mathbb{A})$. In general,

$$\mathrm{GL}_n(\mathbb{A}) \;=\; \Big\{ (g_p) \in \prod_{p \leq \infty} \mathrm{GL}_n(\mathbb{Q}_p) \,\Big|\, g_p \in \mathrm{GL}_n(\mathbb{Z}_p) \text{ for a.a. } p < \infty \Big\}$$

is equipped with a topology, for which

$$K \;=\; \mathrm{O}_n \times \prod_{p < \infty} \mathrm{GL}_n(\mathbb{Z}_p)$$

is a maximal compact subgroup. In particular, $\mathrm{GL}_n(\mathbb{A})$ is a locally compact group and carries a Haar measure, which allows us to form integrals over subgroups.

Automorphic cusp forms

An **(automorphic) cusp form** for GL_n over $\mathbb{Q}$ is a "smooth" function

$$f: \mathrm{GL}_n(\mathbb{A}) \longrightarrow \mathbb{C}$$

such that

- there is a finite index subgroup K' of K such that $f(\gamma g k) = f(g)$ for all $\gamma \in \mathrm{GL}_n(\mathbb{Q})$ and all $k \in K'$;
- the *constant Fourier coefficient*

$$c_0(f)(g) = \int_{U(\mathbb{A})} f(ug)\, du$$

vanishes for every $g \in \mathrm{GL}_n(\mathbb{A})$ and every unipotent subgroup U of GL_n.

Joseph Fourier

Analogy: A modular form $f = \sum_{n=0}^{\infty} a_n\, e^{2\pi i n z/N}$ is a cusp form if and only if $a_0 = 0$.

Automorphic cuspidal representation

The space $\mathcal{A}_0$ of all cusp forms for GL_n over $\mathbb{Q}$ is "almost" a representation of $\mathrm{GL}_n(\mathbb{A})$ with respect to the action $g.f(h) = f(hg)$ where $g, h \in \mathrm{GL}_n(\mathbb{A})$ and $f \in \mathcal{A}_0$.

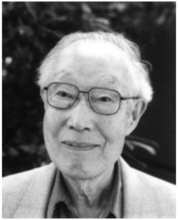
An **automorphic cuspidal representation** of GL_n over $\mathbb{Q}$ is a subrepresentation π of $\mathcal{A}_0$.

Note that $\mathcal{A}_0$ decomposes into a direct sum of irreducible subrepresentations π.

Example: Every modular cusp eigenform generates an irreducible automorphic cuspidal representation π for GL_2 over $\mathbb{Q}$.

L-functions

Langlands defines an L-function $L(\pi, s)$ for every irreducible automorphic cuspidal representation π in terms of the *Satake parameters* of π.



Ichirō Satake

Conjecture (Langlands 1970)

$L(\pi, s)$ is arithmetic.

Example: If

$$f(z) \;=\; \sum_{n=1}^{\infty} a_n\, e^{2\pi i n z/N}$$

is a modular cusp eigenform, then its Satake parameters are the coefficients a_p for p prime. If π is the corresponding automorphic representation, then

$$L(\pi, s) \;=\; L(f, s) \;=\; \prod_{p<\infty} \frac{1}{1 - a_p p^{-s}}.$$

A first Langlands conjecture

Conjecture (Langlands 1970)

For every positive integer n, there is an injection

$$\left\{ \begin{array}{c} \textit{irreducible representations} \\ \rho : G_{\mathbb{Q}} \to \mathrm{GL}_n(\mathbb{C}) \end{array} \right\} \xrightarrow{\Phi} \left\{ \begin{array}{c} \textit{irreducible automorphic} \\ \textit{cuspidal representations} \\ \pi \textit{ of } \mathrm{GL}_n \textit{ over } \mathbb{Q} \end{array} \right\}$$

such that $L(\rho, s) = L(\pi, s)$ *if* $\pi = \Phi(\rho)$ *where* $G_{\mathbb{Q}} = \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ *is the absolute Galois group of* $\mathbb{Q}$*.*

In order to recover the “missing” Galois representation, Langlands suggests the existence of a certain extension $L_{\mathbb{Q}}$ of $G_{\mathbb{Q}}$, coined as the *Langlands group* nowadays.

