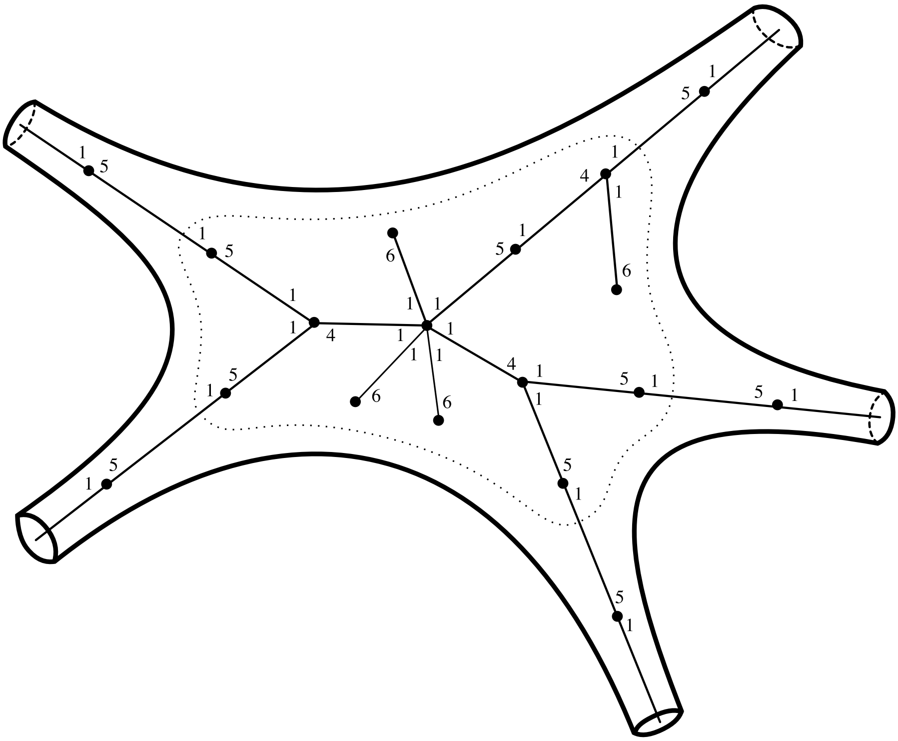
Toroidal Automorphic Forms for Function Fields



Oliver Lorscheid

Toroidal Automorphic Forms for Function Fields

Toroïdale Automorfe Vormen voor Functielichamen
(met een samenvatting in het Nederlands)

Toroidale Automorphe Formen für Funktionenkörper
(mit einer Zusammenfassung in deutscher Sprache)

Proefschrift

ter verkrijging van de graad van doctor aan de Universiteit Utrecht op gezag van de rector magnificus, prof.dr. J.C. Stoof, ingevolge het besluit van het college voor promoties in het openbaar te verdedigen op woensdag 28 mei 2008 des middags te 2.30 uur

door

Oliver Lorscheid

geboren op 8 februari 1978 te Keulen, Duitsland

Promotor: Prof. dr. G.L.M. Cornelissen

Einem Mathematiker erschien der Teufel und fragte, was er jenem für seine Seele bieten könne. Gegen einen Beweis der Riemannschen Vermutung tausche er gerne seine Seele, antwortete der Mathematiker, und der Teufel zog sich zurück um nachzudenken.
Am darauffolgenden Tage unterbreitete der Teufel dem Mathematiker seine Tat: Einen Beweis habe er nicht gefunden, aber eine Umformulierung.

Erzählt von Günter Harder

A mathematician was accosted by the devil with the question what he should be willing to barter his soul for. The mathematician answered that he would gladly trade his soul for a proof of the Riemann hypothesis. The devil withdrew to think about this proposal.
The next day, the devil told the mathematician about his findings: a proof he had been unable to find, but he could offer a reformulation.

Communicated by Günter Harder

Cover illustration:

The graph that is used for the cover illustration is the graph of an unramified Hecke operator Φ_x that belongs to a rational point x of the elliptic curve X over $\mathbf{F}_5$ given by the Weierstrass equation

$$\underline{Y}^2 = \underline{X}^3 + \underline{X}^2 + 1 .$$

This elliptic curve has class number 5. The structure of the graph of Φ_x is determined in Chapter 7; in particular cf. Figure 7.2.

ISBN: 978-90-393-4825-3

Contents

Introduction

Motivation

At the Bombay Colloquium in January 1979, Don Zagier ([83]) observed that if the kernel of certain operators on automorphic forms turns out to be a unitarizable representation, a formula of Hecke implies the Riemann hypothesis. Zagier called elements of this kernel toroidal automorphic forms.

In the language of adeles, an automorphic form f on $\mathrm{PGL}_2(\mathbf{Q}) \backslash \mathrm{PGL}_2(\mathbf{A})$, where **A** denote the adeles of **Q**, is toroidal if for all maximal anisotropic tori T in GL_2 that are defined over **Q** and all $g \in \mathrm{PGL}_2(\mathbf{A})$,

$$\int_{\widetilde{T}(\mathbf{Q}) \backslash \widetilde{T}(\mathbf{A})} f(tg)\, dt \tag{1}$$

vanishes, where $\widetilde{T}$ denotes the image of T in PGL_2. From its definition, the space $\mathcal{A}_{\mathrm{tor}}$ of toroidal automorphic forms is an automorphic representation of $\mathrm{PGL}_2(\mathbf{A})$ by right translation of the argument. If f is an Eisenstein series $E(s)$, the integral (1) equals the product of the completed zeta function $\zeta^*(s+1/2)$ (including the gamma factor) and a function that depends nontrivially on g. (Note that in the original paper, Zagier used a different normalisation of the weight s of the Eisenstein series than we do.) Consequently, for every zero $s+1/2$ of ζ^*, i.e. for every nontrivial zero of the Riemann zeta function, the Eisenstein series $E(s)$ is toroidal. On the other hand, $E(s)$ spans a unitary representation of $\mathrm{PGL}_2(\mathbf{A})$ if and only if $s+1/2 \in (0,1)$ or $\mathrm{Re}(s+1/2) = 1/2$. Since ζ has no zeros on the interval $(0,1)$, cf. [66, Formula (2.12.4)], the Riemann hypothesis follows if $\mathcal{A}_{\mathrm{tor}}$ is a unitarizable representation. Indeed, it suffices to find a $\mathrm{PGL}_2(\mathbf{A})$-invariant hermitian product on the subspace of unramified vectors, since the Eisenstein series in question are unramified automorphic forms.

We briefly review consecutive developments. The monumental work of Waldspurger on the Shimura correspondence ([69], [70], [71] and [72]) includes a formula connecting toroidal integrals of cusp forms (nowadays also called Waldspurger periods) with the value of the L-function of the corresponding cuspidal representation at $1/2$. In [80] and [81] Franck Wielonsky worked out Zagier's ideas and obtained a generalisation to a limited class of Eisenstein series on $\mathrm{PGL}_n(\mathbf{A})$. Lachaud tied up the spaces with Connes' view on the zeta function, cf. [34] and [35]. Clozel and Ullmo ([12]) used both Waldspurger's and Zagier's works to prove a equidistribution result for tori in GL_2, and Lysenko ([44]) translated certain Waldspurger periods into geometric language. Finally, [15] contains easy proofs of the Theorems A, C and D below, when restricted to global function fields of genus less than or equal to 1 whose class number is 1.

Results

On the last page of his paper [83], Zagier asks what happens if $\mathbf{Q}$ is replaced by a global function field. He remarks that the space of unramified toroidal automorphic forms can be expected to be finite dimensional since the zeta function is essentially a polynomial, which marks a difference to the case of number fields. This forms the starting point for the present thesis.

Let F denote a global function field of genus g and class number h. Define an automorphic form on $\mathrm{PGL}_2 F$ to be toroidal if the literal translation for $\mathbf{Q}$ to F holds. The main results are:

Theorem A. *The space of unramified toroidal automorphic forms is finite dimensional.*

Theorem B. *The dimension of the space of unramified toroidal automorphic forms is at least* $(g_F - 1)h_F + 1$.

Theorem C. *There are no nontrivial unramified toroidal automorphic forms for rational function fields.*

Theorem D. *Let F be the function field of an elliptic curve over a finite field with q elements, and $s + 1/2$ a zero of the zeta function of F. If the characteristic is not 2 or $h \neq q + 1$, the space of unramified toroidal automorphic forms is 1-dimensional and spanned by the Eisenstein series of weight s.*

Theorem E. *The irreducible unramified subquotients of the representation space of toroidal automorphic forms are unitarizable, and do not contain a complementary series.*

These are Theorems 6.1.8, 6.2.14, 6.1.10, 8.3.11 and 6.7.5, respectively. The main ingredient of the proofs of Theorems A, C and D is the theory of graphs of Hecke operators as it will be developed in this thesis. It can be seen as a global variant of the quotients of Bruhat-Tits trees by arithmetic subgroups as considered by Serre in [60, II.2]. Theorem B is a consequence of Zagier's paper using the theory of Eisenstein series and class field theory.

The proof of Theorem E makes use of the proof of the Ramanujan-Petersson conjecture for GL_2 ([17]) and the Hasse-Weil theorem ([76]), which is the analogue of the Riemann hypothesis for global function fields. For the function field of an elliptic curve it is possible to prove unitarizability without making use of the Hasse-Weil theorem, but this does not imply the Riemann hypothesis (section 8.4).

Conjectures

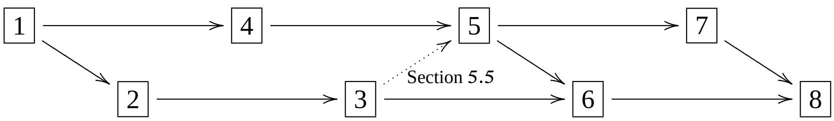
The work of Waldspurger for number fields, cf. [71] and [72], and a theory of double Dirichlet series of Fisher and Friedberg, cf. [18] and [19], lead to Conjectures 6.2.15 and 6.4.3, which can be combined to

Conjecture A. *Let r be the number of isomorphism classes of irreducible unramified cuspidal representations whose L-function vanishes at $1/2$. Then the dimension of the space of unramified toroidal automorphic forms equals $(g-1)h+1+r$.*

Tentative calculations for ramified representations show that the subspace of toroidal automorphic forms with certain fixed ramification type has a finite decomposition series. There are technical obstructions to proving this in general. Nevertheless it suggests Conjecture 6.1.15, which is

Conjecture B. *The space of all toroidal automorphic forms is admissible.*

Leitfaden



Content overview

The present thesis intends to be comprehensible to a reader with a basic knowledge of number theory and algebraic geometry. To realise this goal, it is necessary to give short introductions to various topics adapted to the purposes of the thesis, namely, to certain aspects from the theory of automorphic forms as well as parts from the theory of vector bundles on curves. The philosophy in these introductory parts is that they contain proofs where they are short and instructive or missing from the existing literature, and provide a reference otherwise. As known theory is interwoven with new results, the following overview tries to disentangle the knot.

CHAPTER **1** defines the context of this thesis and gives short introductions to adelic groups, automorphic forms and the Hecke algebra. The last section introduces the object of investigation, the space of toroidal automorphic forms. In particular, there is a definition for split tori that will reproduce the results that Zagier obtained in [83] in classical language for **Q**.

CHAPTER **2** sketches the theory of L-series and Eisenstein series and puts some emphasis on derivatives, which play an important role in the representation theory of toroidal automorphic forms.

CHAPTER **3** inspects unramified admissible representations and their decomposition series. It further provides results for detecting unramified automorphic forms by their eigenvalues under Hecke operators.

CHAPTER **4** introduces the new notion of the graph of a Hecke operator. Examples for rational function fields are accessible by elementary matrix manipulations. Further investigations for general function fields will be done only for generators of the unramified part of the Hecke algebra. The structure of the graphs of these operators inherits many properties from local coverings with Bruhat-Tits trees.

CHAPTER **5** describes the well-known geometric interpretation of automorphic forms as functions on vector bundles and the meaning of Hecke operators in this context. This enables methods from algebraic geometry and in particular reduction theory for vector bundles to enter the investigation of graphs of Hecke operators. Namely, the graph is a union of finitely many half lines, called cusps, that are connected by a finite graph, called the nucleus. The cusps are of simple nature and it is the nucleus that encodes arithmetic information about F. Finally, we will reinterpret automorphic forms as functions on the vertices.

CHAPTER **6** uses the theory developed in the previous chapters to prove Theorems A and C. Theorem A further implies that the space of unramified toroidal automorphic forms decomposes into three parts, which are, roughly speaking, Eisenstein series, residues of Eisenstein series and cusp forms. Zagier's calculation (applied to the first part) proves Theorem B. Applied to the second, it shows that there are no toroidal residues of Eisenstein series. The question of toroidal cusp forms in general can be answered if Waldspurger's work is translated to global function fields. The last sections discuss the history of the Riemann hypothesis and its connections to toroidal automorphic forms. In particular the proof of the Riemann hypothesis for global function fields implies Theorem E.

CHAPTER **7** determines the structure of graphs of certain Hecke operators for function fields of an elliptic curve. This is done by completely geometric methods using the classification of vector bundles on elliptic curves by Atiyah.

CHAPTER **8** applies the graphs of the previous chapter to prove Theorem D. It further contains new proofs of some particular results, such as a dimension formula for the space of unramified cusp forms, and unitarizability results for unramified toroidal automorphic forms that can be proven without making use of the Hasse-Weil theorem.

CHAPTER 1

Definitions and Preliminaries

This chapter recalls relevant notions and facts from the theory of adeles, algebraic groups and automorphic forms. It is not meant to give a complete treatment, but rather discusses the viewpoint of this thesis and settles notation. References will help to find missing facts in the literature. The last section introduces toroidal automorphic forms, the main object of study.

1.1 Notation

As reference for this section, consider [49], [55] or [79].

1.1.1 Let $\mathbf{N}$ be the natural numbers, $\mathbf{Z}$ the integers, $\mathbf{Q}$ the rationals, $\mathbf{R}$ the reals and $\mathbf{C}$ the complex numbers together with the usual absolute value $|\ |_{\mathbf{C}}$ and the usual topology. Fix a field $\mathbf{F}_q$ with q elements and let F be a global function field with constant field $\mathbf{F}_q$, i.e. $F/\mathbf{F}_q$ is a field extension of transcendence degree 1 whose elements of finite multiplicative order are contained in $\mathbf{F}_q$.

1.1.2 A *place* is an equivalence class of nontrivial valuations of F. Let $|X|$ denote the set of all places. For the rest of this paragraph, fix $x \in |X|$. Let F_x be the completion of F at x. Choose a uniformiser $\pi_x \in F$. Then F_x is isomorphic to the field of Laurent series $\mathbf{F}_{q_x}((\pi_x))$ in π_x over $\mathbf{F}_{q_x}$, where $q_x = q^{\deg x}$ for some positive integer $\deg x$, which is called the degree of x. Let $\mathcal{O}_x$ be the ring of integers of F_x, which is isomorphic to the ring of formal power series $\mathbf{F}_{q_x}[[\pi_x]]$. Further let $\mathfrak{m}_x = \pi_x \mathcal{O}_x$ be its maximal ideal and $\kappa_x = \mathcal{O}_x / \mathfrak{m}_x \simeq \mathbf{F}_{q_x}$ its residue field. The field F_x comes with a valuation v_x that satisfies $v_x(\pi_x) = 1$ and an absolute value $|\ |_x = q_x^{-v_x}$, which satisfies $|\pi_x| = q_x^{-1}$.

1.1.3 For every finite subset $S \subset |X|$, define $\mathbf{A}_S$ as $\prod_{x \in S} F_x \times \prod_{x \notin S} \mathcal{O}_x$. When $S = \emptyset$, we also write $\mathcal{O}_{\mathbf{A}}$ for $\mathbf{A}_\emptyset$. The family of all finite subsets of $|X|$ together with inclusions forms a direct system, and the colimit, or union, $\mathbf{A} = \mathbf{A}_F = \bigcup \mathbf{A}_S$ over this system is called the *adele ring* of F. In other words, $\mathbf{A}$ is the subring of $\prod_{x \in |X|} F_x$ consisting of all elements (a_x) such that for all but finitely many $x \in |X|$, $v_x(a_x) \geq 0$. The canonical injections to the full product restrict to $F_x \to \mathbf{A}$, and the canonical projections restricted to $\mathbf{A} \to F_x$ are still surjective.

1.1.4 The *idele group* $\mathbf{A}^\times$ is the group of invertible adeles. An idele $a = (a_x)$ is characterised by the vanishing of $v_x(a_x)$ for all but finitely many places x. The degree

$\deg a = \sum_{x \in |X|} \deg x \cdot v_x(a_x)$ and the norm $|a| = \prod_{x \in |X|} |a_x|_x$ of an idele are thus well-defined functions. Denote by $\mathbf{A}_0^\times$ the ideles of degree 0, or equivalently, of norm 1.

1.1.5 A *divisor* of F is an element

$$D = (D_x) \in \bigoplus_{x \in |X|} \mathbf{Z} \cdot x \;\simeq\; \mathbf{A}^\times / \mathcal{O}_\mathbf{A}^\times$$

with $D_x \in \mathbf{Z}$ for all $x \in |X|$. The latter isomorphism is obtained by sending the divisor x (for $x \in |X|$) to π_x, where we interpret π_x as idele via the inclusion $F^\times \subset F_x^\times \subset \mathbf{A}^\times$. Define the *idele class group* as $F^\times \backslash \mathbf{A}^\times$ and the *divisor class group* $\mathrm{Cl}\, F$ as $F^\times \backslash \mathbf{A}^\times / \mathcal{O}_\mathbf{A}^\times$. If we write $[D] \in \mathrm{Cl}\, F$, then we always mean that D is a divisor that represents the divisor class $[D]$.

By embedding an element $a \in F$ diagonally into $\mathbf{A}$ along the canonical inclusions $F \hookrightarrow F_x$, we may regard F as a subring of $\mathbf{A}$. The product formula $\prod_{x \in |X|} |a|_x = 1$ can be reformulated as $F^\times \subset \mathbf{A}_0^\times$. Since $\mathcal{O}_\mathbf{A}^\times$ consists of the ideles $a = (a_x)$ with $v_x(a_x) = 0$ for all places x, also $\mathcal{O}_\mathbf{A}^\times \subset \mathbf{A}_0^\times$. Thus we can define the the degree of a divisor and divisor class to be the degree of a representing idele. The *class group* $\mathrm{Cl}^0 F = F^\times \backslash \mathbf{A}_0^\times / \mathcal{O}_\mathbf{A}^\times$ is a finite group, whose order h_F is the *class number*. More generally, $\mathrm{Cl}^d F$ denotes the divisor classes of degree d and $\mathrm{Cl}^{\geq d} F$ the divisor classes of degree greater or equal to d.

These groups fit into an exact sequence

$$0 \longrightarrow \mathrm{Cl}^0 F \longrightarrow \mathrm{Cl}\, F \xrightarrow{\deg} \mathbf{Z} \longrightarrow 0 \;\;,$$

which splits non-canonically, cf. paragraph 2.1.2. For surjectivity of the degree map, cf. [57, para. 8.2]. In particular, there are always ideles of degree 1, even when F has no place of degree 1.

An *prime divisor* is a divisor that is represented by π_x for some place x and an *effective divisor* is a divisor that is either trivial or the sum of prime divisors.

The definition of a canonical divisor is somewhat more involved and will therefore be assumed to be known ([79, Ch.VI, Defs. VI.1, VII.4] or [28, p. 295]). A *differental idele* (terminology *sic* due to Weil) is an idele that represents a canonical divisor. All differental ideles have the same degree. Let $\mathfrak{c}$ be a fixed differental idele. The *genus* g_F *of* F satisfies $\deg \mathfrak{c} = 2g_F - 2$.

1.2 Adelic topologies

Local fields and adeles come with a natural topology, which turns them into locally compact rings. Hence all algebraic groups over these rings turn into locally compact groups, which carry a Haar measure. As general reference consider the same books as in the previous section. For the theory of locally compact groups we suggest the classic by Pontryagin ([51]) and [31].

1.2.1 The topology of F_x is given by the neighbourhood basis $\{\pi_x^i \mathcal{O}_x\}_{i \in \mathbf{N}}$ of 0, which turns F_x into a locally compact field, since $\mathcal{O}_x$ is a compact neighbourhood of 0. Remark that F_x is totally disconnected and Hausdorff. By its definition as

$$\mathbf{A} \quad = \quad \bigcup_{\substack{S \subset |X| \\ \text{finite}}} \quad \prod_{x \in S} F_x \times \prod_{x \notin S} \mathcal{O}_x \ ,$$

we can endow $\mathbf{A}$ with a canonical topology, in which the subsets above are embedded as open subspaces carrying the product topology. By Tychonoff's theorem, $\mathcal{O}_{\mathbf{A}}$ is compact as product of compact spaces, and thus $\mathbf{A}$ is locally compact, totally disconnected and Hausdorff.

1.2.2 For every variety V over F_x, i.e. an separable integral F_x-scheme of finite type, the set $V(F_x)$ has a *strong topology*, cf. [43]. It is uniquely determined by the properties that the set of F_x-rational points of the affine line $\mathrm{A}^1(F_x)$ is homeomorphic to F_x, that if $V = V_1 \times V_2$, then $V(F_x)$ is homeomorphic to $V_1(F_x) \times V_2(F_x)$, and that for locally closed embeddings $V' \hookrightarrow V$, the space $V'(F_x)$ has the subspace topology of $V(F_x)$. The strong topology turns $V(F_x)$ into a locally compact, totally disconnected Hausdorff space.

If V is a variety over F that is embedded into affine space, then we can consider for every $x \in |X|$ the $\mathcal{O}_X$-rational points of V and obtain an inclusion $V(\mathcal{O}_x) \subset V(F_x)$. Equipped with the subspace topology, $V(\mathcal{O}_x)$ is compact. Define $V(\mathbf{A}_S)$ as $\prod_{x \in S} V(F_x) \times \prod_{x \notin S} V(\mathcal{O}_x)$, and $V(\mathbf{A})$ as the colimit $\bigcup V(\mathbf{A}_S)$ over all finite $S \subset |X|$. The topological space $V(\mathbf{A})$ does not depend anymore on the embedding into affine space (in contrast to $V(\mathcal{O}_x)$ and $V(\mathbf{A}_S)$). Therefore we can associate to any variety V over F a natural topological space $V(\mathbf{A})$, which does not depend on the choice of an atlas. We call the topology on $V(\mathbf{A})$ the *strong topology*, too. Again, $V(\mathcal{O}_{\mathbf{A}}) = \prod_{x \in |X|} \mathcal{O}_x$ is compact by Tychonoff's theorem, and thus $V(\mathbf{A})$ is locally compact, totally disconnected and Hausdorff.

If $V(\mathbf{A})$ is not empty but compact, then $V(F_x)$ is compact for every $x \in |X|$ since the projection maps $V(\mathbf{A}) \to V(F_x)$ are surjective. The converse implication holds as well if we consider all finite field extensions. More precisely, we can prove the following statement along the lines of the proof of [43, Thm. 1.1].

1.2.3 Theorem. *Let V be a variety over F and $x \in |X|$. Then the following are equivalent.*

(i) *V is complete.*

(ii) *$V(E_x)$ is compact in the strong topology for every finite field extension E_x/F_x.*

(iii) *$V(\mathbf{A}_E)$ is compact in the strong topology for every finite field extension E/F.*

1.2.4 If V is an algebraic group over F, then the group law turns $V(\mathbf{A})$ into a locally compact group. A locally compact group has a *left* and a *right Haar measure*, i.e. a non-trivial measure that is invariant by left or right translations, respectively. Every time that an algebraic group appears we assume the adelic points to carry a Haar measure. A Haar measure is unique up to a constant multiple. Rather than fixing the constant, we point out that constructions are independent of the choice of constant.

The Haar measure defines a Lebesgue integral for measurable functions with compact support. The Haar measure of the product $H_1 \times H_2$ of two locally compact groups equals, up to a multiple, the product of the Haar measures of the factors H_1 and H_2. Thus we can apply Fubini's theorem if we have an isomorphism $H \simeq H_1 \times H_2$ of topological groups, quietly assuming that the Haar measures are suitably normalised.

1.2.5 The group $V(\mathbf{A})^\vee$ of continuous group homomorphisms $V(\mathbf{A}) \to \mathbf{S}^1$ into the unit circle $\mathbf{S}^1 \subset \mathbf{C}$ is called the *character group* of $V(\mathbf{A})$. If $V(\mathbf{A})$ is abelian, then $V(\mathbf{A})^\vee$ is also called the *Pontryagin dual* of $V(\mathbf{A})$. The crucial property of the Pontryagin dual is that $(V(\mathbf{A})^\vee)^\vee = V(\mathbf{A})$.

1.2.6 Let $\mathbf{G}_\mathrm{a}$ be the additive group scheme and $\mathbf{G}_\mathrm{m}$ be the multiplicative group scheme. Then $\mathbf{G}_\mathrm{a}(\mathbf{A})$ is the additive group of $\mathbf{A}$ with the topology that we have defined before. Its Pontryagin dual is isomorphic to $\mathbf{G}_\mathrm{a}(\mathbf{A})$ itself in a non-canonical way. The group $\mathbf{G}_\mathrm{m}(\mathbf{A})$ is isomorphic to the idele group, and endows the ideles with a locally compact topology.

For example, $\mathbf{G}_\mathrm{m}(\mathbf{A}) = \mathrm{GL}_1(\mathbf{A})$ can be realised as the closed subspace defined by $\underline{X}\,\underline{Y} = 1$ of the affine space with coordinates $\underline{X}$ and $\underline{Y}$. The dual of the idele group is somewhat involved, but in the following chapters, we investigate the quasi-characters of $F^\times \backslash \mathbf{A}^\times$, a topological group closely related to the dual of $\mathbf{A}^\times$.

Finally, we warn the reader that the idele topology is finer than the subspace topology of $\mathbf{A}^\times \subset \mathbf{A}$ as the inclusion $\mathrm{GL}_n(\mathbf{A}) \subset \mathrm{Mat}_n(\mathbf{A})$ of the invertible matrices into all n-by-n-matrices is not an embedding of topological spaces but only continuous. Only if we embed GL_n as closed subvariety into affine space A^k of some dimension k, e.g. by sending points g of GL_n to (g, g^{-1}) in $\mathrm{Mat}_n \times \mathrm{Mat}_n \simeq \mathrm{A}^{2n^2}$, the induced map $\mathrm{GL}_n(\mathbf{A}) \to \mathrm{A}^k(\mathbf{A})$ is a topological embedding, which can be used to describe the topology on $\mathrm{GL}_n(\mathbf{A})$.

1.3 Automorphic forms

The concept of an automorphic form used nowadays can be applied to a large class of algebraic groups. Moeglin and Waldspurger describe in [48] the theory for a certain class of extensions of connected reductive groups. Here, however, we will restrict to GL_2. Standard reference books are the classic [32] by Jacquet and Langlands, [11] and [23]. We consider automorphic forms on GL_2 with trivial central character. These are nothing else but automorphic forms on PGL_2, but for technical reasons, it is more convenient to work with GL_2.

1.3.1 Set $G = \mathrm{GL}_2$ and let Z be the centre of G. We will often write $G_\mathbf{A}$ instead of $G(\mathbf{A})$, Z_F instead of $Z(F)$, etc. Let $K = GL_2(\mathcal{O}_\mathbf{A})$, which is the standard maximal compact subgroup of $G_\mathbf{A}$. The topology of $G_\mathbf{A}$ has a neighbourhood basis $\mathcal{V}$ of the identity matrix that is given by all subgroups

$$K' = \prod_{x \in |X|} K'_x < \prod_{x \in |X|} K_x = K$$

such that for all $x \in |X|$ the subgroup K'_x of K_x is open and consequently of finite index and such that K'_x differs from K_x only for a finite number of places.

Consider the space $C^0(G_\mathbf{A})$ of continuous functions $f : G_\mathbf{A} \to \mathbf{C}$. Such a function is called *smooth* if it is locally constant. $G_\mathbf{A}$ acts on $C^0(G_\mathbf{A})$ through the *right regular representation* $\rho : G_\mathbf{A} \to \mathrm{Aut}(C^0(G_\mathbf{A}))$ that is defined by right translation of the argument: $\big(\rho(g)\,f\big)(h) = f(hg)$ for $g, h \in G_\mathbf{A}$ and $f \in C^0(G_\mathbf{A})$. Since we are only concerned with subrepresentations of ρ, we will also write $g.f$ for $\rho(g)f$. A function f is called *K-finite* if the complex vector space that is generated by $\{k.f\}_{k\in K}$ is finite dimensional.

A function f is called *left* or *right H-invariant* for a subgroup $H < G_{\mathbf{A}}$ if for all $h \in H$ and $g \in G$, $f(hg) = f(g)$ or $f(gh) = f(g)$, respectively. If f is right and left H-invariant, it is called *bi-H-invariant*.

1.3.2 Lemma. *A function $f \in C^0(G_{\mathbf{A}})$ is smooth and K-finite if and only if there is a $K' \in \mathcal{V}$ such that f is right K'-invariant.*

Proof. If f is smooth, then we find for every $g \in G_{\mathbf{A}}$ a $K_g \in \mathcal{V}$ such that for all $k \in K_g$, $f(gk) = f(g)$. If f is K-finite, span$\{k.f\}_{k\in K}$ admits a finite basis $\{f_1, \dots, f_r\}$ and K acts on this finite-dimensional space. Let $F = (f_1, \dots, f_r) : G_{\mathbf{A}} \to \mathbf{C}^r$. By the linear independence of basis elements, we find $g_1, \dots, g_r \in G_{\mathbf{A}}$ such that $\{F(g_i)\}_{i=1,\dots,r} \subset \mathbf{C}^r$ is linearly independent. Put $K' = K_{g_1} \cap \dots \cap K_{g_r}$, then for all $k \in K'$ and $i = 1, \dots, r$, we have $k.F(g_i) = F(g_i k) = F(g_i)$. Thus K' acts trivially on span$\{k.f\}_{k\in K}$, and in particular, f is right K'-invariant. The reverse implication is obvious. $\square$

1.3.3 Let f be a smooth function that is K-finite and left $G_F Z_{\mathbf{A}}$-invariant. We say that f is of *moderate growth* if for every $c > 0$ and all compact subsets $K' \subset G_{\mathbf{A}}$, there are constants C and N such that for all $k \in K'$ and $a \in \mathbf{A}^\times$ with $|a| > c$,

$$f\left(\left(\begin{smallmatrix} a & \\ & 1 \end{smallmatrix}\right)k\right) \le C\,|a|^N \ .$$

1.3.4 Remark. The condition of moderate growth can be restated as follows, cf. [9, paragraph 1.6] and [11, p. 300]. Choose a closed embedding $G \to \mathrm{A}^k$ into affine space, e.g. the embedding described in paragraph 1.2.6, and consider the maximum norm $|\ |_{\max}$ on $\mathrm{A}^k(\mathbf{A})$, which restricts to $G_{\mathbf{A}}$. A function $f \in C^0(G_{\mathbf{A}})$ is of moderate growth if and only if there are numbers N and C such that for all $g \in G_{\mathbf{A}}$,

$$|f(g)|_{\mathbf{C}} \le C\,|g|_{\max}^N \ .$$

This notion is in fact independent of the chosen embedding $G \to \mathrm{A}^k$ and it is consistent with the definition of the previous paragraph, cf. loc. cit.

1.3.5 Definition. The *space $\mathcal{A}$ of automorphic forms (with trivial central character)* is the complex vector space of all smooth functions $f : G_{\mathbf{A}} \to \mathbf{C}$ that are K-finite, of moderate growth and left $G_F Z_{\mathbf{A}}$-invariant. Its elements are called *automorphic forms*.

1.3.6 Note that for $g \in G_{\mathbf{A}}$ and f smooth, K-finite, of moderate growth, or left $G_F Z_{\mathbf{A}}$-invariant, $g.f$ is also smooth, K-finite, of moderate growth, or left $G_F Z_{\mathbf{A}}$-invariant, respectively. Thus the right regular representation restricts to $\mathcal{A}$.

For every subspace $V \subset \mathcal{A}$, let $V^{K'}$ be the subspace of all $f \in V$ that are right K'-invariant. The functions in $\mathcal{A}^{K'}$ can be identified with the functions on $G_F Z_{\mathbf{A}} \backslash G_{\mathbf{A}} / K'$ that satisfy an appropriate growth condition, cf. paragraph 5.5.1. $\mathcal{A}^K$ is called the space of *unramified automorphic forms*. Lemma 1.3.2 implies

1.3.7 Proposition. $V = \bigcup_{K' \in \mathcal{V}} V^{K'}$ *for every subspace $V \subset \mathcal{A}$.* $\square$

1.3.8 Remark. A subrepresentation of $G_{\mathbf{A}}$ on $C^0(G_{\mathbf{A}})$ is called smooth if the stabiliser of each element of $C^0(G_{\mathbf{A}})$ is open in $G_{\mathbf{A}}$. Lemma 1.3.2 implies that $f \in C^0(G_{\mathbf{A}})$ is contained in a smooth subrepresentation if and only f is smooth and K-finite and the last proposition states that every subrepresentation of $\mathcal{A}$ is smooth.

1.4 The Hecke algebra

Hecke algebras are convolution algebras of functions with compact support on the adelic points of the group under consideration. A representation of the group in a complex vector space corresponds to a representation of the Hecke algebra by assigning an integral operator to each of its elements, which are called Hecke operators. We make this precise for $G = \mathrm{GL}_2$. Fix a choice of Haar measure for $G_{\mathbf{A}}$. (Note that $G_{\mathbf{A}}$ is unimodular, i.e. the left and right Haar measures coincide.)

1.4.1 Definition. The complex vector space $\mathcal{H}$ of all smooth compactly supported functions $\Phi : G_{\mathbf{A}} \to \mathbf{C}$ together with the convolution product

$$\Phi_1 * \Phi_2 : g \mapsto \int_{G_{\mathbf{A}}} \Phi_1(gh^{-1})\Phi_2(h)\, dh$$

for $\Phi_1, \Phi_2 \in \mathcal{H}$ is called the *Hecke algebra for* $G_{\mathbf{A}}$. Its elements are called *Hecke operators*.

1.4.2 The zero element of $\mathcal{H}$ is the zero function, but there is no multiplicative unit. For $K' \in \mathcal{V}$, we define $\mathcal{H}_{K'}$ to be the subalgebra of all bi-K'-invariant elements, i.e. all $\Phi \in \mathcal{H}$ that are left and right K'-invariant. These subalgebras, however, have multiplicative units, to wit, the normalised characteristic function $\epsilon_{K'} := (\mathrm{vol}\, K')^{-1}\, \mathrm{char}_{K'}$ acts as the identity on $\mathcal{H}_{K'}$ by convolution.

1.4.3 Lemma. *Every $\Phi \in \mathcal{H}$ is bi-K'-invariant for some $K' \in \mathcal{V}$.*

Proof. Since Φ is locally constant and $\mathcal{V}$ is a system of neighbourhoods of the identity, we can cover the support of Φ with sets of the form $g_i K_i$ with $g_i \in G_{\mathbf{A}}$ and $K_i \in \mathcal{V}$, where i varies in some index set, such that Φ is constant on each $g_i K_i$. But the support of Φ is compact, so we may restrict to a finite index set. Then $K'' = \bigcap_i K_i \in \mathcal{V}$, and Φ is right K''-invariant. In the same manner, we find a $K''' \in \mathcal{V}$ such that Φ is left K'''-invariant. Then $K' = K'' \cap K'''$ satisfies the assertion of the lemma. □

1.4.4 Proposition. $\mathcal{H} = \bigcup_{K' \in \mathcal{V}} \mathcal{H}_{K'}$. □

1.4.5 Lemma. *If $\Phi_1 \in \mathcal{H}$ is left K_1-invariant and $\Phi_2 \in \mathcal{H}$ is right K_2-invariant for $K_1, K_2 \in \mathcal{V}$, then $\Phi_1 * \Phi_2$ is left K_1-invariant and right K_2-invariant.*

Proof. We calculate for $g \in G_{\mathbf{A}}$, $k_1 \in K_1$, and $k_2 \in K_2$ that

$$\Phi_1 * \Phi_2(k_1 g k_2) = \int_{G_{\mathbf{A}}} \Phi_1(k_1 g k_2 h^{-1}) \Phi_2(h)\, dh = \int_{G_{\mathbf{A}}} \Phi_1(g h'^{-1}) \Phi_2(h' k_2)\, dh' = \Phi_1 * \Phi_2(g)$$

by the change of variables $h' = h k_2^{-1}$. □

1.4.6 The right regular representation ρ of $G_{\mathbf{A}}$ on $\mathcal{A}$ induces the *right regular representation of $\mathcal{H}_{\mathbf{A}}$ on $\mathcal{A}$* by

$$\rho(\Phi) f : g \mapsto \int_{G_{\mathbf{A}}} \Phi(h) \rho(h) f(g)\, dh\,,$$

which we also denote by $\Phi(f)$. We have that $\Phi_1 * \Phi_2(f) = \Phi_1(\Phi_2(f))$. Restriction gives a representation of $\mathcal{H}_{K'}$ on $\mathcal{A}_{K'}$ for each $K' \in \mathcal{V}$.

Note that the right regular representation is not trivial since for $f \in \mathcal{A}^{K'}$,

$$\epsilon_{K'}(f)(g) = \int_{G_{\mathbf{A}}} \epsilon_{K'}(h) f(gh)\, dh = \mathrm{vol}(K')^{-1} \int_{K'} f(gh)\, dh = f(g)\,.$$

1.4.7 Lemma. *For every $f \in C^0(G_{\mathbf{A}})$ and every $\Phi \in \mathcal{H}_{K'}$, $\Phi(f)$ is right K'-invariant.*

Proof. Let $g \in G_{\mathbf{A}}$ and $k \in K'$, then

$$\Phi(f)(gk) = \int_{G_{\mathbf{A}}} \Phi(h) f(gkh)\, dh \underset{(h'=kh)}{=} \int_{G_{\mathbf{A}}} \Phi(k^{-1}h') f(gh')\, dh' = \Phi(f)(g)\,. \quad \square$$

1.4.8 We call a subspace of $C^0(G_{\mathbf{A}})$ that is invariant under $G_{\mathbf{A}}$ or $\mathcal{H}$ briefly an *invariant subspace*. It is also called a *$G_{\mathbf{A}}$-module* or an *$\mathcal{H}$-module*. This is nothing else but a subrepresentation of $G_{\mathbf{A}}$ or $\mathcal{H}$, also called an $\mathcal{H}$-submodule of $\mathcal{A}$. An *irreducible subspace* or *simple $\mathcal{H}$-submodule* is a non-zero invariant subspace that has no other invariant subspaces than the zero-space and itself. We call an *$\mathcal{H}_{K'}$-submodule* of $V^{K'}$ an $\mathcal{H}_{K'}$-invariant subspace for every $K' \in \mathcal{V}$, and we call it *irreducible* if it is non-zero and contains no proper $\mathcal{H}_{K'}$-submodule different from the trivial subspace.

For any $K' \in \mathcal{V}$, let

$$\mathcal{H}(V) = \{\Phi(f) \mid \Phi \in \mathcal{H}, f \in V\} \quad \text{and} \quad \mathcal{H}_{K'}(V) = \{\Phi(f) \mid \Phi \in \mathcal{H}_{K'}, f \in V\}$$

be the $\mathcal{H}$-module and the $\mathcal{H}_{K'}$-module, respectively, generated by V, and let

$$G_{\mathbf{A}}.V = \{g.f \mid g \in G_{\mathbf{A}}, f \in V\}$$

be the $G_{\mathbf{A}}$-module generated by V. Write $G_{\mathbf{A}}.f := G_{\mathbf{A}}.\{f\}$, $\mathcal{H}(f) := \mathcal{H}(\{f\})$ and $\mathcal{H}_{K'}(f) := \mathcal{H}_{K'}(\{f\})$ for $f \in \mathcal{A}$ and $K' \in \mathcal{V}$.

1.4.9 Proposition. *If $V \subset C^0(G_{\mathbf{A}})$ is an invariant subspace, then $\mathcal{H}_{K'}(V) = V^{K'}$ for each $K' \in \mathcal{V}$.*

Proof. The inclusion $\mathcal{H}_{K'}(V) \subset V^{K'}$ follows from Lemma 1.4.7, the inclusion $V^{K'} \subset \mathcal{H}_{K'}(V)$ from that $\mathcal{H}_{K'}$ has a unit, see paragraph 1.4.2. □

1.4.10 Lemma. *For every K', every right K'-invariant $f \in C^0(G_{\mathbf{A}})$ and every $g \in G_{\mathbf{A}}$, there is a $\Phi \in \mathcal{H}_{K'}$ such that $\Phi(f) = g.f$.*

Proof. Put $\Phi = (\mathrm{vol}\, K')^{-1}\, \mathrm{char}_{gK'}$, then for all $g' \in \mathcal{A}$,

$$\Phi(f)(g') \;=\; \int_{G_{\mathbf{A}}} \Phi(h) f(g'h)\, dh \;=\; (\mathrm{vol}\, K')^{-1} \int_{K'} f(g'gk)\, dk \;=\; g.f(g')\,. \quad \square$$

1.4.11 Lemma. *For every $K' \in \mathcal{V}$ and every $\Phi \in \mathcal{H}_{K'}$, there are $h_1, \ldots, h_r \in G_{\mathbf{A}}$ and $m_1, \ldots, m_r \in \mathbf{C}$ for some integer r such that for all $g \in G_{\mathbf{A}}$ and all $f \in \mathcal{A}^{K'}$,*

$$\Phi(f)(g) \;=\; \sum_{i=1}^{r} m_i \cdot f(gh_i)\,.$$

Proof. Since Φ is K'-bi-invariant and compactly supported, it is a finite linear combination of characteristic functions on double cosets of the form $K'hK'$ with $h \in G_{\mathbf{A}}$. So we may reduce the proof to $\Phi = \mathrm{char}_{K'hK'}$. Again, since $K'hK'$ is compact, it equals the union of a finite number of pairwise distinct cosets $h_1K', \ldots, h_rK'$, and thus

$$\int_{G_{\mathbf{A}}} \mathrm{char}_{K'hK'}(h') f(gh')\, dh' \;=\; \sum_{i=1}^{r} \int_{G_{\mathbf{A}}} \mathrm{char}_{h_iK'}(h') f(gh')\, dh \;=\; \sum_{i=1}^{r} \mathrm{vol}(K') f(gh_i)$$

for arbitrary $g \in G_{\mathbf{A}}$. □

1.4.12 Proposition. *A subspace of $C^0(G_{\mathbf{A}})$ is invariant under $G_{\mathbf{A}}$ if and only if it is invariant under $\mathcal{H}$, both acting via the right regular representation.*

Proof. Lemma 1.4.10 implies that a subspace invariant under $\mathcal{H}$ is also invariant under $G_{\mathbf{A}}$. The converse follows from Lemma 1.4.11. □

1.4.13 Lemma. *A subspace $V \subset \mathcal{A}$ is irreducible if and only if $V^{K'}$ is irreducible as $\mathcal{H}_{K'}$-module for all sufficiently small $K' \in \mathcal{V}$.*

Proof. Let $V \subset \mathcal{A}$ be irreducible. If $W \subset V^{K'}$ is an $\mathcal{H}_{K'}$-submodule for some $K' \in \mathcal{V}$, then $\mathcal{H}(W)$ is an invariant subspace of V. Assume that W is a proper subspace of $V^{K'}$. For $\Phi \in \mathcal{H}$ and $f \in W$ such that $\Phi(f) \in V^{K'}$,

$$\Phi(f) \;=\; \epsilon_{K'}(\Phi(f)) \;=\; \epsilon_{K'}(\Phi(\epsilon_{K'}(f))) \;=\; \epsilon_{K'} * \Phi * \epsilon_{K'}(f)\,,$$

but by Lemma 1.4.5, $\epsilon_{K'} * \Phi * \epsilon_{K'} \in \mathcal{H}_{K'}$, and thus $\Phi(f) \in W$. This shows that W equals $\mathcal{H}(W) \cap V^{K'}$, and as V is irreducible, both $\mathcal{H}(W)$ and W are trivial, so $V^{K'}$ is zero or irreducible for every $K' \in \mathcal{V}$. Since V is non-zero, $V^{K'}$ is non-zero for all sufficiently small $K' \in \mathcal{V}$.