A first Langlands conjecture

Conjecture (Langlands 1970)

For every positive integer n, there is a bijection

$$\left\{ \begin{array}{c} \textit{irreducible representations} \\ \rho : L_{\mathbb{Q}} \to \mathrm{GL}_n(\mathbb{C}) \end{array} \right\} \xrightarrow{\Phi} \left\{ \begin{array}{c} \textit{irreducible automorphic} \\ \textit{cuspidal representations} \\ \pi \textit{ of } \mathrm{GL}_n \textit{ over } \mathbb{Q} \end{array} \right\}$$

such that $L(\rho, s) = L(\pi, s)$ if $\pi = \Phi(\rho)$ where $L_{\mathbb{Q}}$ is the (conjectural) Langlands group of $\mathbb{Q}$.

In order to recover the "missing" Galois representation, Langlands suggests the existence of a certain extension $L_{\mathbb{Q}}$ of $G_{\mathbb{Q}}$, coined the Langlands group nowadays.

Automorphic L-functions

In fact, Langlands proposes a common generalization of Artin L-functions and L-functions of automorphic representations for any reductive group G over any local or global field.

To give an idea: the Satake parameters of an automorphic representation π of $\mathrm{GL}_n(\mathbb{A})$ are the entries of certain diagonal matrices $A_{\pi,p}$ in $\mathrm{GL}_n(\mathbb{C})$ (one for each $p < \infty$).

The **Langlands dual** ${}^L G$ of $G = \mathrm{GL}_n(\mathbb{A})$ is an extension of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ by $\mathrm{GL}_n(\mathbb{C})$:

$$1 \longrightarrow \mathrm{GL}_n(\mathbb{C}) \longrightarrow {}^L G \longrightarrow \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \longrightarrow 1.$$

The **automorphic L-function** $L(\pi, \rho, s)$ is defined for every irreducible automorphic cuspidal representation π of G and every representation $\rho : {}^L G \to \mathrm{GL}_N(\mathbb{C})$.

Conjecture (Langlands 1970)

$L(\pi, \rho, s)$ is arithmetic.

Langlands functoriality

Let k be a local or global field (e.g. a p-adic field or a number field) and G and G' reductive groups over k (e.g. GL_n, SO_n or Sp_{2n}) with respective Langlands duals ${}^L G$ and ${}^L G'$.

An **L-morphism** is a continuous group homomorphism $\varphi : {}^L G \to {}^L G'$ that commutes with the respective projections to G_k.

Conjecture (Langlands 1970)

Let $\varphi : {}^L G \to {}^L G'$ be an L-morphism and π an irreducible automorphic cuspidal representation of G. Then

1. *there is an irreducible automorphic cuspidal representation π' of G' whose Satake parameters are the images of the Satake parameters of π, and*
2. *for every representation $\rho' : {}^L G' \to \mathrm{GL}_N(\mathbb{C})$ and $\rho = \rho' \circ \varphi$, we have $L(\pi', \rho', s) = L(\pi, \rho, s)$.*

Part 5: What has been done?

1970 Jaquet, Langlands: correspondence for GL_2 over p-adic fields ($p \neq 2$)

1973 Langlands: correpondence for GL_2 over $\mathbb{R}$ and $\mathbb{C}$

1977 Drinfeld: global correspondence for GL_2 over global function fields (awarded with a Fields medal)

1980 Kutzko: correspondence for GL_2 over all local fields

1993 Laumon, Rapoport, Stuhler: correspondence for GL_n over local fields of positive characteristic

1995 Wiles: modularity theorem (awarded with the Abel prize)

2000 L. Lafforgue: global correspondence for GL_n over global function fields (awarded with a Fields medal)

2001 Harris, Taylor: correspondence for GL_n over local fields of characteristic 0

2000 Henniart: alternative proof for GL_n over local fields of characteristic 0

2008 Ngô: fundamental lemma (awarded with a Fields medal)

2013 Scholze: alternative proof for GL_n over local fields of characteristic 0