If, on the other hand, V contains a proper nontrivial invariant subspace W, then there is a $K' \in \mathcal{V}$ such that $W^{K'}$ is a proper nontrivial subspace of $V^{K'}$. This proves the reverse direction. □

1.4.14 We call $\mathcal{H}_K$ the *unramified part of* $\mathcal{H}$. Its elements are called unramified Hecke operators. For x a place of F, let Φ_x be the characteristic function of $K\left(\begin{smallmatrix}\pi_x & \\ & 1\end{smallmatrix}\right)K$ divided by $\operatorname{vol} K$, and $\Phi_{x,0}$ the characteristic function of $K\left(\begin{smallmatrix}\pi_x & \\ & \pi_x\end{smallmatrix}\right)K = \left(\begin{smallmatrix}\pi_x & \\ & \pi_x\end{smallmatrix}\right)K$. Both are elements of $\mathcal{H}_K$.

1.4.15 Lemma. *Identifying* ϵ_K *with* $1 \in \mathbf{C}$ *yields* $\mathcal{H}_K \simeq \mathbf{C}[\Phi_x, \Phi_{x,0}, \Phi_{x,0}^{-1}]_{x\in|X|}$. *In particular,* $\mathcal{H}_K$ *is commutative. For all* $f \in \mathcal{A}^K$, *one has* $\Phi_{x,0}(f) = f = \Phi_{x,0}^{-1}(f)$.

Proof. The first assertion follows immediately from Proposition 4.6.2 and Theorem 4.6.1 in [11], the last from the fact that f is $Z_{\mathbf{A}}$-invariant. □

1.4.16 Remark. We are actually considering automorphic forms on PGL_2, and the unramified part of the Hecke algebra for PGL_2 is nothing else but $\mathbf{C}[\Phi_x]_{x\in|X|}$. For technical reason, however, we will work with automorphic forms on GL_2 that are $Z_{\mathbf{A}}$-invariant, and thus have to consider the Hecke algebra for GL_2. The latter statement of the lemma can be expressed by saying that the representation of the Hecke algebra for GL_2 on $\mathcal{A}$ factors through the representation of the Hecke algebra for PGL_2.

1.5 Toroidal automorphic forms

This section introduces the object of investigation in this thesis, the space of toroidal automorphic forms. We first collect some facts about maximal tori in $G = \mathrm{GL}_2$.

1.5.1 Definition. A *maximal torus of* G is an algebraic subgroup T of G defined over F such that $T(F^{\mathrm{sep}}) \simeq \mathbf{G}_{\mathrm{m}}(F^{\mathrm{sep}}) \times \mathbf{G}_{\mathrm{m}}(F^{\mathrm{sep}})$ over a separable closure F^{sep}In particular, it is abelian. A torus is called *split over* E if $T(E) \simeq \mathbf{G}_{\mathrm{m}}(E) \times \mathbf{G}_{\mathrm{m}}(E)$ for an field extension E of F, and *anisotropic over* E otherwise. We say that T is a split or anisotropic torus if it is a maximal torus that is split or anisotropic, respectively, over F.

1.5.2 Let E/F be a separable quadratic algebra extension. Then E is either a separable field extension of degree 2 of F or isomorphic to $F \oplus F$ ([6, §1] or [47, §26]). Choosing a basis of E as vector space over F defines an inclusion of algebras

$$\Theta_E : E \simeq \mathrm{End}_E(E) \subseteq \mathrm{End}_F(E) \xrightarrow{\sim} \mathrm{Mat}_2 F \ ,$$

and $\Theta_E(E^\times)$ is a maximal torus of $G_F \subset \mathrm{Mat}_2 F$, isomorphic to $E^\times$. Note that $F \subset E$ implies $Z_F \subset T_F$. It is split if and only if $E \simeq F \oplus F$.

On the other hand, every torus T is given by an embedding of this form, because over the separable closure $T_{F^{\mathrm{sep}}} \simeq (F^{\mathrm{sep}})^\times \oplus (F^{\mathrm{sep}})^\times$, which is the multiplicative group of $F^{\mathrm{sep}} \oplus F^{\mathrm{sep}}$, a separable quadratic algebra extension of F^{sep}. Taking invariants under the action of $\mathrm{Gal}(F^{\mathrm{sep}}/F)$ yields the embedding of $F^\times \oplus F^\times$ into a separable quadratic algebra extension of F as muliplicative group.

Since changing the basis that we used to define Θ_E conjugates the torus, we obtain:

1.5.3 Proposition. *The map*

$$\left\{ \begin{array}{c} \textit{subfields } E \subset F^{\mathrm{sep}} \\ \textit{quadratic over } F \end{array} \right\} \cup \{F \oplus F\} \quad \overset{1:1}{\longrightarrow} \quad \left\{ \begin{array}{c} \textit{conjugation classes} \\ \textit{of maximal} \\ \textit{tori in } G_F \end{array} \right\}$$

$$E \qquad\longmapsto\qquad \Theta_E(E^\times)$$

is a bijection. The inverse map sends a maximal torus T to the field E whose group of units $E^\times$ is isomorphic to T_F. □

1.5.4 Lemma. *If T is an anisotropic torus, then $T_F Z_{\mathbf{A}} \backslash T_{\mathbf{A}}$ is a compact abelian group.*

Proof. As quotient of an abelian group, $T_F Z_{\mathbf{A}} \backslash T_{\mathbf{A}}$ is abelian. Concerning compactness, observe that if $T_F \simeq E^\times$, then $T_{\mathbf{A}} \simeq \mathbf{A}_E^\times$, $Z_{\mathbf{A}} \simeq \mathbf{A}_F^\times$, and $T_F \simeq E^\times$ as topological groups, and thus $T_F Z_{\mathbf{A}} \backslash T_{\mathbf{A}} \simeq E^\times \mathbf{A}_F^\times \backslash \mathbf{A}_E^\times$. Look at the exact sequence of topological groups

$$1 \longrightarrow \mathcal{O}_{\mathbf{A}_E}^\times \longrightarrow E^\times \backslash \mathbf{A}_E^\times \longrightarrow \mathrm{Cl}\, E \longrightarrow 0\,,$$

The action of the Galois group $\mathrm{Gal}(E/F)$ on E extends to an action on $\mathbf{A}_E$ with invariants precisely $\mathbf{A}_F$. We have that $E^\times \cap \mathbf{A}_F^\times = F^\times$ and $\mathcal{O}_{\mathbf{A}_E}^\times \cap \mathbf{A}_F^\times = \mathcal{O}_{\mathbf{A}_F}^\times$. Dividing out by $\mathbf{A}_F^\times$ yields

$$1 \longrightarrow \mathcal{O}_{\mathbf{A}_F}^\times \backslash \mathcal{O}_{\mathbf{A}_E}^\times \longrightarrow E^\times \mathbf{A}_F^\times \backslash \mathbf{A}_E^\times \longrightarrow \mathrm{Cl}\, E \;/\; j(\mathrm{Cl}\, F) \longrightarrow 0\,,$$

where $j : \mathrm{Cl}\, F \to \mathrm{Cl}\, E$ is the canonical map. Since the left and the right term are both compact groups, so is the middle one. □

1.5.5 Lemma. *If T is an split torus, then $T_F Z_{\mathbf{A}} \backslash T_{\mathbf{A}} \simeq F^\times \backslash \mathbf{A}^\times$.*

Proof. There is an isomorphism $T \simeq \mathbf{G}_{\mathrm{m}} \times \mathbf{G}_{\mathrm{m}}$ over F that induces $T_F \simeq F^\times \times F^\times$ and $T_{\mathbf{A}} \simeq \mathbf{A}^\times \times \mathbf{A}^\times$. This gives

$$\begin{aligned} T_F Z_{\mathbf{A}} \backslash T_{\mathbf{A}} &\simeq (Z_F \backslash Z_{\mathbf{A}}) \backslash (T_F \backslash T_{\mathbf{A}}) \\ &\simeq (F^\times \backslash \mathbf{A}^\times) \backslash \big((F^\times \backslash \mathbf{A}^\times) \times (F^\times \backslash \mathbf{A}^\times) \big) \simeq F^\times \backslash \mathbf{A}^\times\,. \end{aligned}$$ □

1.5.6 Lemma. *If T is an anisotropic torus with $T_F \simeq E^\times$, then it splits over E.*

Proof. We defined T_F as the image of $E^\times$ under the injective algebra homomorphism $\Theta_E : E \to \mathrm{Mat}_2(F)$ over F. Identifying T_F with $E^\times$ yields

$$T_E \simeq (E \otimes_F E)^\times \simeq \bigoplus_{\sigma \in \mathrm{Gal}(E/F)} \sigma(E)^\times \simeq \bigoplus_{\sigma \in \mathrm{Gal}(E/F)} \mathbf{G}_{\mathrm{m}}(E)\,.$$ □

1.5.7 We recall some facts about Borel subgroups from the theory of linear algebraic groups, cf. [8, §§11.1–11.3] or [74, §10.5]. A *Borel subgroup* B of G is an algebraic subgroup of G defined over F such that the quotient variety $B \backslash G$ is isomorphic to $\mathbf{P}^1$ over F, and the *standard Borel subgroup* is the subgroup of invertible upper triangular matrices.

Every Borel subgroup B of G contains a maximal split torus T, and is conjugated in G to the standard Borel subgroup such that T conjugates to the diagonal torus, i.e. the group of invertible diagonal matrices. On the other hand, every split torus T is contained in precisely two different Borel subgroups, and if we call the one B, then we call the other B^T.

A Borel subgroup B contains a unique *unipotent radical* N, i.e. a subgroup of maximal size whose elements are the sum of the identity matrix with a nilpotent matrix. If T is a split torus contained in B, then $B = TN$. We denote the unipotent radical of B^T by N^T.

Since the following constructions are invariant under conjugation, it suffices to keep in mind the standard Borel subgroup B together with the diagonal torus T. Then B^T is the group of invertible lower triangular matrices, which is conjugated to B by $\left(\begin{smallmatrix} & 1 \\ 1 & \end{smallmatrix}\right)$, leaving T invariant, but interchanging the values on the diagonal. The algebraic groups N and N^T are nothing else but the matrix groups $\left\{\left(\begin{smallmatrix} 1 & * \\ & 1 \end{smallmatrix}\right)\right\}$ and $\left\{\left(\begin{smallmatrix} 1 & \\ * & 1 \end{smallmatrix}\right)\right\}$, respectively.

1.5.8 Remark. For better readability we leave a blank space where matrix entries are zero, and asterisks stand for arbitrary elements of the ring under consideration that make the matrix invertible. So $\left(\begin{smallmatrix} 1 & \\ & 1 \end{smallmatrix}\right)$ has to be read as $\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$, and $\left\{\left(\begin{smallmatrix} * & * \\ & 1 \end{smallmatrix}\right)\right\}$ stands for the algebraic subgroup H of G that gives the subgroup $H_R = \left\{\left(\begin{smallmatrix} a & b \\ 0 & 1 \end{smallmatrix}\right) \middle| a \in R^\times, b \in R\right\}$ of G_R for every algebra R over F .

1.5.9 ([11, Thm. 3.5.5], [48, §§I.2.6–I.2.7]). Let B be a Borel subgroup with split torus T and unipotent radical N. Then define the *constant term f_N (with respect to N)* of an automorphic form $f \in \mathcal{A}$ as the following function on $G_{\mathbf{A}}$:

$$f_N(g) \;:=\; \mathrm{vol}(N_F \backslash N_{\mathbf{A}})^{-1} \int\limits_{N_F \backslash N_{\mathbf{A}}} f(ng)\, dn \,.$$

Since N is invariant under conjugation by B, f_N is a function that is left $B_F Z_{\mathbf{A}}$-invariant.

If $f_N(g)$ vanishes for all $g \in G_{\mathbf{A}}$, we call f a *cusp form*, a notion that does not depend on the chosen Borel subgroup, since for $B_\gamma = \gamma B \gamma^{-1}$ with unipotent radical $N_\gamma = \gamma^{-1} N \gamma$,

$$f_{N_\gamma}(g) \;=\; \int\limits_{N_\gamma(F) \backslash N_\gamma(\mathbf{A})} f(ng)\, dn \;=\; \int\limits_{N_F \backslash N_{\mathbf{A}}} f(\gamma^{-1} n \gamma g)\, dn \;=\; \int\limits_{N_F \backslash N_{\mathbf{A}}} f(n g_\gamma)\, dn \;=\; f_N(g_\gamma)$$

with $g_\gamma = \gamma g$ running through $G_{\mathbf{A}}$ as g varies in $G_{\mathbf{A}}$. We denote the whole space of cusp forms by $\mathcal{A}_0$.

Every automorphic form has an *approximation by constant terms*:

1.5.10 Theorem ([48, I.2.9]). *For every $f \in \mathcal{A}$, the function $f - f_N$ has compact support as a function on $B_F Z_{\mathbf{A}} \backslash G_{\mathbf{A}}$.*

Let $e \in G_{\mathbf{A}}$ denote the identity matrix.

1.5.11 Proposition. *Cusp forms have compact support modulo $G_F Z_{\mathbf{A}}$, and for the unipotent radical N of any Borel subgroup,*

$$\mathcal{A}_0 = \{ f \in \mathcal{A} \mid \forall \Phi \in \mathcal{H}, \Phi(f)_N(e) = 0 \} \,.$$

Proof. The first claim follows from theorem 1.5.10. The second claim follows from Lemmas 1.4.10 and 1.4.11. □

1.5.12 Let T be an anisotropic torus, and endow $Z_{\mathbf{A}}$ and $T_{\mathbf{A}}$ with Haar measures such that $Z_{\mathbf{A}} \simeq \mathbf{A}_F^\times$ and $T_{\mathbf{A}} \simeq \mathbf{A}_E^\times$ as measure spaces. Endow T_F with the discrete measure. This defines a Haar measure on $T_F Z_{\mathbf{A}} \backslash T_{\mathbf{A}}$ as quotient measure. We call

$$f_T(g) := \int\limits_{T_F Z_{\mathbf{A}} \backslash T_{\mathbf{A}}} f(tg)\, dt$$

the *toroidal integral of* T *(evaluated at* g*)*. By Lemma 1.5.4 the integral converges for all $f \in \mathcal{A}$ and $g \in G_{\mathbf{A}}$.

If T is a split torus, then endow $T_{\mathbf{A}} \simeq \mathbf{A}^\times \oplus \mathbf{A}^\times$ with the product measure of $\mathbf{A}^\times$. Further let $Z_{\mathbf{A}}$ carry the same measure as before and let T_F carry the discrete measure. This defines a quotient measure on $T_F Z_{\mathbf{A}} \backslash T_{\mathbf{A}}$. Let B and B^T be the Borel subgroups that contain T, and let N and N^T, respectively, be their unipotent radicals. Note that $T_F Z_{\mathbf{A}} \backslash T_{\mathbf{A}}$ is not compact, but according to Theorem 1.5.10, both $f - f_N$ and $f - f_{N^T}$ have compact support as functions on $B_F Z_{\mathbf{A}} \backslash G_{\mathbf{A}}$ and $B_F^T Z_{\mathbf{A}} \backslash G_{\mathbf{A}}$, respectively. The *toroidal integral of* T *(evaluated in* g*)* is

$$f_T(g) := \int\limits_{T_F Z_{\mathbf{A}} \backslash T_{\mathbf{A}}} \Big(f - \frac{1}{2}(f_N + f_{N^T})\Big)(tg)\, dt\ ,$$

which converges for all $f \in \mathcal{A}$ and any choice of Haar measure on $T_F Z_{\mathbf{A}} \backslash T_{\mathbf{A}}$.

1.5.13 Definition. Let T be a maximal torus of G corresponding to a separable quadratic algebra extension E/F. Then define

$$\mathcal{A}_{\text{tor}}(E) = \{ f \in \mathcal{A} \mid \forall g \in G_{\mathbf{A}},\ f_T(g) = 0 \}\ ,$$

the space of E*-toroidal automorphic forms,* and

$$\mathcal{A}_{\text{tor}} = \bigcap_{\substack{\text{separable quadratic} \\ \text{algebra extensions } E/F}} \mathcal{A}_{\text{tor}}(E)\ ,$$

the space of *toroidal automorphic forms*.

1.5.14 Remark. The spaces $\mathcal{A}_{\text{tor}}(E)$ indeed do not depend on the choice of torus in the conjugacy class corresponding to E, because a calculation similar to that for f_N proves that for a conjugate $T_\gamma = \gamma^{-1} T \gamma$ with $\gamma \in G_F$, we have $f_{T_\gamma}(g) = f_T(g_\gamma)$, where $g_\gamma = \gamma g$. Note that the definition is also independent of the choices of Haar measures.

1.5.15 Proposition. *For all* T *and* E *as above,*

$$\mathcal{A}_{\text{tor}}(E) = \{ f \in \mathcal{A} \mid \forall \Phi \in \mathcal{H}, \Phi(f)_T(e) = 0 \}\ ,$$

and

$$\mathcal{A}_{\text{tor}} = \{ f \in \mathcal{A} \mid \forall \textit{ maximal tori } T < G, \forall \Phi \in \mathcal{H},\ \Phi(f)_T(e) = 0 \}\ .$$

Proof. This follows from Lemmas 1.4.10 and 1.4.11. $\square$

CHAPTER 2

L-series and Eisenstein series

This chapter introduces the notions and reviews the results from the theory of L-series and Eisenstein series that are needed in the subsequent chapters.

2.1 Quasi-characters

Quasi-characters on $\mathbf{A}^\times$ that are continuous and trivial on $F^\times$ have a simple description in terms of invariants of $\mathbf{A}^\times$. This section gives an overview with short proofs. For alternative and more detailed introductions, cf. Tate's thesis [65] (for number fields only) and [79, VII.3].

2.1.1 A continuous group homomorphism $\chi : \mathbf{A}^\times \to \mathbf{C}^\times$ that is trivial on $F^\times$ is called a *quasi-character* (on the idele class group $F^\times \backslash \mathbf{A}^\times$). If $\chi(\mathbf{A}^\times) \subset \mathbf{S}^1 = \{z \in \mathbf{C} \mid |z|_\mathbf{C} = 1\}$, then we call it a *character*. The product $\chi_1\chi_2 : a \mapsto \chi_1(a) \cdot \chi_2(a)$ defines a group structure on the set of quasi-characters Ξ. Together with the compact-open topology, Ξ becomes a topological group.

For an effective divisor $D = (D_x)$, let

$$\Xi_D \;=\; \left\{ \chi \in \Xi \;\middle|\; \begin{array}{l} \chi(a) = 1 \text{ for all } a \in \mathcal{O}_\mathbf{A}^\times \text{ such that} \\ \forall x \in |X|,\ a_x \equiv 1 \pmod{\mathfrak{m}_x^{D_x}} \end{array} \right\} .$$

A quasi-character χ is called *unramified* if $\chi \in \Xi_0$.

2.1.2 The choice of an idele a_1 of degree 1 defines a section for the sequence in paragraph 1.1.5 (by identifying $\mathbf{Z}$ with the subgroup $A_1 = \langle a_1 \rangle$). We obtain a decomposition

$$\mathbf{A}^\times = \mathbf{A}_0^\times \times A_1 .$$

Observe that the natural logarithm

$$\ln : \mathbf{C}^\times \longrightarrow \mathbf{C}/2\pi\mathrm{i}\mathbf{Z}$$

is well defined as inverse to the exponential map since it factors over $\mathbf{C}/2\pi\mathrm{i}\mathbf{Z}$. Let e denote the Euler number. In the quasi-character $|\;|^s$, where the idele norm $|\;|$ satisfies $|a_1| = q^{-1} = \mathrm{e}^{-\ln q}$, the complex number s is determined up to $(2\pi\mathrm{i}/\ln q)\mathbf{Z}$.

2.1.3 Lemma. *For any $\chi \in \Xi$, and A_1 as above, the restriction $\chi|_{\mathbf{A}_0^\times}$ is of finite order, and $\chi|_{A_1} = |\,|^s$ with*

$$s \equiv -\frac{\ln\chi(a_1)}{\ln q} \pmod{\frac{2\pi\mathrm{i}}{\ln q}\mathbf{Z}} .$$

Proof. Since $F^\times\backslash\mathbf{A}_0^\times$ is compact, the image of $\chi|_{\mathbf{A}_0^\times}$ is compact. Since $\mathbf{A}^\times$ is totally disconnected, every quotient of $\mathbf{A}^\times$ is totally disconnected. The only totally disconnected compact subgroups of $\mathbf{S}^1$ are the finite subgroups, hence $\chi|_{\mathbf{A}_0^\times}$ is of finite order. To prove the second statement, observe that A_1 is generated by a_1, hence $\chi(a_1)$ determines the group homomorphism $\chi|_{A_1}$, which then has the form as described as in the lemma. □

2.1.4 Proposition. *With the notation of Lemma 2.1.3, there is a unique character ω of finite order such that $\omega|_{\mathbf{A}_0^\times} = \chi|_{\mathbf{A}_0^\times}$ and $\omega(a_1) = 1$. Consequently, we have*

$$\chi(a) = \omega(a)\,|a|^s . \quad \square$$

2.1.5 Both ω and s depend on the choice of a_1. For a different choice a_1', one obtains

$$s' \equiv s + \frac{\ln\chi(a_1) - \ln\chi(a_1')}{\ln q} \pmod{\frac{2\pi\mathrm{i}}{\ln q}\mathbf{Z}} \qquad \text{and} \qquad \omega' = \omega\,|\,|^{s'-s} .$$

Conversely, for two characters ω and ω' of finite order, there is a $s \in \mathbf{C}$ with $\omega' = \omega\,|\,|^s$ if and only if $\omega|_{\mathbf{A}_0^\times} = \omega'|_{\mathbf{A}_0^\times}$.

Define the *real part* $\operatorname{Re}\chi$ *of* χ as $\operatorname{Re}s$. For different choices of a_1 and a_1', we have that $|\,|^{s'-s} = \omega'\omega^{-1}$ is a character and thus $\operatorname{Re}|\,|^{s'-s} = 0$. This shows that the real part of χ does not depend on the choice of a_1.

2.1.6 Proposition. *The assignment*

$$\begin{array}{ccc} \Xi_0 & \xrightarrow{\sim} & (\mathrm{Cl}^0 F)^\vee \times \mathbf{C}/\frac{2\pi\mathrm{i}}{\ln q}\mathbf{Z} \\ \chi & \longmapsto & (\omega|_{\mathbf{A}_0^\times}, s) \end{array}$$

given by the decomposition $\chi = \omega\,|\,|^s$ of Corollary 2.1.4 is an isomorphism of topological groups, and endows Ξ_0 with the structure of a Lie group.

Proof. By Proposition 2.1.4 and the previous paragraph, every $\chi \in \Xi_0$ is in one-to-one correspondence with an $s \in \mathbf{C}/\frac{2\pi\mathrm{i}}{\ln q}\mathbf{Z}$ and a quasi-character ω of $F^\times\backslash\mathbf{A}_0^\times$ that satisfies $\omega(\mathcal{O}_{\mathbf{A}}^\times) = 1$. But $F^\times\backslash\mathbf{A}_0^\times / \mathcal{O}_{\mathbf{A}}^\times \simeq \mathrm{Cl}^0 F$.

Concerning the topologies, note that the group of character group of A_1 (as in Lemma 2.1.3) is homeomorphic to the unit circle $\mathbf{S}^1$. Consequently the group of quasi-characters restricted to A_1 is homeomorphic to $\mathbf{C}/\frac{2\pi\mathrm{i}}{\ln q}\mathbf{Z}$. □

2.1.7 Since $\mathcal{O}_{\mathbf{A}}^\times$ is compact and totally disconnected, it is a profinite group ([25, §1.4, Thm. 1]). By [25, §1.4, Cor. 1], we can describe $\mathcal{O}_{\mathbf{A}}^\times$ as inverse limit as follows. Define for every effective divisor $D = (D_x)$ the subgroup

$$U_D = \{a \in \mathcal{O}_{\mathbf{A}}^\times \mid \forall x \in |X|,\ a_x \equiv 1 \pmod{\mathfrak{m}_x^{D_x}}\}$$

of $\mathcal{O}_{\mathbf{A}}^{\times}$ and the finite quotient group $Q_D = \mathcal{O}_{\mathbf{A}}^{\times} / U_D$. Together with canonical projections, the groups Q_D for varying effective divisor D form a projective system. The group $\mathcal{O}_{\mathbf{A}}^{\times}$ is the projective limit of this system.

By Pontryagin duality ([51, §37, Thms. 39, 40 and 46]), the character group of the idele class group $F^{\times} \backslash \mathbf{A}^{\times}$ is the union of the character groups of $F^{\times} \backslash \mathbf{A}^{\times} / U_D$. By Lemma 2.1.3, every quasi-character is the product of a character and $|\ |^s$ for some $s \in \mathbf{C}$. Since $|\ |^s \in \Xi_D$ for all $s \in \mathbf{C}$ and all effective divisors D, we obtain:

2.1.8 Proposition. $\Xi = \bigcup_{\substack{\text{effective} \\ \text{divisors } D}} \Xi_D$. □

2.2 L-series

As reference for this section, consider [65] and [79], but also [61] and [59].

2.2.1 Let $\chi \in \Xi$ and $S = \{x \in |X| \mid \exists a_x \in \mathcal{O}_x^{\times},\ \chi(a_x) \neq 1\}$, then define

$$L_F(\chi, s) = \prod_{x \in |X| - S} \frac{1}{1 - \chi(\pi_x)\,|\pi_x|^s}$$

whenever the product converges. If no confusion arises, we omit the subscript F and write $L(\chi, s)$.

Recall from paragraph 1.1.5 that $\mathrm{c} \in \mathbf{A}^{\times}$ is a differental idele.

2.2.2 Theorem ([79, VII, §§ 6-7], [55, Prop. 9.26]). *The expression $L(\chi, 1/2 + s)$ converges if* $\operatorname{Re} s > 1/2 - \operatorname{Re}\chi$*, is analytic in s, and has a meromorphic continuation to all $s \in \mathbf{C}$, which we denote by the same symbol $L(\chi, 1/2 + s)$. It has poles in those s for which $\chi|\ |^s = |\ |^{\pm 1/2}$, and the poles are of order* 1*. Furthermore, it satisfies a* functional equation

$$L(\chi, 1/2 + s) = \epsilon(\chi, s) L(\chi^{-1}, 1/2 - s)$$

for a certain non-zero factor $\epsilon(\chi, s)$. If $\chi \in \Xi_0$, then $\epsilon(\chi, s) = \chi(\mathrm{c})\,|\mathrm{c}|^s$.

2.2.3 Remark. We chose to formulate the theorem for $L(\chi, 1/2 + s)$ instead of $L(\chi, s)$ to emphasise the analogy with the corresponding statements for Eisenstein series as introduced in the next section.

2.2.4 Definition. We call $L(\chi, s)$ the *L-series of the quasi character χ*, and define the *zeta function* of F as $\zeta_F(s) := L(1, s)$.

2.2.5 An alternative expression for the zeta function is

$$\zeta_F(s) = \sum_{\substack{\text{effective} \\ \text{divisors } D}} \frac{1}{\mathrm{N}(D)^s},$$

a sum that converges for $\mathrm{Re}\, s > 1$, where $\mathrm{N}(D) = q^{\deg D}$. If ω is a finite unramified character, i.e. ω factors through the divisor class group, then

$$L_F(\omega, s) = \sum_{\substack{\text{effective} \\ \text{divisors } D}} \frac{\omega(D)}{\mathrm{N}(D)^s} ,$$

if $\mathrm{Re}\, s > 1$.

2.2.6 Let $\psi : \mathbf{A} \to \mathbf{C}$ be a Schwartz-Bruhat function, i.e. a locally constant function with compact support. Choose a Haar measure on $\mathbf{A}^\times$ and define the *Tate integral*

$$L(\psi, \chi, s) = \int_{\mathbf{A}^\times} \psi(a) \chi(a) \, |a|^s \, da ,$$

whenever the integral converges. Define the Schwartz-Bruhat function ψ_0 by

$$\psi_0 \; = \; h_F \, (q-1)^{-1} \, (\mathrm{vol}\, \mathcal{O}_{\mathbf{A}})^{-1} \, \mathrm{char}_{\mathcal{O}_{\mathbf{A}}} .$$

2.2.7 Theorem ([79, VII, Thm. 2 and §§ 6-7]). *The expression $L(\psi, \chi, 1/2 + s)$ converges if $\mathrm{Re}\, s > 1/2 - \mathrm{Re}\, \chi$. For every Schwartz-Bruhat function ψ and $\chi \in \Xi$, it is a holomorphic multiple of $L(\chi, 1/2 + s)$ as function of $s \in \mathbf{C}$. For every $\chi \in \Xi$ there is a Schwartz-Bruhat function ψ such that $L(\psi, \chi, s) = L(\chi, s)$. In particular if $\chi \in \Xi_0$, then $L(\psi_0, \chi, s) = L(\chi, s)$.*

2.2.8 Theorem ([79, Thms. VII.4 and VII.6]).

(i) *The zeta function of F is of the form*

$$\zeta_F(s) \; = \; \frac{\mathfrak{L}(q^{-s})}{(1 - q^{-s})(1 - q^{1-s})}$$

with $\mathfrak{L}(T) = \mathfrak{L}(1, T)$ a polynomial of degree $2g_F$ with integer coefficients that has no zero at $T = 1$ or $T = q^{-1}$. In particular, ζ_F has simple poles at $s = 0$ and $s = 1$.

(ii) *For every $s' \in \mathbf{C}$ and $\chi = |\;|^{s'}$, there is a polynomial $\mathfrak{L}(\chi, T)$ with complex coefficients of degree $2g_F$ such that*

$$L(\chi, s) \; = \; \frac{\mathfrak{L}(\chi, q^{-s})}{(1 - q^{-(s+s')})(1 - q^{1-(s+s')})} .$$

This L-series has simple poles at $s = -s'$ and $s = 1 - s'$.

(iii) *For every unramified character χ that cannot be written as $|\;|^s$ for some $s \in \mathbf{C}$, there is a polynomial $\mathfrak{L}(\chi, T)$ with complex coefficients of degree $2g_F - 2$ such that*

$$L(\chi, s) \; = \; \mathfrak{L}(\chi, q^{-s}) .$$

2.2.9 Let E/F be a finite Galois extension and $\mathrm{N}_{E/F} : \mathbf{A}_E \to \mathbf{A}_F$ the norm map. Then the reciprocity homomorphism $r_{E/F} : \mathrm{Gal}(E/F) \to F^\times \mathrm{N}_{E/F}(\mathbf{A}_E^\times)\backslash \mathbf{A}_F^\times$ induces an isomorphism

$$r_{E/F}^* : \mathrm{Hom}(\, F^\times \mathrm{N}_{E/F}(\mathbf{A}_E^\times)\backslash \mathbf{A}_F^\times,\ \mathbf{S}^1\,) \longrightarrow \mathrm{Hom}(\mathrm{Gal}(E/F),\ \mathbf{S}^1\,)\,.$$

If ω is a character of $\mathrm{Gal}(E/F)$, then denote by $\tilde{\omega}$ the corresponding character of $\mathbf{A}_F^\times$ that is trivial on $F^\times$ and $\mathrm{N}_{E/F}(\mathbf{A}_E^\times)$. In particular, since E/F is unramified if and only if $\mathcal{O}_{\mathbf{A}}^\times \subset \mathrm{N}_{E/F}(\mathbf{A}_E^\times)$, we see that $\tilde{\omega}$ is unramified if E/F unramified is.

2.2.10 Lemma. *Let E/F be a finite abelian Galois extension and $\chi \in \Xi$. Then*

$$L_E(\chi \circ \mathrm{N}_{E/F}, s) \;=\; \prod_{\omega \in \mathrm{Hom}(\mathrm{Gal}(E/F), \mathbf{S}^1)} L_F(\chi\tilde{\omega}, s)$$

as meromorphic functions of s.

Proof. This follows easily from a formal calculation with Euler products. Since these converge for large $\mathrm{Re}\, s$, this yields an equality of meromorphic functions. □

2.2.11 Proposition. *Let $\chi \in \Xi$ be of finite order n. Then there is an abelian Galois extension E/F of order n such that $\chi(\mathrm{N}_{E/F}(\mathbf{A}_E^\times)) = 1$, and*

$$\prod_{\omega \in \mathrm{Hom}(\mathrm{Gal}(E/F), \mathbf{S}^1)} L_F(\chi \circ \tilde{\omega}, s) = \zeta_E(s)$$

as meromorphic functions of s. If χ is an unramified character, then E/F is an unramified field extension.

Proof. The existence of E/F such that the equation holds follows from class field theory together with Lemma 2.2.10 since $\chi(\mathrm{N}_{E/F}(\mathbf{A}_E^\times)) = 1$ implies that $L_E(\chi \circ \mathrm{N}_{E/F}, s) = L_E(1, s) = \zeta_E(s)$. The last assertion follows from paragraph 2.2.9. □

2.2.12 Corollary. *If $\chi \in \Xi$ is of finite order and not of the form $|\ |^s$ for some $s \in \mathbf{C}$, then*

$$L(\chi, 0) \neq 0 \quad \textit{and} \quad L(\chi, 1) \neq 0\,.$$

Proof. This follows from the equation of the proposition, since both ζ_F and ζ_E have simple poles at $s = 0$ and $s = 1$, and ζ_F occurs precisely once in the product on the left hand side, so all other factors do not vanish at $s = 0$ and $s = 1$ and in particular $L(\chi, \cdot)$ does not. □

2.2.13 Corollary. *Let E/F be a finite Galois extension, and $a \in \mathbf{A}^\times$ an idele of degree* 1. *If s is a n-fold zero of*

$$\prod_{\substack{\omega \text{ unram. char.} \\ \text{of } \mathrm{Cl}\, F,\ \omega(a)=1}} L_F(\omega, s)\,,$$

then s also is at least an n-fold zero of

$$\prod_{\substack{\omega \text{ unram. char.} \\ \text{of } \mathrm{Cl}\, F,\ \omega(a)=1}} L_E(\omega \circ \mathrm{N}_{E/F}, s)\,.$$

Proof. This follows immediately from Theorem 2.2.8 and Lemma 2.2.10. □

2.3 Eisenstein series

Originally, Eisenstein series were defined as modular forms on the upper half plane, given by explicit infinite sums. With the development of the theory of automorphic forms, these sums found generalisation in different directions. This section introduces the notion of Eisenstein series that we will use and states the most important facts in the form we need it in. As references consider [11], [23], [32] and [41], where the theory is explained for GL_2, or [10], [26] and [48] for more general approaches.

2.3.1 Let B be the standard Borel subgroup of upper triangular matrices, and $\chi_1, \chi_2 \in \Xi$. The *principal series representation* $\mathcal{P}(\chi_1, \chi_2)$ *(of* χ_1 *and* χ_2*)* is the space of all smooth and K-finite $f \in C^0(G_{\mathbf{A}})$ that for all $\left(\begin{smallmatrix} a & b \\ & d \end{smallmatrix}\right) \in B_{\mathbf{A}}$ and all $g \in G_{\mathbf{A}}$ satisfy

$$f\left(\begin{pmatrix} a & b \\ & d \end{pmatrix} g\right) = \left|\frac{a}{d}\right|^{1/2} \chi_1(a)\,\chi_2(d)\,f(g)\,.$$

The right regular representation ρ of $G_{\mathbf{A}}$ on $C^0(G_{\mathbf{A}})$ as defined in paragraph 1.3.1 restricts to $\mathcal{P}(\chi_1, \chi_2)$. By the equivalence of $G_{\mathbf{A}}$- and $\mathcal{H}$-modules (Proposition 1.4.12), $\mathcal{P}(\chi_1, \chi_2)$ is also an $\mathcal{H}$-module.

2.3.2 Theorem. *Let* $\chi_1, \chi_2 \in \Xi$. *The principal series representation* $\mathcal{P}(\chi_1, \chi_2)$ *is irreducible unless* $\chi_1 \chi_2^{-1} = |\ |^{\pm 1}$.

Proof. This can be proven by reduction to the representation theory over local fields, cf. [23, chapter 4.B] or [11, section 4.5], as well as the comment on page 355 of the same book. □

2.3.3 Theorem. *Let* $\mathcal{P}(\chi_1, \chi_2)$ *be irreducible. Then* $\mathcal{P}(\chi_1, \chi_2) \simeq \mathcal{P}(\chi'_1, \chi'_2)$ *as* $\mathcal{H}$*-modules if and only if either*

(i) $\chi_1 = \chi'_1$ *and* $\chi_2 = \chi'_2$ *or* (ii) $\chi_1 = \chi'_2$ *and* $\chi_2 = \chi'_1$.

Proof. See the references in the proof of the previous theorem. □

2.3.4 Proposition. $f \in \mathcal{P}(\chi_1, \chi_2)$ *is uniquely determined by its restriction to* K.

Proof. This follows immediately from the Iwasawa decomposition $G_{\mathbf{A}} = B_{\mathbf{A}} K$ (also cf. paragraph 4.2.2), and the definition of $\mathcal{P}(\chi_1, \chi_2)$. □

2.3.5 Since we consider only automorphic forms with trivial central character, it suffices to restrict to $\chi = \chi_1 = \chi_2^{-1}$, and we briefly write $\mathcal{P}(\chi)$ for $\mathcal{P}(\chi, \chi^{-1})$.

Let $\chi \in \Xi$. A *flat section* is a map $f_\chi : \mathbf{C} \to C^0(G_{\mathbf{A}})$ that assigns to each $s \in \mathbf{C}$ an element $f_\chi(s) \in \mathcal{P}(\chi|\ |^s)$ such that $f_\chi(s)|_K$ is independent of s.

2.3.6 Proposition ([11, Prop. 3.7.1]). *For every* $f \in \mathcal{P}(\chi)$, *there exists an unique flat section* f_χ *such that* $f = f_\chi(0)$. *We say* f *is* embedded in the flat section f_χ.

2.3.7 For the remainder of this section, fix $\chi \in \Xi$, $f \in \mathcal{P}(\chi)$, and $g \in G_{\mathbf{A}}$. Since χ is trivial on $F^\times$, $f \in \mathcal{P}(\chi)$ is left B_F-invariant, and we may define

$$E(g,f) \;:=\; L(\chi^2,1) \cdot \sum_{\gamma \in B_F \backslash G_F} f(\gamma g)\,,$$

provided the sum converges. If f is embedded in the flat section f_χ, then put

$$E(g,f,s) = E(g,f_\chi(s))$$

for those s for which the right hand side is defined. If $\chi \in \Xi_0$ and $\chi^2 \neq |\;|^{\pm 1}$, then $\mathcal{P}(\chi)^K$ is 1-dimensional by Schur's lemma (cf. Lemma 3.1.10) and contains thus a unique *spherical vector,* i.e. an f^0 such that $f^0(k) = 1$ for all $k \in K$. Then define

$$E(g,\chi,s) = E(g,f^0,s)\,.$$

2.3.8 Theorem ([41, Thm. 2.3]). *The function $E(g,f,s)$ converges for every $g \in G_{\mathbf{A}}$ and $\operatorname{Re} s > 1/2 - \operatorname{Re}\chi$, and is analytic as a function of s. It is an automorphic form as a function of g.*

2.3.9 Let N be the unipotent radical of the standard Borel subgroup $B \subset G$. The constant term $E_N(\,\cdot\,,f,s)$ of $E(\,\cdot\,,f,s)$ as defined in paragraph 1.5.9 is

$$E_N(g,f,s) \;=\; L(\chi^2,1+2s)\,\big(f(g) + M_\chi(s)f(g)\big)$$

with

$$M_\chi(s)f(g) \;=\; \int_{N_{\mathbf{A}}} f\big(\big(\begin{smallmatrix} & 1 \\ 1 & b\end{smallmatrix}\big)g\big)\,db\,,$$

cf. [11, pp. 352–353]. The operator $M_\chi(s)$ is an intertwining operator, i.e. an isomorphism of $G_{\mathbf{A}}$-modules

$$M_\chi(s):\; \mathcal{P}(\chi|\;|^s) \;\longrightarrow\; \mathcal{P}(\chi^{-1}|\;|^{-s})\,.$$

2.3.10 Theorem ([41, Thm. 3.2]). (i) *As a function of s, $E(g,f,s)$ has a meromorphic continuation to all $s \in \mathbf{C}$. It has simple poles in those s for which $\chi^2|\;|^{2s} = |\;|^{\pm 1}$.*

(ii) *The function $M_\chi(s) : \mathcal{P}(\chi|\;|^s) \to \mathcal{P}(\chi^{-1}|\;|^{-s})$ extends to all s except for those which satisfy $\chi^2|\;|^{2s} = 1$. If $\chi^2|\;|^{2s} \neq |\;|^{\pm 1}$, then $M_\chi(s)$ is an isomorphism.*

2.3.11 Definition. The meromorphic continuation of $E(\,\cdot\,,f) = E(\,\cdot\,,f,0)$ is called the *Eisenstein series associated to f*. If $\chi \in \Xi_0$, then $E(\,\cdot\,,\chi) = E(\,\cdot\,,\chi,0)$ is called the *Eisenstein series associated to χ*.

2.3.12 Remark. In the literature there is a difference in the normalisation of s. While classical Eisenstein series for the complex upper half plane were originally defined such that the centre of symmetry of the functional equation lies at $s = 1/2$, the literature on automorphic forms on adele groups usually defines Eisenstein series such that the centre of symmetry lies at $s = 0$. We stick to the latter, whence the L-factor $L(\chi^2,1) = L((\chi|\;|^{1/2})^2,0)$ in the definition of the Eisenstein series.

2.3.13 Theorem ([41, Thm. 3.1]). *Let $\chi \in \Xi$, let $f \in \mathcal{P}(\chi)$ be embedded in the flat section $f_\chi(s)$. Define $\hat{f} = M_\chi(0) f \in \mathcal{P}(\chi^{-1})$ embedded into the flat section $\hat{f}_{\chi^{-1}}(s)$. Then there is a function $c(\chi,s)$ that is holomorphic in $s \in \mathbf{C}$ such that*

$$M_\chi(s)\, f \;=\; c(\chi,s)\, \hat{f}_{\chi^{-1}}(-s)$$

for all $\chi \in \Xi$ and $s \in \mathbf{C}$ unless $\chi^2 |\ |^{2s} = 1$.

2.3.14 Theorem ([41, Thm. 5.2]). *For every $f \in \mathcal{P}(\chi)$, the* functional equation

$$E(\,\cdot\,, f, s) \;=\; c(\chi,s)\; E(\,\cdot\,, \hat{f}, -s)$$

holds if $\chi^2 |\ |^{2s} \neq |\ |^{\pm 1}$, where $\hat{f} \in \mathcal{P}(\chi^{-1})$ and $c(\chi,s)$ are as in the previous theorem. If $\chi \in \Xi_0$, then

$$E(g,\chi,s) \;=\; \chi^2(\mathrm{c})\; |\mathrm{c}|^{2s}\; E(g,\chi^{-1},-s)\,.$$

2.3.15 Proposition ([11, Prop. 3.7.3]). *Let $\chi \in \Xi$ such that $\chi^2 \neq |\ |^{\pm 1}$, and $f \in \mathcal{P}(\chi)$. Then $E(\,\cdot\,, f)$ is an automorphic form as a function of the first argument.*

2.3.16 Since the Eisenstein series $E(\,\cdot\,, f)$ is a sum over left translates of $f(\,\cdot\,)$, and this sum does not interfere with the action of $\mathcal{H}$ (which is defined in terms of right translates), the map

$$\begin{array}{ccc} \mathcal{P}(\chi) & \longrightarrow & \mathcal{A} \\ f & \longmapsto & E(\,\cdot\,, f) \end{array}$$

is a morphism of $\mathcal{H}$-modules.

2.3.17 Let $a \in \mathbf{A}^\times$ and $t = \begin{pmatrix} a & \\ & 1 \end{pmatrix}$. Let $\chi^2 \neq |\ |^{\pm 1}$. Since $f \in \mathcal{P}(\chi)$ and $\hat{f} \in \mathcal{P}(\chi^{-1})$, Theorem 2.3.13 implies that for every $g \in G_{\mathbf{A}}$,

$$E_N(tg, f) \;=\; L(\chi^2,1)\,\big(\chi(a)\; |a|^{1/2}\; f(g) \;+\; c(\chi,0)\; \chi(a)^{-1}\; |a|^{1/2}\; \hat{f}(g)\big)\,,$$

which equals $E(tg, f)$ if $\deg a$ is large enough, see Theorem 1.5.10.

In particular, if $\chi \in \Xi_0$, then [41, eq. (3.7)] says that

$$E_N(t,\chi) \;=\; |a|^{1/2}\; \big(L(\chi^2,1)\,\chi(a) + \chi^2(\mathrm{c})\, L(\chi^{-2},1)\,\chi^{-1}(a)\big)\,.$$

If $\chi^2 = 1$, each L-series on the right hand side has poles and thus the equation is not defined. However one can calculate with help of the functional equation and [41, eq. (3.7)] that in this case $E_N(t,\chi) = 2\chi(a)\; |a|^{1/2}$.

2.3.18 Proposition. *Let $\chi \in \Xi$ such that $\chi^2 \notin \{1, |\ |^{\pm 1}\}$ or let $\chi \in \Xi_0$ with $\chi^2 = 1$. If $f \in \mathcal{P}(\chi)$ is nontrivial, then $E(\,\cdot\,, f)$ is nontrivial.*

Proof. Choose a $g \in G_{\mathbf{A}}$ such that $f(g) \neq 0$. First, let $\chi \neq \chi^{-1}$. Let now $a \in \mathbf{A}^\times$ be of degree 1 such that $\chi(a) \neq \pm 1$. For arbitrary $c_1, c_2 \in \mathbf{C}$ that do not vanish both, there are arbitrarily large n such that $c_1\chi(a)^n + c_2\chi(a)^{-n} \neq 0$. Put $t = \begin{pmatrix} a & \\ & 1 \end{pmatrix}$. Then by paragraph 2.3.17, there is a large n such that

$$E(t^n g, f) \;=\; E_N(t^n g, f) \;=\; L(\chi^2,1)\, |a|^{n/2}\, \big(f(g)\chi(a)^n \;+\; \hat{f}(g)\chi(a)^{-n}\big) \;\neq\; 0\,.$$

If $\chi = \chi^{-1}$ is unramified, we may assume that $f = f^0$ since we only have to show that the irreducible $G_{\mathbf{A}}$-module $\mathcal{P}(\chi)$ maps nontrivially to $\mathcal{A}$. Then by paragraph 2.3.17, we have for large n,

$$E(t^n g, \chi) = E_N(t^n g, \chi) = 2\chi(a)^n\, |a|^{n/2} f(g) \neq 0\,. \quad \square$$

2.3.19 Corollary. *Let $\chi \in \Xi$ such that $\chi^2 \notin \{1, |\ |^{\pm 1}\}$ or let $\chi \in \Xi_0$ with $\chi^2 = 1$. If $f \in \mathcal{P}(\chi)$ is nontrivial, then $E(\,\cdot\,, f)$ has non-compact support.*

Proof. This follows from the last proposition. The constant term $E_N(\,\cdot\,, f)$ of an Eisenstein series $E(\,\cdot\,, f)$ has non-compact support, and differs from the Eisenstein series only on a compact set (Theorem 1.5.10). $\square$

2.3.20 Let $\varphi : \mathbf{A}^2 \to \mathbf{C}$ be a Schwartz-Bruhat function, i.e. a locally constant function with compact support. Choose a Haar measure on $Z_{\mathbf{A}}$ and define

$$f_{\varphi,\chi}(s) : g \longmapsto \int_{Z_{\mathbf{A}}} \varphi((0,1)zg)\,\chi(\det zg)\,|\det zg|^{s+1/2}\; dz\,.$$

This is a Tate integral and converges for $\operatorname{Re} s > 1/2 - \operatorname{Re}\chi$ (Theorem 2.2.7). The definition of φ ensures us that $f_{\varphi,\chi}(s)$ is smooth and K-finite, and because

$$f_{\varphi,\chi}(s)\left(\begin{pmatrix} a & b \\ & d \end{pmatrix} g\right) = \chi(ad^{-1})\,\left|ad^{-1}\right|^{s+1/2} f_{\varphi,\chi}(s)(g)\,,$$

we have $f_{\varphi,\chi}(s) \in \mathcal{P}(\chi|\ |^s)$. Define the particular Schwartz-Bruhat function

$$\varphi_0 = h_F\,(q-1)^{-1}\,(\operatorname{vol}\mathcal{O}_{\mathbf{A}}^2)^{-1}\,\operatorname{char}_{\mathcal{O}_{\mathbf{A}}^2}\,.$$

2.3.21 Proposition. *Let* $\operatorname{Re}\chi > 1$.

(i) *For all $f \in \mathcal{P}(\chi)$, there exists a Schwartz-Bruhat function $\varphi : \mathbf{A}^2 \to \mathbf{C}$ such that $f = f_{\varphi,\chi}(0)$.*

(ii) *If $\chi \in \Xi_0$ and $f = f^0$ is the spherical vector, then $f_{\varphi_0,\chi}(0) = L(\chi^2, 2s+1)\, f^0$.*

Proof. In [79, VII.6–VII.7], Weil constructs for every $\chi \in \Xi$ a Bruhat-Schwartz function φ such that $f_{\varphi,\chi}(0)$ is nontrivial. For a proof of (ii) observe that for $g = e$,

$$f_{\varphi_0,\chi}(0)(e) = \int_{Z_{\mathbf{A}}} \varphi_0((0,1)z)\,\chi(\det z)\,|\det z|^{s+1/2}\; dz\,,$$

which is the Tate integral for $L(\psi_0, \chi, s) = L(\chi^2, 2s+1)$, cf. Theorem 2.2.7.

For a proof of (i) observe that $\varphi_g = \varphi(\,\cdot\, g)$ is still a Schwartz-Bruhat function for every $g \in G_{\mathbf{A}}$, and $g.f_{\varphi,\chi}(0) = f_{\varphi_g,\chi}(0)$ is still a function in $\mathcal{P}(\chi)$. By Theorem 2.3.2, $\operatorname{Re}\chi > 1$ implies that $\mathcal{P}(\chi)$ is irreducible, and thus $G_{\mathbf{A}}.f_{\varphi,\chi}(0) = \mathcal{P}(\chi)$. $\square$

2.3.22 Define

$$E(g,\varphi,\chi,s) \;=\; \sum_{\gamma\in B_F\backslash G_F} f_{\varphi,\chi}(s)(\gamma g)$$

for $\mathrm{Re}\,s > 1/2 - \mathrm{Re}\,\chi$. This definition extends to a meromorphic function of $s \in \mathbf{C}$. Put $E(g,\varphi,\chi) = E(g,\varphi,\chi,0)$. The last proposition implies that the class of Eisenstein series of the form $E(\,\cdot\,,\varphi,\chi)$ is the same as the class of Eisenstein series of the form $E(\,\cdot\,,f)$. For a $\chi \in \Xi_0$, we obtain the equality $E(\,\cdot\,,\varphi_0,\chi,s) = E(\,\cdot\,,\chi,s)$.

2.4 Residues of Eisenstein series

Where the Eisenstein series have poles, automorphic forms are hidden as the residues at these poles.

2.4.1 Let $\chi \in \Xi$ with $\chi^2 = |\;|^{\pm 1}$, $f \in \mathcal{P}(\chi)$, and $g \in G_{\mathbf{A}}$. Then $E(g,f,s)$ as a function of s has a pole at $s = 0$, which is order 1. Thus the Eisenstein series has a nontrivial residue

$$R(g,f) \;:=\; \mathrm{Res}_{s=0}\; E(g,f,s) \;=\; \lim_{s\to 0}\; s\cdot E(g,f,s)\;,$$

which is itself an automorphic forms since manipulations of the first argument g commute with the limit and multiplication by s. Moderate growth (paragraph 1.3.3) will be clear from Theorem 2.4.2. Define

$$R(\,\cdot\,,\chi) = \mathrm{Res}_{s=0}\; E(\,\cdot\,,\chi)$$

if χ is unramified. The functional equation has a natural extension to residues of Eisenstein series. In particular, for unramified χ, it becomes

$$R(\,\cdot\,,\chi) \;=\; -\chi^2(\mathrm{c})\, R(\,\cdot\,,\chi^{-1})\;.$$

Let $\varphi : \mathbf{A}^2 \to \mathbf{C}$ be a Schwartz-Bruhat function. Then one can also define

$$R(\,\cdot\,,\varphi,\chi) = \mathrm{Res}_{s=0}\; E(\,\cdot\,,\varphi,\chi)\;.$$

From the result for Eisenstein series, one obtains that for every φ, there is a $f \in \mathcal{P}(\chi)$ such that $R(\,\cdot\,,\varphi,\chi) = R(\,\cdot\,,f)$, and vice versa.

2.4.2 Theorem ([24, Thm. 4.19]). *Let $\chi = \omega|\;|^{\pm 1/2}$ be a quasi-character with $\omega^2 = 1$ and $f \in \mathcal{P}(\chi)$, then $R(\,\cdot\,,f) \;=\; R(e,f)(\omega\circ\det)$ as functions on $G_{\mathbf{A}}$.*

2.4.3 Corollary. *Let $\chi = \omega|\;|^{\pm 1/2}$ be a quasi-character with $\omega^2 = 1$. Then the $\mathcal{H}$-submodule $\{R(\,\cdot\,,f)\}_{f\in\mathcal{P}(\chi)} \subset \mathcal{A}$ is 1-dimensional.* □

2.5 Derivatives

The space of automorphic forms contains more interesting functions, namely, derivatives of Eisenstein series. Similarly, there are also functions that play the role of derivatives of residues and which we simply call derivatives of residues by abuse of terminology.

2.5.1 For $i \geq 0$, Schwartz-Bruhat functions $\psi : \mathbf{A} \to \mathbf{C}$ and $\varphi : \mathbf{A}^2 \to \mathbf{C}$ and $\chi \in \Xi$ define the *derivative of an L-series* and the *derivative of an Eisenstein series* as

$$\begin{aligned} L^{(i)}(\psi,\chi,s) &= \frac{d^i}{ds^i}\, L(\psi,\chi,s) \quad \text{and} \\ E^{(i)}(g,\varphi,\chi,s) &= \frac{d^i}{ds^i}\, E(g,\varphi,\chi,s) \end{aligned}$$

in the sense of derivatives of meromorphic functions of s. Define the *derivative of the residue of an Eisenstein series* as

$$R^{(i)}(g,\varphi,\chi) = \lim_{s\to 0} \frac{d^i}{ds^i}\, s\cdot E(g,\varphi,\chi,s)$$

if $\chi^2 = |\;|^{\pm 1}$.

2.5.2 Lemma. *For* $\operatorname{Re} s > 1 - \operatorname{Re}\chi$,

$$L^{(i)}(\psi,\chi,s) = \int_{\mathbf{A}^\times} \psi(a)\chi(a)\big(\ln|a|\big)^i\, |a|^s\, da\,.$$

Proof. Since $|a|^s = e^{\ln|a|^s} = e^{s\ln|a|}$, we have

$$\frac{d^i}{ds^i}\,|a|^s = \big(\ln|a|\big)^i\,|a|^s\,.$$

We have to show that $\frac{d^i}{ds^i}$ commutes with the integral. Since this is a local question, we may restrict to a compact neighbourhood of s.

We apply standard results from analysis in two steps. First observe that

$$\mathbf{A}^\times = \bigcup_{\substack{S \text{ finite set} \\ \text{of divisors}}} U(S) \qquad \text{with} \qquad U(S) = \bigcup_{D=(D_x)\in S} (\pi_x^{D_x})\mathcal{O}_{\mathbf{A}}^\times\,.$$

All subsets $U(S)$ are compact, thus

$$\int_{U(S)} \frac{d^i}{ds^i}\psi(a)\chi(a)\,|a|^s\, da = \frac{d^i}{ds^i}\int_{U(S)} \psi(a)\chi(a)\,|a|^s\, da\,.$$

(This standard result can be found, for example, in [37, Thm. XIII.8.1]. Note that replacing the compact interval $[a,b]$ in loc. cit. by the compact measure space $U(S)$ does not change the proof.)

Secondly, we choose a sequence $\{S_n\}_{n\geq 0}$ of finite sets of divisors such that $S_n \subset S_{n+1}$ for all $n \geq 0$ and $\mathbf{A}^\times = \bigcup_{n\geq 0} U(S_n)$. Write

$$f_n(s) \;=\; \int\limits_{U(S_n)} \psi(a)\chi(a)\,|a|^s\,da$$

for short. Then $\{f_n\}$ converges to $L(\psi,\chi,s)$. Since we restricted to a compact domain for s, the sequence

$$\left\{\frac{d^i}{ds^i}\,f_n\right\} \quad \underset{n\to\infty}{\longrightarrow} \quad \int\limits_{\mathbf{A}^\times} \frac{d^i}{ds^i}\psi(a)\chi(a)\,|a|^s\,da$$

converges uniformly. From a standard result about exchanging limits and derivatives ([37, Thm. XIII.9.1]) the lemma follows. □

2.5.3 Lemma. *For* $\operatorname{Re} s > 1/2 - \operatorname{Re}\chi$*,*

$$E^{(i)}(g,\varphi,\chi,s) \;=\; \sum_{B_F\backslash G_F}\int\limits_{Z_{\mathbf{A}}} \varphi((0,1)zg)\,\chi(\det zg)\big(\ln|\det zg|\big)^i\,|\det zg|^{s+1/2}\,dz\;.$$

Proof. The proof is completely analogous to the one of the previous lemma. □

2.5.4 Lemma. *For* $\operatorname{Re} s > 1/2 - \operatorname{Re}\chi$*,*

$$E^{(i)}(g,\varphi,\chi,s) \;=\; \int\limits_{Z_F\backslash Z_{\mathbf{A}}}\;\sum_{u\in F^2-\{0\}} \varphi(uzg)\,\chi(\det zg)\big(\ln|\det zg|\big)^i\,|\det zg|^{s+1/2}\,dz\;.$$

Proof. Let G_F act on $\mathbf{P}^1(F)$ by multiplication from the right. Then B_F is the stabiliser of $[0:1]$, and thus we have a bijection

$$\begin{array}{ccc} B_F\backslash G_F & \overset{1:1}{\longrightarrow} & \mathbf{P}^1(F) \;=\; Z_F\backslash(F^2-\{0\})\;. \\ g & \longmapsto & [0:1]g \end{array}$$

Since $\sum_{\gamma\in B_F\backslash G_F} f(\gamma g)$ is absolutely convergent for every $f \in \mathcal{P}(\chi|\;|^s)$ and $g \in G_{\mathbf{A}}$, ([41, Thm. 2.3]), the lemma follows by Fubini's theorem from Lemma 2.5.3. □

2.5.5 Lemma. *For* $\chi = \omega|\;|^{\pm 1/2}$ *with* $\omega^2 = 1$ *and* $i \geq 1$*,*

$$R^{(i)}(g,\varphi,\chi) = \lim_{s\to 0}\big(i\cdot E^{(i-1)}(g,\varphi,\chi,s) + s\cdot E^{(i)}(g,\varphi,\chi,s)\big)\,. \quad \square$$

2.5.6 Lemma. *Let* $\chi \in \Xi_0$ *satisfy* $\chi^2 = 1$*. If* $L(\chi,1/2) = 0$*, then* $1/2$ *is a zero of even multiplicity.*

Proof. Since the divisor class represented by $\mathfrak{c}$ is a square in the divisor class group, cf. [79, XIII.12, thm. 13], $\chi(\mathfrak{c}) = 1$. Let $L^{(i)}(\chi,\cdot)$ vanish at $1/2$ for all $i = 0,\ldots,n-1$, for some odd n. We will show that in this case the multiplicity of $1/2$ as a zero must be strictly larger than n. Taking into account the vanishing of lower derivatives and $\chi(\mathfrak{c}) = 1$, the n-th derivatives of both sides of the functional equation are

$$L^{(n)}(\chi,1/2) = (-1)^n\,L^{(n)}(\chi^{-1},1/2)\,.$$

Thus $L^{(n)}(\chi,s) = 0$ as $(-1)^n = -1$. □

CHAPTER 3

Admissible automorphic forms

Admissible representations of $G_{\mathbf{A}}$ are one of the most important objects in the theory of automorphic forms. This class of representations is large enough to contain interesting representations, but it is still small enough to guarantee that every admissible representation decomposes as an algebraic sum into well-known components. This chapter describes all possible unramified admissible subrepresentations of the space of automorphic forms along with the action of the unramified part of the Hecke algebra on these subrepresentations. The last section characterises simultaneous eigenfunctions of all unramified Hecke operators by their eigenvalues.

3.1 Admissible representations

3.1.1 Let V be a subset of the space $\mathcal{A}$ of automorphic forms. We use the neighbourhood basis $\mathcal{V}$ of e in $G_{\mathbf{A}}$ as introduced in paragraph 1.3.1 and we use the convention of paragraph 1.4.8 in that we call a subspace $V \subset \mathcal{A}$ invariant if it is invariant under the action of $\mathcal{H}$.

3.1.2 Definition. An invariant subspace V is called an *admissible representation* if the complex vector space $\mathcal{H}_{K'}(V) = V^{K'}$ is finite-dimensional for all $K' \in \mathcal{V}$. An automorphic form $f \in \mathcal{A}$ is called *admissible* if $\mathcal{H}(f)$ is admissible, or equivalently, if for all $K' \in \mathcal{V}$, $\mathcal{H}_{K'}(f)$ is finite-dimensional. The *admissible part of* V is the subrepresentation

$$V_{\text{adm}} = \{ f \in V \mid f \text{ is admissible} \} .$$

The *unramified part of* V is the subrepresentation

$$V^{\text{nr}} = \mathcal{H}(V^K) .$$

If $V^{\text{nr}} = V$, the representation V is called *unramified*.

3.1.3 Note that subrepresentations and finite sums of admissible representations are admissible. Thus every element of an admissible representation is admissible and

$$V_{\text{adm}} = \bigcup_{\substack{\text{admissible} \\ \text{representations } W \subset V}} W .$$

Be aware that arbitrary unions of admissible representations are in general not admissible as $\mathcal{A}_{\mathrm{adm}}$ is not.

Further we have an alternative description of the unramified part of V,

$$V^{\mathrm{nr}} = \bigcap_{\substack{\text{subrepresentations} \\ W \subset V \text{ with } W^K = V^K}} W ,$$

which follows from the fact that $\mathcal{H}(V^K)$ is contained in an invariant subspace $W \subset \mathcal{A}$ if W contains V^K.

We make the following convention: If an invariant subspace $V \subset \mathcal{A}$ is decorated by more then one label, it is the intersection of the spaces with single label. For example, $V^{\mathrm{nr}}_{\mathrm{adm}} = V_{\mathrm{adm}} \cap V^{\mathrm{nr}}$ and $\mathcal{A}^K_{\mathbf{0},\mathrm{tor}} = \mathcal{A}_{\mathbf{0}} \cap \mathcal{A}_{\mathrm{tor}} \cap \mathcal{A}^K$.

3.1.4 Remark. There is a satisfactory theory of admissible representations. We add to it the new terminology "admissible automorphic form" and "admissible part" for the following reasons.

Although $\mathcal{A}_{\mathrm{adm}}$ itself is not an admissible representation, it is a natural subspace of $\mathcal{A}$ that has a decomposition as a direct sum of subrepresentations, for which we can give explicit bases and the matrix form of Hecke operators relative to these bases. We will work out a decomposition for $\mathcal{A}^{\mathrm{nr}}_{\mathrm{adm}}$ from results in the literature.

We will show that the space of unramified toroidal automorphic forms is contained in the admissible part and it inherits the decomposition. This allows us to investigate it part by part. One may ask: is the space of all toroidal automorphic forms contained in the admissible part?

From the theory in Chapter 6, it will follow that a positive answer would imply that the space of toroidal automorphic forms is admissible. However, this implication does not hold in the corresponding theory for number fields. There, the space of toroidal automorphic forms is far from admissible. But it is still interesting to put the question: is the space of toroidal automorphic forms for a number field contained in the admissible part?

3.1.5 One of the crucial observations in the representation theory of $G_{\mathbf{A}}$ is that every irreducible admissible representation factors into a restricted tensor product of local representations. To this end, we recall what the restricted tensor of representations is, where—to keep it simple—we restrict the discussion to unramified representations.

Define $G_x = G(F_x)$ and $K_x = G(\mathcal{O}_x)$. Choose for every $x \in |X|$ a G_x-representation V_x and a non-zero vector $v^0_x \in V_x^{K_x}$. In particular, $V_x^{K_x}$ is nontrivial. For finite sets $S \subset |X|$, define V_S as linear combinations of expressions of the form (v_x), where $v_x \in V_x$ for $x \in S$ and $v_x = v^0_x$ for $x \in |X| - S$, and which satisfy the relations for finite tensor products. Then the restricted tensor product of all V_x relative to (v^0_x) is defined as

$$\bigotimes_{x \in |X|}{}' V_x = \bigcup_{S \subset |X| \text{ finite}} V_S .$$

Note that $G_{\mathbf{A}}$ acts on $\bigotimes' V_x$ by $(g_x).(v_x) := (g_x v_x)$ and that the isomorphism type of $\bigotimes' V_x$ as a $G_{\mathbf{A}}$-representation does not depend on the choice of (v^0_x) since for a different choice (w^0_x), we obtain an isomorphism of $G_{\mathbf{A}}$-representations by sending (v_x) to $(v_x -$

$v_x^0 + w_x^0$). We will not mention the vector (v_x^0) anymore if we are only interested in the isomorphism type of $\bigotimes' V_x$.

An important question is: when is the restricted tensor product $\bigotimes' V_x$ an automorphic representation, i.e. when is $\bigotimes' V_x$ isomorphic to a subquotient of $\mathcal{A}$? We will recall the answer for one series of examples, namely, the principal series representations.

Let χ_x be a quasi-character of $F_x^\times$, i.e. a continuous group homomorphism $F_x^\times \to \mathbf{C}^\times$. Define the *principal series representation* $\mathcal{P}_x(\chi_x)$ of G_x as the space of all locally constant functions $f : G_x \to \mathbf{C}$ such that for all $\left(\begin{smallmatrix} a & b \\ & d \end{smallmatrix}\right) \in G_x$ and all $g \in G_x$,

$$f\left(\begin{pmatrix} a & b \\ & d \end{pmatrix} g\right) = \left|ad^{-1}\right|^{1/2} \chi_x(ad^{-1})\, f(g)\,,$$

together with the representation of G_x by right translation of the argument.

If now $\chi \in \Xi_0$ is an unramified quasi-character of $\mathbf{A}^\times$, then the restriction of χ to $F_x^\times \subset \mathbf{A}^\times$ defines unramified quasi-characters $\chi_x : F_x^\times \to \mathbf{C}^\times$, i.e. χ_x is trivial on $\mathcal{O}_x^\times$, for every $x \in |X|$. In this case, $\mathcal{P}_x(\chi_x)$ contains a unique right K_x-invariant vector f_x^0 with $f_x^0(e) = 1$, and we can form the restricted tensor product $\bigotimes' \mathcal{P}_x(\chi_x)$ over all $x \in |X|$ with respect to (f_x^0). Since χ is trivial on $F^\times$, the restricted tensor product $\bigotimes' \mathcal{P}_x(\chi_x)$ is left G_F-invariant, and furthermore the map

$$\begin{array}{ccc} \bigotimes' \mathcal{P}_x(\chi_x) & \longrightarrow & \mathcal{P}(\chi) \\ (f_x) & \longmapsto & \big((g_x) \mapsto \prod_{x \in |X|} f_x(g_x)\big) \end{array}$$

is an isomorphism of $G_{\mathbf{A}}$-modules.

An G_x-representation V_x is called *unramified* if $V_x = \{g.v \mid g \in G_x, v \in V_x^{K_x}\}$. In particular, $\mathcal{P}_x(\chi_x)$ is unramified if χ_x is unramified.

3.1.6 If V_x is a G_x-representation for a place x, then the Hecke algebra $\mathcal{H}$ acts on V_x by

$$\Phi(v) = \int_{G_x} \Phi(j(h))\, h.v\, dh\,,$$

where $\Phi \in \mathcal{H}$, $v \in V_x$ and $j : G_x \to G_{\mathbf{A}}$ is the canonical inclusion.

3.1.7 Theorem ([11, Thm. 3.3.3]). *Let $V \subset \mathcal{A}_{\mathrm{adm}}^{\mathrm{nr}}$ be an invariant subspace that is irreducible. Then there exist irreducible unramified G_x-representations V_x with* $\dim V_x^{K_x} = 1$ *for all $x \in |X|$ such that $V \simeq \bigotimes' V_x$.*

3.1.8 Theorem ([11, Thm. 4.6.4]). *Let V_x be an irreducible G_x-representation such that $V_x^{K_x}$ is finite-dimensional, but not trivial. Then there is an unramified quasi-character $\chi_x : F_x^\times \to \mathbf{C}^\times$ so that either $V_x \simeq \mathcal{P}_x(\chi_x)$ or V_x is* 1*-dimensional and $g.v = \chi_x(\det g) \cdot v$ for all $g \in G_x$ and $v \in V_x$.*

3.1.9 Theorem ([11, Thm. 3.4.3]). *Let V_1 and V_2 be irreducible subrepresentations of $\mathcal{A}$. Then $V_1^{\mathrm{nr}} \simeq V_2^{\mathrm{nr}}$ as $\mathcal{H}$-modules if and only if $V_1^K \simeq V_2^K$ as $\mathcal{H}_K$-modules.*

3.1.10 Lemma. *For every irreducible representation V of $\mathcal{H}$, the space V^K is either* 0*-dimensional,* 1*-dimensional, or infinite-dimensional.*

Proof. Theorem 3.1.9 implies that V^K is zero or irreducible as $\mathcal{H}_K$-module. Since $\mathcal{H}_K$ is commutative, we can apply Schur's lemma ([21, 1.7]), which says that every irreducible finite-dimensional $\mathcal{H}_K$-module is 1-dimensional. Hence the proposition. □

3.1.11 Lemma 3.1.10 implies that the K-invariant subspace of an irreducible unramified admissible representation is 1-dimensional, and thus the basis vector is an eigenvector for all $\Phi \in \mathcal{H}_K$. We assume eigenvectors to be nontrivial. We call a $f \in \mathcal{A}$ an $\mathcal{H}_K$-*eigenfunction with eigencharacter* λ_f if it is an eigenvector for every $\Phi \in \mathcal{H}_K$ with eigenvalue $\lambda_f(\Phi)$. Note that $\lambda_f : \mathcal{H}_K \to \mathbf{C}$ is a homomorphism of $\mathbf{C}$-algebras and thus indeed an additive character.

3.2 $\mathcal{H}_K$-eigenfunctions

The article [41] of Li describes a decomposition of the space of the $\mathcal{H}_K$-eigenfunctions in $\mathcal{A}$. We state this result and a lemma that is the key for a generalisation of the theorem to a decomposition of the admissible part. Recall the definition of Φ_x for $x \in |X|$ as defined in paragraph 1.4.2, and recall that we write $q_x = q^{\deg x}$.

3.2.1 Let the *Eisenstein part* $\mathcal{E}$ be the vector space spanned by all Eisenstein series and their derivatives, the *residual part* $\mathcal{R}$ be the vector space spanned by the residues of Eisenstein series and their derivatives in the sense of paragraph 2.5.1, and the *cuspidal part* $\mathcal{A}_0$ be the space of cusp forms as defined in paragraph 1.5.9. We shall refer to $\widetilde{\mathcal{E}} := \mathcal{E} \oplus \mathcal{R}$ as the *completed Eisenstein part*. It follows from Theorem 3.2.2 that the sum is direct.

For $\lambda \in \mathbf{C}$, and $\Phi \in \mathcal{H}_K$, define the *space of* Φ*-eigenfunctions with eigenvalue* λ:

$$\mathcal{A}(\Phi,\lambda) \;=\; \{f \in \mathcal{A} \mid \Phi(f) = \lambda f\}\,,$$

and for an invariant subspace $V \subset \mathcal{A}$, define $V(\Phi,\lambda) = V \cap \mathcal{A}(\Phi,\lambda)$.

Recall the definitions of the standard Borel subgroup B and its unipotent radical N from paragraph 1.5.7, and define $r_{K'} = \#\big(K \,/\, K'(Z_{\mathbf{A}} N_{\mathbf{A}} B_{\mathbf{F}_q} \cap K)\big)$ for $K' \in \mathcal{V}$.

3.2.2 Theorem (Li). *Let $\lambda \in \mathbf{C}$, x a place of degree d_x and $K' \in \mathcal{V}$. Then*

$$\mathcal{A}(\Phi_x,\lambda)^{K'} \;=\; \mathcal{E}(\Phi_x,\lambda)^{K'} \oplus \mathcal{R}(\Phi_x,\lambda)^{K'} \oplus \mathcal{A}_0(\Phi_x,\lambda)^{K'}$$

and

$$\dim\, \widetilde{\mathcal{E}}(\Phi_x,\lambda)^{K'} \;=\; h_F \cdot \deg x \cdot r_{K'}\,.$$

Proof. This is [41, Thm. 7.1], but one should note that Li uses different conventions to those in this thesis. First of all, Li writes from right to left from our point of view, i.e. G_F operates from the right while K and the Hecke algebra operate from the left. Thus, elements of $G_{\mathbf{A}}$ need to be inverted.

Secondly, in [41], the family of Eisenstein series is multiplied with a certain polynomial such that the poles get resolved and the residues lie within the new family. As a result, no distinction between Eisenstein series and residues occurs. If one only considers the unramified part as $\mathcal{H}_K$-module, there indeed is no difference between subrepresentations in the Eisenstein part and in the residual part, and they can be described as a continuous family as will be seen in the following sections. □

3.2.3 Theorem.

(i) *Let $\chi \in \Xi_0$ and $\chi^2 \neq |\,|^{\pm 1}$. The Eisenstein series $E(\,\cdot\,,\chi)$ generates an admissible representation of $\mathcal{H}$.*

(ii) *Let $\chi \in \Xi_0$ and $\chi^2 = |\,|^{\pm 1}$. The residue $R(\,\cdot\,,\chi)$ generates an admissible representation of $\mathcal{H}$.*

(iii) *Let $f \in \mathcal{A}_0^K$ be an $\mathcal{H}_K$-eigenfunction. The cusp form f generates an admissible representation of $\mathcal{H}$.*

Proof. For (i), the $\mathcal{H}$-module generated by $E(\,\cdot\,,\chi)$ is isomorphic to $\mathcal{P}(\chi)$. By Proposition 2.3.4 every $f \in \mathcal{P}(\chi)$ is determined by its values at elements of K. If $K' \in \mathcal{V}$, then the index of K' in K is finite and thus $\mathcal{P}(\chi)^{K'}$ is finite dimensional.

Statement (ii) follows from Theorem 2.4.2.

Statement (iii) is [32, Prop. 10.5]. □

3.2.4 Corollary. *If $f \in \mathcal{A}$ is unramified and $\mathcal{H}(f)$ is irreducible, then f is admissible if and only if f is an $\mathcal{H}_K$-eigenfunction.* □

One may ask what happens if the condition that $\mathcal{H}(f)$ is irreducible is dropped. Theorem 3.6.2 below will give a complete description of unramified admissible automorphic forms.

3.2.5 Lemma. *Let V be a finite dimensional complex vector space and $\Phi : V \to V$ a linear map such that there exists no nontrivial decomposition of V into Φ-invariant subspaces. Then there is precisely one Φ-invariant subspace of V of every given dimension smaller then* $\dim V$.

Proof. This is a consequence of the Jordan decomposition ([21, §9.3]): Φ is the sum of a diagonalisable linear map Φ_{ss} and a nilpotent linear map Φ_{nil}, which commute with each other, and this decomposition is unique. Because Φ_{ss} and Φ_{nil} commute and the images of a non-zero vector under these two operators are linearly independent, a subspace of V is Φ-invariant if and only if it is Φ_{ss}- and Φ_{nil}-invariant, but since Φ_{ss} is diagonalisable, the Φ_{ss}-invariance follows from the Φ_{nil}-invariance.

The sequence $\Phi_{nil}^k(V)$ for $k \geq 0$ is a filtration of V whose subquotients have shrinking dimension, and every sequence of Φ_{nil}-invariant (or Φ-invariant) subspaces of V must be a subsequence. Since V has no nontrivial decomposition into Φ-invariant subspaces, all subquotients of this filtration are at most 1-dimensional, and thus there is a unique sequence of Φ-invariant subspaces whose dimensions increase by 1. □

3.2.6 This lemma together with Theorem 3.2.2 implies that for every $x \in |X|$, the Hecke operator Φ_x decomposes the admissible part into a direct sum of subspaces that are the (possibly infinite-dimensional) generalised eigenspaces of the Φ_x-eigenfunctions. In the subsequent sections we shall investigate these generalised eigenspaces for the Eisenstein, residual and cuspidal part, respectively, which will turn out to be independent of the choice of x.

3.3 The Eisenstein part

We give explicit formulas for the action of $\mathcal{H}_K$ on Eisenstein series and their derivatives. They determine the $\mathcal{H}_K$-module structure of the spaces that are generated by these functions.

3.3.1 In paragraph 2.3.7, we already saw that for $\chi \in \Xi_0$ with $\chi^2 \neq |\ |^{\pm 1}$, there is a distinguished Eisenstein series $E(\,\cdot\,,\chi,s) = E(\,\cdot\,,f^0,s)$, where $f^0 \in \mathcal{P}(\chi)$ is the spherical vector. Up to a constant multiple, these are the only unramified Eisenstein series. We denote their derivatives in the sense of paragraph 2.5.1 by $E^{(i)}(\,\cdot\,,\chi,s)$.

Define for all $\chi \in \Xi_0$, $x \in |X|$ and $l \geq 0$ the value

$$\lambda_x^{(l)}(\chi) \;:=\; q_x^{1/2}\big(\chi^{-1}(\pi_x) + (-1)^l \chi(\pi_x)\big)\,.$$

Note that the value of $\lambda_x^{(l)}(\chi)$ only depends on the parity of l. Define $\lambda_x(\chi) = \lambda_x^{(l)}(\chi)$ if l is even and $\lambda_x^{-}(\chi) = \lambda_x^{(l)}(\chi)$ if l is odd.

3.3.2 Lemma. *If $\chi \in \Xi_0$ with $\chi^2 \neq |\ |^{\pm 1}$, then for every $x \in |X|$,*

$$\Phi_x E(g,\chi) \;=\; \lambda_x(\chi)\, E(g,\chi)\,.$$

Proof. Since $\mathcal{P}(\chi)$ is irreducible for $\chi^2 \neq |\ |^{\pm 1}$, the K-invariants form a one-dimensional subspace, cf. Proposition 3.1.10. Hence the spherical vector $f^0 \in \mathcal{P}(\chi)$ is a Φ_x-eigenfunction for every x. The action of Φ_x on unramified automorphic forms is described in [23, §3 Lemma 3.7] or Proposition 4.2.4. With this, we derive

$$\begin{aligned}
\Phi_x f^0(e) \;&=\; f^0\big(\big(\begin{smallmatrix}1 & \\ & \pi_x\end{smallmatrix}\big)\big) + \sum_{b\in\kappa_x} f^0\big(\big(\begin{smallmatrix}\pi_x & b \\ & 1\end{smallmatrix}\big)\big) \\
&=\; q_x^{1/2}\chi^{-1}(\pi_x)\, f^0(e) + q_x \cdot q_x^{-1/2}\chi(\pi_x)\, f^0(e) \\
&=\; q_x^{1/2}\big(\chi(\pi_x) + \chi^{-1}(\pi_x)\big)\, f^0(e) \\
&=\; \lambda_x(\chi)\, f^0(e)
\end{aligned}$$

where $e = \big(\begin{smallmatrix}1 & \\ & 1\end{smallmatrix}\big)$ is the identity matrix. Since the Eisenstein series is a map of $\mathcal{H}$-modules, $E(\,\cdot\,,\chi) = E(\,\cdot\,,f^0)$, has the same eigenvalue as f^0. □

3.3.3 Proposition. *If $\chi \in \Xi_0$ with $\chi^2 \neq |\ |^{\pm 1}$, then for every $x \in |X|$,*

$$\Phi_x E^{(i)}(g,\chi) \;=\; \sum_{k=0}^{i} \binom{i}{k} \big(\ln q_x\big)^{i-k}\, \lambda_x^{(i-k)}(\chi)\, E^{(k)}(g,\chi)\,.$$

Proof. Observe that

$$\frac{d}{ds}\lambda_x^{(l)}(\chi|\ |^s) \;=\; (\ln q_x)\,\lambda_x^{(l+1)}(\chi|\ |^s)\,.$$

The formula is obtained by taking derivatives on both sides of the equation in Lemma 3.3.2 and applying the Leibniz rule to the right hand side. □

3.3.4 Lemma. *Let $\chi \in \Xi_0$. Then $\chi^2 = 1$ if and only if $\lambda_x^-(\chi)$ vanishes for all places x.*

Proof. Observe that for every π_x, we have

$$q_x^{-1/2}\lambda_x^-(\chi) = \chi^{-1}(\pi_x) - \chi(\pi_x) = 0 \iff \chi(\pi_x) = \chi^{-1}(\pi_x) \iff \chi^2(\pi_x) = 1 .$$

Since the π_x's generate $F^\times \backslash \mathbf{A}^\times / \mathcal{O}_\mathbf{A}^\times$, the quasi-character χ is determined by its values on the π_x's. □

3.3.5 Proposition. *Let $\chi \in \Xi_0$ with $\chi^2 \notin \{1, |\,|^{\pm 1}\}$. Then*

$$\big\{E(\,\cdot\,,\chi),\ E^{(1)}(\,\cdot\,,\chi),\ E^{(2)}(\,\cdot\,,\chi), \dots\big\}$$

is linearly independent and spans a vector space on which $\mathcal{H}_K$ acts. In particular none of these functions vanishes.

Proof. By Proposition 3.3.3, it is clear that the span of the functions is an $\mathcal{H}_K$-module. We do induction on $n = \#\big\{E(\,\cdot\,,\chi), E^{(1)}(\,\cdot\,,\chi), \dots, E^{(n-1)}(\,\cdot\,,\chi)\big\}$.

The case $n = 1$ is established in Proposition 2.3.18.

For $n > 1$, assume that there exists a relation

$$E^{(n)}(\,\cdot\,,\chi) \;=\; c_{n-1} E^{(n-1)}(\,\cdot\,,\chi) + \ldots + c_0 E(\,\cdot\,,\chi) .$$

We derive a contradiction as follows. For every place x, we have on the one hand,

$$\begin{aligned}\Phi_x\, E^{(n)}(\,\cdot\,,\chi) \;&=\; c_{n-1}\,\Phi_x\, E^{(n-1)}(\,\cdot\,,\chi) + \ldots + c_0\,\Phi_x\, E(\,\cdot\,,\chi) \\ &\underset{3.3.3}{=} c_{n-1}\,\lambda_x(\chi)\, E^{(n-1)}(\,\cdot\,,\chi) \;+\; \big(\text{terms in lower derivatives of } E(\,\cdot\,,\chi)\big) ,\end{aligned}$$

and on the other hand,

$$\begin{aligned}\Phi_x\, E^{(n)}(\,\cdot\,,\chi) &\underset{3.3.3}{=} \lambda_x(\chi)\, E^{(n)}(\,\cdot\,,\chi) + n\,(\ln q_x)\,\lambda_x^-(\chi)\, E^{(n-1)}(\,\cdot\,,\chi) \;+\; \big(\text{lower terms}\big) \\ &=\; \big(c_{n-1}\,\lambda_x(\chi) + n\,(\ln q_x)\,\lambda_x^-(\chi)\big)\, E^{(n-1)}(\,\cdot\,,\chi) \;+\; \big(\text{lower terms}\big) .\end{aligned}$$

By the induction hypothesis, $\big\{E(\,\cdot\,,\chi), E^{(1)}(\,\cdot\,,\chi), \dots, E^{(n-1)}(\,\cdot\,,\chi)\big\}$ is linearly independent, and therefore

$$c_{n-1}\,\lambda_x(\chi) \;=\; c_{n-1}\,\lambda_x(\chi) + n\,(\ln q_x)\,\lambda_x^-(\chi) ,$$

which implies that $\lambda_x^-(\chi) = 0$ for every place x. But this contradicts Lemma 3.3.4. □

3.3.6 Corollary. *Let $\chi \in \Xi_0$ with $\chi^2 \neq |\,|^{\pm 1}$. Then the following are equivalent.*

(i) $\chi^2 = 1$.

(ii) *$\lambda_x^-(\chi)$ vanishes for all places x.*

(iii) *$E^{(1)}(\,\cdot\,,\chi)$ is an $\mathcal{H}_K$-eigenfunction or trivial.*

Proof. The equivalence of (i) and (ii) is Lemma 3.3.4. For the equivalence of (ii) and (iii), note that since the elements Φ_x for $x \in |X|$ generate $\mathcal{H}_K$, the function $E^{(1)}(\,\cdot\,,\chi)$ is an $\mathcal{H}_K$-eigenfunction if and only if it is an eigenfunction of Φ_x for all $x \in |X|$. But by Proposition 3.3.3, this only happens if $\lambda_x^-(\chi)$ vanishes for all $x \in |X|$. □

3.3.7 Lemma. *Let $\chi \in \Xi_0$ such that $\chi^2 = 1$. Then*

$$E^{(1)}(\,\cdot\,,\chi) \;=\; (\ln q)\;(2g_F - 2)\; E(\,\cdot\,,\chi)\,.$$

Proof. Since $\chi^2 = 1$, the functional equation looks like

$$E(g,\chi,s) \;=\; |\mathfrak{c}|^{2s}\;\; E(g,\chi,-s)\,.$$

Using $|\mathfrak{c}| = q^{-(2g_F-2)}$ and taking derivatives in s of both sides yields

$$E^{(1)}(g,\chi,s) \;=\; -\;|\mathfrak{c}|^{2s}\;\; E^{(1)}(g,\chi,-s) \;+\; 2\;(\ln q)\;(2g_F-2)\;|\mathfrak{c}|^{2s}\;\; E(g,\chi,-s)\,,$$

and filling in $s = 0$ results in the desired equation. □

3.3.8 Proposition. *Let $\chi \in \Xi_0$ with $\chi^2 = 1$. Both*

$$\big\{E(\,\cdot\,,\chi),\; E^{(2)}(\,\cdot\,,\chi),\; E^{(4)}(\,\cdot\,,\chi),\ldots\big\} \quad \textit{and} \quad \big\{E^{(1)}(\,\cdot\,,\chi),\; E^{(3)}(\,\cdot\,,\chi),\; E^{(5)}(\,\cdot\,,\chi),\ldots\big\}$$

span a vector space on which $\mathcal{H}_K$ acts. If $g_F \neq 1$, then both are linearly independent, but they span the same space. If $g_F = 1$, then the former set is linearly independent and all functions in the latter set vanish.

Proof. That both sets span $\mathcal{H}_K$-modules follows from Proposition 3.3.3 since by Corollary 3.3.6, for all $x \in |X|$, the value $\lambda_x^-(\chi)$ vanishes.

The linear independence of the former set can be shown by the same calculation as in the proof of Proposition 3.3.5, provided one knows that $\lambda_x(\chi) \neq 0$ for some $x \in |X|$. This holds since otherwise

$$0 \;=\; \lambda_x(\chi) - \lambda_x^-(\chi) \;=\; 2q_x\,\chi(\pi_x)$$

for all $x \in |X|$, which contradicts the nature of χ.

If $g_F \neq 1$, then Lemma 3.3.7 implies that $E^{(1)}(\,\cdot\,,\chi)$ is a non-vanishing multiple of $E(\,\cdot\,,\chi)$ and spans thus the same vector space as $E(\,\cdot\,,\chi)$. Consequently the latter set in the Proposition is linearly independent for the same reasons as for the former set. By Lemma 3.2.5, the two sets in question generate the same space.

If $g_F = 1$, the vanishing of all $E^{(i)}(\,\cdot\,,\chi)$ for odd i follows from the i-th derivative of the functional equation at $s = 0$, which looks like

$$E^{(i)}(\,\cdot\,,\chi) \;=\; (-1)^i\,E^{(i)}(\,\cdot\,,\chi) \;+\; \underbrace{(2g_F-2)}_{=\,0}\;\text{(terms in lower derivatives)}\,. \qquad \square$$

3.4 The residual part

The $\mathcal{H}_K$-module structure of the spaces generated by residues of Eisenstein series and their derivatives behaves completely analogous to the case of Eisenstein series. We extend the results of previous section to those quasi-characters at which the Eisenstein series have their poles.

3.4.1 Recall from paragraph 2.4.1 that for $\chi = \omega|\ |^{\pm 1/2} \in \Xi_0$ with $\omega^2 = 1$, there is the residue of an Eisenstein series

$$R(\,\cdot\,,\chi) \;=\; \lim_{s\to 0}\; s\cdot E(\,\cdot\,,\chi,s)\,.$$

It is a non-vanishing function. More precisely, by Theorem 2.4.2, it is a multiple of $\omega\circ\det$. We denote the derivatives in the sense of paragraph 2.5.1 by $R^{(i)}(\,\cdot\,,\chi)$.

Recall the definition of $\lambda_x(\chi)$, $\lambda_x^-(\chi)$ and $\lambda_x^{(l)}(\chi)$ form paragraph 3.3.1.

3.4.2 Lemma. *If $\chi = \omega|\ |^{\pm 1/2} \in \Xi_0$ with $\omega^2 = 1$, then for every $x \in |X|$,*

$$\Phi_x\, R(g,\chi) \;=\; \lambda_x(\chi)\, R(g,\chi) \;=\; \omega(\pi_x)\,(q_x+1)\, R(g,\chi)\,.$$

Proof. We make use of the corresponding result for Eisenstein series (Lemma 3.3.2). Note that $E(\,\cdot\,,\chi,s) = E(\,\cdot\,,\chi|\ |^s)$ if $\chi^2|\ |^{2s} \neq |\ |^{\pm 1}$. Since the Hecke operator only manipulates the first argument of R, it commutes with the variation in s. We calculate for any place x:

$$\begin{aligned}\Phi_x R(\,\cdot\,,\chi) \;&=\; \lim_{s\to 0}\; s\cdot\Phi_x E(\,\cdot\,,\chi,s) \;=\; \lim_{s\to 0}\; s\cdot\lambda_x(\chi|\ |^s) E(\,\cdot\,,\chi,s)\\ &=\; \lim_{s\to 0}\;\lambda_x(\chi|\ |^s)\lim_{s\to 0}\; s\cdot E(\,\cdot\,,\chi,s) \;=\; \lambda_x(\chi)\, R(g,\chi)\,.\end{aligned}$$

The second equality in the lemma follows from the fact that $\chi = \omega|\ |^{\pm 1/2}$ and from the fact that $\omega(\pi_x) = \omega^{-1}(\pi_x) = \pm 1$. □

3.4.3 Lemma. *Let $\chi \in \Xi_0$ with $\chi^2 = |\ |^{\pm 1}$. Then $\lambda_x^-(\chi) \neq 0$ for all $x \in |X|$.*

Proof. For every $x \in |X|$, $1 \neq q_x^{\pm 1}$, so

$$\lambda_x^-(\chi) \;=\; q_x^{1/2}\big(\chi^{-1}(\pi_x) - \chi(\pi_x)\big) \;=\; q_x^{1/2}\chi(\pi_x)(|\pi_x|^{\pm 1}-1) \;\neq\; 0\,. \quad \square$$

3.4.4 Proposition. *If $\chi \in \Xi_0$ with $\chi^2 = |\ |^{\pm 1}$, then*

$$\Phi_x R^{(i)}(g,\chi) \;=\; \sum_{k=0}^{i}\binom{i}{k}\big(\ln q_x\big)^{i-k}\,\lambda_x^{(i-k)}(\chi)\, R^{(k)}(g,\chi)$$

for every $x \in |X|$, where $\lambda_x^{(l)}(\chi)$ are defined as in Proposition 3.3.3.

Proof. The proof is the same as for Proposition 3.3.3. Note that the function $s\cdot E(\,\cdot\,,\chi)$ is holomorphic at $s=0$, so the limit in the definition of the residue and the limit in the definition of the derivative with regard to s commute. □

3.4.5 Corollary. *Let $\chi \in \Xi_0$ with $\chi^2 = |\ |^{\pm 1}$. Then $R^{(1)}(\,\cdot\,,\chi)$ is not an eigenfunction of Φ_x for any $x \in |X|$.* □

3.4.6 Proposition. *Let $\chi \in \Xi_0$ with $\chi^2 = |\ |^{\pm 1}$. Then*

$$\big\{R(\,\cdot\,,\chi), R^{(1)}(\,\cdot\,,\chi), R^{(2)}(\,\cdot\,,\chi),\ldots\big\}$$

is linearly independent and spans a vector space on which $\mathcal{H}_K$ acts. In particular none of these functions vanishes.

Proof. The proof is completely analogous to that of Proposition 3.3.5. Lemma 3.4.3 ensures us of the fact that $\lambda_x^-(\chi) \neq 0$ for some $x \in |X|$. □

3.5 The cuspidal part

We collect some general facts about the cuspidal part.

3.5.1 Theorem ([26, Cor. 1.2.3]). *For every $K' \in \mathcal{V}$, there exists a left $G_F Z_{\mathbf{A}}$ and right K'-invariant subset $\Omega \subset G_{\mathbf{A}}$ such that $G_F Z_{\mathbf{A}} \backslash \Omega / K'$ is finite and for every $f \in \mathcal{A}_0^{K'}$,* $\operatorname{supp} f \subset \Omega$.

3.5.2 Theorem ([11, Section 3.3]). *For every $K' \in \mathcal{V}$, $\mathcal{A}_0^{K'}$ decomposes into a finite direct sum of irreducible $\mathcal{H}_{K'}$-modules.*

3.5.3 Theorem (Multiplicity one, [11, Thm. 3.3.6]).
If $V_1, V_2 \subset \mathcal{A}_0$ are isomorphic $\mathcal{H}$-modules, then $V_1 = V_2$.

3.5.4 Corollary. *$\mathcal{A}_0^K$ admits a finite basis of $\mathcal{H}_K$-eigenfunctions, which is unique up to multiples of the basis vectors.* □

3.6 Main theorem on admissible automorphic forms

We summarise the discussion as follows.

3.6.1 For $\chi \in \Xi_0$, define

$$\tilde{E}^{(i)}(\,\cdot\,,\chi) \;=\; \begin{cases} E^{(i)}(\,\cdot\,,\chi) & \text{if } \chi^2 \notin \{1, |\;|^{\pm 1}\}, \\ R^{(i)}(\,\cdot\,,\chi) & \text{if } \chi^2 = |\;|^{\pm 1}, \\ E^{(2i)}(\,\cdot\,,\chi) & \text{if } \chi^2 = 1. \end{cases}$$

and $\tilde{E}(\,\cdot\,,\chi) = \tilde{E}^{(0)}(\,\cdot\,,\chi)$. Let $\widetilde{\mathcal{E}}(\chi)^K \subset \widetilde{\mathcal{E}} = \mathcal{E} \oplus \mathcal{R}$ be the span of $\{\tilde{E}^{(i)}(\,\cdot\,,\chi)\}_{i \geq 0}$.

Note that by the functional equations for Eisenstein series and their residues, the linear spaces spanned by the set

$$\big\{\tilde{E}^{(0)}(\,\cdot\,,\chi), \ldots,\; \tilde{E}^{(n)}(\,\cdot\,,\chi)\big\} \quad \text{and} \quad \big\{\tilde{E}^{(0)}(\,\cdot\,,\chi^{-1}), \ldots,\; \tilde{E}^{(n)}(\,\cdot\,,\chi^{-1})\big\}$$

are the same for all $\chi \in \Xi_0$. In particular, $\widetilde{\mathcal{E}}(\chi)^K = \widetilde{\mathcal{E}}(\chi^{-1})^K$.

3.6.2 Theorem. *The unramified vectors of the admissible part of $\mathcal{A}$ decompose as an $\mathcal{H}_K$-module into*

$$\mathcal{A}^K_{\text{adm}} \;=\; \mathcal{A}_0^K \;\oplus \bigoplus_{\{\chi,\chi^{-1}\} \subset \Xi_0} \widetilde{\mathcal{E}}(\chi)^K\,.$$

The finite-dimensional vector space $\mathcal{A}_0^K$ admits a basis of $\mathcal{H}_K$-eigenfunctions, and thus every Hecke operators acts as a diagonal matrix on this basis. For every $\chi \in \Xi_0$ and $n \geq 0$, $\{\tilde{E}(\,\cdot\,,\chi), \ldots\, \tilde{E}^{(n-1)}(\,\cdot\,,\chi)\}$ is a basis of the unique $\mathcal{H}_K$-submodule of dimension n in $\widetilde{\mathcal{E}}(\chi)^K$. For every $x \in |X|$, the Hecke operator Φ_x acts as follows in this basis:

- *if* $\chi^2 \neq 1$,

$$\Phi_x = \begin{pmatrix} \lambda_x(\chi) & \binom{1}{1}(\ln q_x)\,\lambda_x^-(\chi) & \cdots & \binom{n}{n}(\ln q_x)^n\,\lambda_x^{(n)}(\chi) \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \lambda_x(\chi) & \binom{n}{1}(\ln q_x)\lambda_x^-(\chi) \\ 0 & \cdots & 0 & \lambda_x(\chi) \end{pmatrix}$$

- *if* $\chi^2 = 1$,

$$\Phi_x = \begin{pmatrix} \lambda_x(\chi) & \binom{2}{2}(\ln q_x)^2\,\lambda_x(\chi) & \cdots & \binom{2n}{2n}(\ln q_x)^{2n}\,\lambda_x(\chi) \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \lambda_x(\chi) & \binom{2n}{2}(\ln q_x)\lambda_x(\chi) \\ 0 & \cdots & 0 & \lambda_x(\chi) \end{pmatrix}$$

Proof. From Theorem 2.3.3, it follows that there is no other linear relation of Eisenstein series than the one that is given by the functional equation. Thus the direct sum in the theorem is well-defined as subspace of $\mathcal{A}^K_{\mathrm{adm}}$.

Propositions 3.3.3 and 3.4.4 imply that for every $\chi \in \Xi_0$, $\widetilde{\mathcal{E}}(\chi)^K$ is an $\mathcal{H}_K$-module and that Φ_x operates as described in the theorem. Propositions 3.3.5, 3.4.6 and 3.3.8 ensure that the described bases are indeed linearly independent.

Lemma 3.2.5 proves the uniqueness of the n-dimensional subspaces in the theorem and furthermore that $\{\tilde{E}^{(i)}(\,\cdot\,,\chi)\}_{\{\chi,\chi^{-1}\}\subset\Xi_0, i\geq 0}$ is linearly independent. Finally, it follows from Propositions 3.3.5, 3.4.6 and 3.3.8 together with Theorem 3.2.2 that the decomposition exhausts $\mathcal{A}^K_{\mathrm{adm}}$. □

3.6.3 Theorem. *Let $V \subset \mathcal{A}^{\mathrm{nr}}$ be an invariant subspace. Then*

$$V^K = (V \cap \mathcal{A}^K_0) \oplus \bigoplus_{\{\chi,\chi^{-1}\}\subset\Xi_0} (V \cap \widetilde{\mathcal{E}}(\chi)^K)\,.$$

The representation V is admissible if and only if V^K is finite dimensional.

Proof. Note that every irreducible subrepresentation $\mathcal{A}^{\mathrm{nr}}_{\mathrm{adm}}$ is determined by its isomorphism class. The subrepresentations of $\mathcal{A}^{\mathrm{nr}}_0$ are characterised by the vanishing of the constant terms of all its elements and uniquely determined by their isomorphism type by Theorem 3.5.3. The subrepresentations of $\mathcal{R}^{\mathrm{nr}}$ are 1-dimensional and uniquely determined by their isomorphism type by Corollary 2.4.3. The subrepresentations of $\mathcal{E}^{\mathrm{nr}}$ uniquely determined by their isomorphism type by Theorem 2.3.3. By Theorem 3.1.9 every irreducible subrepresentation $\mathcal{A}^K_{\mathrm{adm}}$ is thus determined by its isomorphism class as $\mathcal{H}_K$-representation. Since V^K decomposes into a direct sum of simultaneous generalised eigenspaces of elements of $\mathcal{H}_K$, this yields the claimed decomposition.

The latter statement follows from the decomposition together with Theorem 3.2.3. □

3.7 Eigencharacters

If $f \in \mathcal{A}^K$ is an $\mathcal{H}_K$-eigenfunction with eigencharacter λ_f, then Theorem 3.1.9 implies that f is determined by λ_f. As $\mathcal{H}_K$ is generated by elements of the form Φ_x and those that act trivially on $\mathcal{A}$, cf. Lemma 1.4.15, it suffices to know the values $\lambda_f(\Phi_x)$. In fact, we will describe a finite set of places such that f is determined by $\lambda_f(\Phi_x)$ for those places x.

3.7.1 Since $\mathcal{A}_0^K$ is finite dimensional, there are only finitely many Hecke operators necessary to distinguish the generating $\mathcal{H}_K$-eigenfunctions. The support of cusp forms is contained in a bounded set and we shall see in Section 5.5 how to make use of this to distinguish cusp forms from Eisenstein series.

So we may concentrate on $\widetilde{\mathcal{E}}^K$. The $\mathcal{H}_K$-eigenfunctions in $\widetilde{\mathcal{E}}^K$ are parametrised by Ξ_0, and Ξ_0 is identified with quasi-characters on $\operatorname{Cl} F = F^\times \backslash \mathbf{A}^\times / \mathcal{O}_{\mathbf{A}}^\times$, so $\chi \in \Xi_0$ can be seen as a group homomorphism $\operatorname{Cl} F \to \mathbf{C}^\times$. For $x \in |X|$, we define $\chi(x) = \chi(\pi_x)$ thinking of places as prime divisors. All expressions of the form $\langle D \rangle$ for a divisor D will be considered as the subgroup of $\operatorname{Cl} F$ generated by the divisor class of D.

Now consider the $\mathcal{H}_K$-eigenfunction $f = \tilde{E}(\,\cdot\,,\chi)$ with eigencharacter λ_f. We have $\lambda_f(\Phi_x) = \lambda_x(\chi)$ for every $x \in |X|$ and λ_x can be seen as a function $\Xi_0 \to \mathbf{C}$. Furthermore, λ_x factors into $l_x \circ \mathrm{ev}_x$, where $\mathrm{ev}_x : \Xi_0 \to \mathbf{C}^\times$ is the group homomorphism $\chi \mapsto \chi(x)$ and $l_x : \mathbf{C}^\times \to \mathbf{C}$ is defined by $z \mapsto q_x^{1/2}(z + z^{-1})$. We will determine the fibres of λ_x by looking at the fibres of the factors ev_x and l_x.

3.7.2 Lemma. *Let $z \in \mathbf{C}^\times$, $x \in |X|$ and $s \in \mathbf{C}$ such that $q_x^{-s} = z$. Then*

$$\mathrm{ev}_x^{-1}(z) \;=\; \{\omega |\ |^s \mid \omega \in \Xi_0,\ \textit{with}\ \omega(x) = 1\}\,.$$

In particular, $\#(\ker \mathrm{ev}_x) \;=\; h_F\, d_x$.

Proof. Since $|\ |^s \in \mathrm{ev}_x^{-1}(z)$, we have that $\mathrm{ev}_x^{-1}(z) = \ker \mathrm{ev}_x \cdot |\ |^s$. The kernel of ev_x are all $\chi \in \Xi_0$ with $\chi(x) = 1$, and these are nothing else but the characters of $\operatorname{Cl} F / \langle x \rangle$, and this group is an extension of $\operatorname{Cl}^0 F$ by a finite group of order $d_x = \deg x$. □

3.7.3 Lemma. *The map $l_x : \mathbf{C}^\times \to \mathbf{C}$ is a rational map that is a double cover ramified exactly over ± 1. Its fibres are of the form $\{z, z^{-1}\}$.*

Proof. Rationality is clear from the definition. By defining

$$\begin{aligned} l_x(0) &= \lim_{z\to 0} q_x^{1/2}(z + z^{-1}) = \infty\,, \quad \text{and} \\ l_x(\infty) &= \lim_{z\to \infty} q_x^{1/2}(z + z^{-1}) = \infty\,, \end{aligned}$$

the map l_x extends to a rational map $l_x : \mathbf{P}^1(\mathbf{C}) \to \mathbf{P}^1(\mathbf{C})$. Now, l_x is ramified if and only if $\frac{d}{dz} l_x(z)$ vanishes. Since

$$\lim_{z\to\infty} \frac{d}{dz} q_x^{1/2}(z + z^{-1}) \;=\; \lim_{z\to\infty} q_x^{1/2}(1 - z^{-2}) \;=\; q_{x_j}^{1/2} \neq 0\,,$$

l_x is not ramified above ∞, and since it has a fibre of cardinality 2 above ∞, l_x is a $2:1$-covering. For $z \in \mathbf{C}$, $q_{x_i}^{1/2}(1-z^{-2})$ vanishes precisely when $z = \pm 1$. The form of the fibres is now clear since they cannot be larger by the preceding. □

3.7.4 Lemma. *Let $x \in |X|$ and $\chi \in \Xi_0$. Then the following are equivalent.*

(i) $\chi(x) = \pm 1$.

(ii) *χ factors through the finite group* $\mathrm{Cl}\, F \,/\, \langle 2x \rangle$.

(iii) $\mathrm{ev}_x(\chi) = \mathrm{ev}_x(\chi^{-1})$.

(iv) *l_x ramifies in* $\mathrm{ev}_x(\chi)$.

(v) $\lambda_x(\chi)^2 = 4q_x$.

(vi) *$\tilde{E}^{(1)}(\,\cdot\,,\chi)$ is an eigenfunction of Φ_x.*

Proof. The equivalence of (i), (ii) and (iii) is obvious, the equivalence of (i) and (iv) follows from Lemma 3.7.3. For the equivalence with (v), one calculates

$$\lambda_x(\chi)^2 \;=\; \big(\chi(x)^{-1} + \chi(x)\big)^2 q_x$$

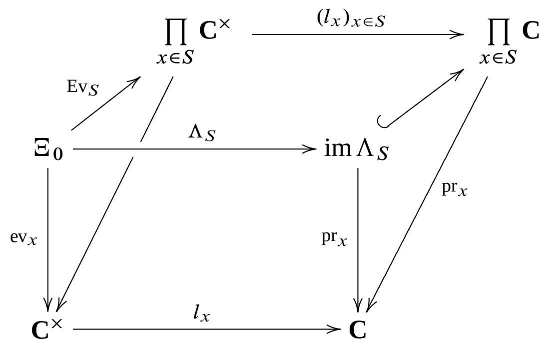
and $\chi(x)^{-1} + \chi(x) = \pm 2$ if and only if $\chi(x) = \pm 1$. Regarding (vi), observe that Proposition 3.3.3 and Corollary 3.4.5 imply that $E^{(1)}(\,\cdot\,,\chi)$ is an eigenfunction of Φ_x if and only if $\lambda_x^-(\chi) = 0$. But since $\lambda_x^-(\chi) = q_x^{1/2}\big(\mathrm{ev}_x(\chi^{-1}) - \mathrm{ev}_x(\chi)\big)$, this is equivalent to (iii). □

3.7.5 Let $S \subset |X|$ be a set of places, finite or infinite, and define

$$\begin{array}{rccc} \Lambda_S: & \Xi_0 & \longrightarrow & \prod_{x\in S} \mathbf{C} \\ & \chi & \longmapsto & \big(\lambda_x(\chi)\big)_{x\in S} \end{array}$$

If a fibre of Λ_S contains a quasi-character χ, then it contains also χ^{-1} since the fibres of every λ_x do so. The question as to whether the Hecke operators Φ_x with $x \in S$ can separate functions in $\widetilde{\mathcal{E}}^K$ is equivalent to asking whether the non-empty fibres of Λ_S are not larger than $\{\chi, \chi^{-1}\}$.

Define $\mathrm{Ev}_S = (\mathrm{ev}_x)_{x\in S}$ and consider the commutative diagram:



The map Ev_S has kernel

$$\ker \mathrm{Ev}_S \;=\; \bigcap_{x\in S} \ker \mathrm{ev}_x\,,$$

which is trivial if and only if the classes of the places in S generate $\mathrm{Cl}\,F = F^\times \backslash \mathbf{A}^\times / \mathcal{O}_\mathbf{A}^\times$. Since all elements that differ by an element in the kernel of Ev_S lie in the same fibre of $(l_x)_{x\in S} \circ \mathrm{Ev}_S$, and therefore in the same fibre of Λ_S, this shows that S should at least generate $\mathrm{Cl}\,F$ for Λ_S to have small fibres.

Define

$$C_{S'}(\chi) \quad = \quad \langle 2x\rangle_{x\in S'} \,/\, \big(\ker\chi \cap \langle 2x\rangle_{x\in S'}\big)$$

for $S' \subset S$ and $\chi \in \Xi_0$, where $\langle 2x\rangle_{x\in S'}$ is considered as subgroup of $\mathrm{Cl}\,F$.

3.7.6 Theorem. *Let S be a set of places that generates $\mathrm{Cl}\,F$ and let $\chi \in \Xi_0$. The fibre of $\Lambda_S(\chi)$ is*

$$\left\{ \chi' \in \Xi_0 \;\middle|\; \begin{array}{ll} \multicolumn{2}{l}{\text{There is a partition } S = S_+ \cup S_- \text{ such that}} \\ \chi'(x) = \chi(x) & \text{for } x \in S_+ \text{ and} \\ \chi'(x) = \chi^{-1}(x) & \text{for } x \in S_-. \end{array} \right\}.$$

It equals $\{\chi, \chi^{-1}\}$ unless if there is a partition $S = S_+ \cup S_-$ such that

$$C_S(\chi) \;=\; C_{S_+}(\chi) \,\oplus\, C_{S_-}(\chi)$$

is a direct sum with nontrivial factors. This can only happen when χ is of finite order.

Proof. Since the kernel of Ev_S is trivial, this means that χ' is in the same fibre of Λ_S as χ if and only if for each $x \in S$,

$$\chi'(x) = \chi(x), \quad \text{or} \quad \chi'(x) = \chi^{-1}(x).$$

This allows us to choose a partition $S = S_+ \cup S_-$ such that $\chi'(x) = \chi(x)$ if $x \in S_+$ and $\chi'(x) = \chi^{-1}(x)$ if $x \in S_-$. Thus the first statement.

We are left to prove that if there exists a χ' in the fibre of χ that neither equals χ nor χ^{-1}, then $C_S = C_{S_+} \oplus C_{S_-}$ is a nontrivial decomposition and χ is of finite order. Observe that for such a χ', neither S_+ nor S_- is empty.

Define for every $S' \subset S$ the subgroup $H_{S'} = \langle x\rangle_{x\in S'}$ of $H_S = \mathrm{Cl}\,F$. Then, restricted to H_{S_+}, we have $\chi' = \chi$, and restricted to H_{S_-}, we have $\chi' = \chi^{-1}$. Hence χ^2 is trivial on $H_{S_+} \cap H_{S_-}$, or in other words, $H_{S_+} \cap H_{S_-} \subset \ker\chi^2$. Since $H_{S_+} \cup H_{S_-}$ generates H_S, we obtain a decomposition

$$\Big(H_S \,/\, (\ker\chi^2 \cap H_S)\Big) \;=\; \Big(H_{S_+} \,/\, (\ker\chi^2 \cap H_{S_+})\Big) \,\oplus\, \Big(H_{S_-} \,/\, (\ker\chi^2 \cap H_{S_-})\Big)\,.$$

Observe that the assignment $h \mapsto h^2$, induces an isomorphism

$$H_{S'} \,/\, (\ker\chi^2 \cap H_{S'}) \xrightarrow{\sim} 2H_{S'} \,/\, (\ker\chi \cap 2H_{S'}) \;=\; C_{S'}(\chi)\,.$$

for every subset $S' \subset S$. Thus we have the decomposition

$$C_S(\chi) \;=\; C_{S_+}(\chi) \,\oplus\, C_{S_-}(\chi)\,.$$

On the other hand, each such decomposition that is nontrivial allows us to choose a χ' as above which neither equals χ nor χ^{-1}. Then the fibre of $\Lambda_S(\chi)$ contains more than 2 elements.

Finally note that $H_{S_+} \cap H_{S_-}$ has finite index in $\mathrm{Cl}\,F$. Hence χ^2 is of finite order, and so is χ. □

3.7.7 Corollary. *Let $S \subset |X|$ generate $\mathrm{Cl}\,F$. Then for $\chi \in \Xi_0$ the following are equivalent.*

(i) $\chi = \chi^{-1}$.

(ii) *The fibre of $\Lambda_S(\chi)$ contains only χ.*

(iii) *l_x ramifies in $\mathrm{ev}_x(\chi)$ for every $x \in S$.*

(iv) *$\lambda_x(\chi) = 2q_x^{1/2}\chi(x)$ for every $x \in S$.*

(v) *$\tilde{E}^{(1)}(\,\cdot\,,\chi)$ is an $\mathcal{H}_K$-eigenfunction or trivial.*

Proof. Observe that if $\chi = \chi^{-1}$, the fibres described in the theorem contain only $\chi = \chi^{-1}$. The equivalence of (i) and (ii) follows.

The equivalence of (i) and (iii) follows from Lemma 3.7.4, bearing in mind that χ is determined by its values at all $x \in S$.

The implication from (i) to (iv) follows by the definition of $\lambda_x(\chi)$. The converse implication follows from the theorem.

The equivalence of (i) and (v) follows from Corollary 3.3.6. □

3.7.8 Corollary. *Let $S \subset |X|$ generate $\mathrm{Cl}\,F$ and let $\chi \in \Xi_0$. Then $\tilde{E}(\,\cdot\,,\chi) \in \mathcal{R}$ if and only if there is a $\omega \in \Xi_0$ with $\omega^2 = 1$ such that for all $x \in S$, $\lambda_x(\chi) = \omega(\pi_x)(q_x + 1)$. In this case, $\chi = \omega|\;|^{\pm 1/2}$.*

Proof. If $\tilde{E}(\,\cdot\,,\chi)$ lies in the residual part, then there is an $\omega \in \Xi_0$ with $\omega^2 = 1$ such that $\chi = \omega|\;|^{\pm 1}$, and Lemma 3.4.2 describes the eigenvalues of residues as desired.

For the converse implication, note that if $\chi' \in \Xi_0$ is of finite order, then $\mathrm{im}\,\chi' \subset \mathbf{S}^1$, and $\chi'^{-1}(x)$ is the complex conjugate of $\chi'(x)$. Thus

$$\lambda_x(\chi') = q_x^{1/2}\big(\chi'^{-1}(x) + \chi'(x)\big) \in [-2q_x^{1/2}, 2q_x^{1/2}]\,.$$

But $q_x > 1$, hence $q_x + 1 > 2q_x^{1/2}$ and χ, which by assumption has $|\lambda_x(\chi)| = q_x + 1$, is not of finite order. Thus by Theorem 3.7.6 the fibre of $\lambda_x(\chi)$ contains only $\omega|\;|^{1/2}$. □

3.7.9 Proposition. *If $\{x_1, \ldots, x_{h_F}\} \subset |X|$ represents the divisor classes of a fixed degree d, then we have for $\chi \in \Xi_0$ that*

$$(\Phi_{x_1} + \ldots + \Phi_{x_{h_F}})\,\tilde{E}(\,\cdot\,,\chi) = \begin{cases} q^{d/2}h_F\,(q^{ds} + q^{-ds})\,\tilde{E}(\,\cdot\,,\chi) & \textit{if } \chi = |\;|^s \textit{ for a } s \in \mathbf{C}, \\ 0 & \textit{otherwise.} \end{cases}$$

Proof. We choose an idele a of degree 1, and write $\chi = \omega|\;|^s$ with $\omega(a) = 1$. Then

$$\begin{aligned} \sum_{i=1}^{h_F}\lambda_{x_i}(\chi) &= \sum_{i=1}^{h_F} q_{x_i}^{1/2}\big(\chi^{-1}(\pi_{x_i}) + \chi(\pi_{x_i})\big) \\ &= q^{d/2}\Big(\sum_{i=1}^{h}\omega^{-1}(a^{-d}\pi_{x_i})\,\big|\pi_{x_i}\big|^{-s} + \sum_{i=1}^{h_F}\omega^{-1}(a^{-d}\pi_{x_i})\,\big|\pi_{x_i}\big|^{s}\Big) \\ &= \begin{cases} q^{d/2}h_F\,(q^{ds} + q^{-ds}) & \text{if } \omega = 1, \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

The last equation follows from the general fact that for a character $\widetilde{\omega}$ of a finite group $\widetilde{G}$, $\sum_{g \in \widetilde{G}}\widetilde{\omega}(g)$ equals $\#\widetilde{G}$ if $\widetilde{\omega}$ is trivial and equals 0 otherwise. □

3.7.10 The rest of this chapter is devoted to a description of finite sets $S \subset |X|$ such that the corresponding Hecke operators are able to distinguish $\mathcal{H}_K$-eigenfunctions in the completed Eisenstein part. The fibres of Theorem 3.7.6 that are larger than $\{\chi, \chi^{-1}\}$ for some $\chi \in \Xi_0$ can be prevented if S generates $\mathrm{Cl}\, F$ and satisfies the property:

$$\text{For every partition } S = S_+ \cup S_-, \text{ either } 2\,\mathrm{Cl}\, F = 2\langle S_+\rangle \text{ or } 2\,\mathrm{Cl}\, F = 2\langle S_-\rangle. \qquad (*)$$

We need some group theoretic preparation.

3.7.11 Lemma. *Let H be an finite abelian group. Then for every partition $H = S_+ \cup S_-$ there is either an $x_+ \in S_+$ such that*

$$2H = 2\langle S_+ - \{x_+\}\rangle$$

or an $x_- \in S_-$ such that

$$2H = 2\langle S_- - \{x_-\}\rangle\ .$$

Proof. The structure theorem for finite abelian groups states that H is isomorphic to a product of cyclic groups of prime power order, which is unique up to permutation of the components. In particular the number n of cyclic factors is an invariant of H. We will do induction on n.

If $n = 0$, note that the trivial group satisfies the lemma for trivial reasons.

If $n > 0$, H is isomorphic to $H' \times (\mathbf{Z}/m\mathbf{Z})$ for some integer $m \geq 2$ and some subgroup H', which has $n-1$ factors, which we assume to satisfy the lemma. If $m = 2$, then $2H = 2H'$ and the induction step is established. If $m > 2$, note that $\mathbf{Z}/m\mathbf{Z}$ has at least two generators, namely, 1 and -1. Either S_+ or S_- must contain elements that satisfy the lemma for $H' \times \{0\}$, say S_+ does so with respect to some $x_+ \in S_+$.

If S_+ further contains an element of $H' \times \{\pm 1\}$, then $2H = 2\langle S_+ - \{x_+\}\rangle$ because H is generated by the union of H' with an arbitrary element of $H' \times \{\pm 1\}$.

If not, then $H' \times \{\pm 1\} \subset S_-$. In both cases that H' is trivial and that H' is not trivial, one sees that $H' \times \{\pm 1\}$ with one element excluded generates H, what in particular implies the assertion of the lemma. Thus we have completed the induction. □

3.7.12 Remark. Jakub Byszewski found the following alternative proof of Lemma 3.7.11. First observe that it holds for trivial reasons for the groups $\mathbf{Z}/2\mathbf{Z}$ and $(\mathbf{Z}/2\mathbf{Z}) \times (\mathbf{Z}/2\mathbf{Z})$. For all other groups it follows from the following more general lemma.

3.7.13 Lemma. *Let H be a finite group (not necessarily abelian) that is not isomorphic to $\mathbf{Z}/2\mathbf{Z}$ or $(\mathbf{Z}/2\mathbf{Z}) \times (\mathbf{Z}/2\mathbf{Z})$. Let $H = S_1 \cup S_2$ be a partition. Then there exists either an $x_1 \in S_1$ such that $H = \langle S_1 - \{x_1\}\rangle$ or an $x_2 \in S_2$ such that $H = \langle S_2 - \{x_2\}\rangle$.*

Note that both $\mathbf{Z}/2\mathbf{Z}$ and $(\mathbf{Z}/2\mathbf{Z}) \times (\mathbf{Z}/2\mathbf{Z})$ do not satisfy the lemma if partitioned into subsets of equal cardinality.

Proof. The majority of cases is excluded by the observation that a subset S of cardinality $\#S > \frac{1}{2}\#H$ necessarily generates H by Lagrange's theorem. There are only three cases left, which we will consider separately. Without loss of generality, we may assume that $\#S_1 \geq \#S_2$.

Case (i): $\#H = 2n+1$ *is odd and* $\#S_1 = n+1$.
Since 2 is not a divisor of the group order, the largest possible subgroup has $\frac{1}{3}\#H$ elements. If $n = \#S_1 - 1 > \frac{1}{3}\#H$, then the lemma holds by Lagrange's theorem. If not, then $n \leq \frac{2n+1}{3}$. This is the case if and only if $n \leq 1$, which in turn means that $\#H \leq 3$. If H has 1 element, then the lemma follows trivially. If H has 3 elements, then S_1 contains 2 elements, one of which generates H.

Case (ii): $\#H = 2n$ *is even and* $\#S_1 = n+1$.
If there is an $x_1 \in S_1$ such that $H' = S_1 - \{x_1\}$ is precisely a subgroup of index 2 in H, then S_1 has to contain the neutral element $e \in H$. But $S_1 - \{e\}$ is not contained in any proper subgroup of H.

Case (iii): $\#H = 2n$ *is even and* $\#S_1 = \#S_2 = n$.
Without loss of generality we may assume that $e \in S_2$. Then S_1 cannot be contained in a proper subgroup and must generate H. If there is an $x_0 \in S_1$ such that $S_1 - \{x_0\}$ generates a subgroup H_0 of index 2 in H, then $H_0 = (S_1 - \{x_0\}) \cup \{e\}$ by counting elements. If further $n = \#H_0 \geq 3$, then there would be an $x_1 \in S_1 - \{x_0\} \subset H_0$ such that we have $H_0 = \langle S_1 - \{x_0, x_1\}\rangle$. But then $H = \langle S_1 - \{x_1\}\rangle$. There are only two possibilities left: $n = \#S_1 = 1$ or $n = \#S_1 = 2$. If $n = 1$, then $H \simeq \mathbf{Z}/2\mathbf{Z}$, which we excluded. If $n = 2$, then either $H \simeq (\mathbf{Z}/2\mathbf{Z}) \times (\mathbf{Z}/2\mathbf{Z})$, which we excluded, or $H \simeq \mathbf{Z}/4\mathbf{Z}$, generated by some element $a \in H$. In the latter case S_2 is the unique subgroup $\{e, a^2\}$, thus $S_1 = \{a, a^3\}$. But both a and a^3 generate H. □

3.7.14 Proposition. *If* $\mathrm{Cl}^d F = S_+ \cup S_-$, *then as subsets of* $\mathrm{Cl}\, F$, *either* $2\langle \mathrm{Cl}^d F\rangle = 2\langle S_+\rangle$ *or* $2\langle \mathrm{Cl}^d F\rangle = 2\langle S_-\rangle$.

Proof. If $d = 0$, then the proposition follows immediately from the last lemma. Assume $d \neq 0$. Choosing a $z_0 \in \mathrm{Cl}^d F$, we obtain a bijection $\mathrm{Cl}^d F \to \mathrm{Cl}^0 F$ by subtracting z_0. This induces a partition $\mathrm{Cl}^0 F = S'_+ \cup S'_-$. By possibly exchanging S_+ and S_-, the lemma implies that there is a $z'_+ \in S'_+$ such that $2\,\mathrm{Cl}^0 F = 2\langle S'_+ - \{z'_+\}\rangle$. If $z_+ = z'_+ + z_0$, then $2\,\mathrm{Cl}^0 F \oplus 2\langle z_+\rangle = 2\langle \mathrm{Cl}^d F\rangle$. □

A field extension E/F is called *geometric* if the constant field of E has the same number of elements as the constant field of F. The following is a consequence of Chebotarev's density theorem.

3.7.15 Theorem ([55, Thm. 9.13B]). *Let* E/F *be a finite abelian separable geometric field extension and* $\mathrm{N}_{E/F} : \mathrm{Cl}\, E \to \mathrm{Cl}\, F$ *the norm of* E *over* F *extended to the divisor class group. Then for every element in* $\mathrm{Cl}\, F / \mathrm{N}_{E/F}(\mathrm{Cl}\, E)$, *there is an integer* d_0 *such that for every* $d \geq d_0$, *there is a prime divisor over* F *of degree* d *that represents this element.*

3.7.16 Theorem. *There is an integer* d_0 *such that every divisor class in* $\mathrm{Cl}\, F$ *of degree larger than* d_0 *is represented by a prime divisor.*

Proof. Let D denote a divisor class of degree 1. Then by class field theory, there is an finite abelian separable geometric field extension E/F with Galois group $\mathrm{Cl}\, F / \langle D\rangle$, and everything follows from Theorem 3.7.15. □

3.7.17 We can now choose a set of generators S with property $(*)$ as follows. Begin with a finite set of generators of $\mathrm{Cl}\, F$ and add places that represent $\mathrm{Cl}^d F$ such that d is coprime to the degree of each of the previously chosen generators. Theorem 3.7.16 ensures us of the fact that there are such places provided d is large enough.

This set S satisfies $(*)$ because for every partition $S = S_+ \cup S_-$, one of both $2\langle S_+\rangle$ and $2\langle S_-\rangle$ contains $2\langle \mathrm{Cl}^d\rangle$, say $2\langle S_+\rangle$ does. If S_+ further contains any other other element of S, then $2\,\mathrm{Cl}\,F = 2\langle S_+\rangle$, otherwise S_- generates $\mathrm{Cl}\,F$. This establishes $(*)$.

3.7.18 If F is a rational function field, then $\mathrm{Cl}\,F \simeq \mathbf{Z}$. Let x be a place of degree 1. Then $\{x\}$ generates $\mathrm{Cl}\,F$, and obviously it fulfills $(*)$. Thus it suffices to calculate only the Φ_x-eigenvalue to recognise an $\mathcal{H}_K$-eigenfunction in the completed Eisenstein part.

If F is the function field of an elliptic curve, then the set of all places of degree 1, which represent precisely $\mathrm{Cl}^1\,F$, generates $\mathrm{Cl}\,F$. Proposition 3.7.14 implies that $(*)$ holds. This makes it possible to distinguish the $\mathcal{H}_K$-eigenfunctions in the completed Eisenstein part by the action of the Hecke operators Φ_x where x varies through the degree one places. We will see in Chapter 8, however, that these Hecke operators cannot distinguish cusp forms, and it will be necessary to consider the operators Φ_x for places x of degree 2.

CHAPTER 4

Graphs of Hecke operators

To each Hecke operator we associate a certain graph with extra structure that will be one of the main tools for the theory of toroidal automorphic forms. Automorphic forms can be reinterpreted as functions on the vertices, and the edges together with a weight function symbolise the action of the Hecke operator on automorphic forms. We investigate the graphs associated to generators of the unramified Hecke algebra in more detail and apply the theory of Bruhat-Tits trees to these graphs.

4.1 Definition

Let $G = \mathrm{GL}_2$ and $K' \subset G_{\mathbf{A}}$ be a compact and open subgroup. We will write $[g] \in G_F Z_{\mathbf{A}} \backslash G_{\mathbf{A}} / K'$ for the class that is represented by $g \in G_{\mathbf{A}}$. Other cosets will also occur but it will be clear from the context what kind of class the square brackets relate to.

4.1.1 Proposition. *For all $\Phi \in \mathcal{H}_{K'}$ and $[g] \in G_F Z_{\mathbf{A}} \backslash G_{\mathbf{A}} / K'$, there is a unique set of pairwise distinct classes $[g_1], \dots, [g_r] \in G_F Z_{\mathbf{A}} \backslash G_{\mathbf{A}} / K'$ and numbers $m_1, \dots, m_r \in \mathbf{C}^\times$ such that for all $f \in \mathcal{A}^{K'}$,*

$$\Phi(f)(g) \;=\; \sum_{i=1}^{r} m_i\, f(g_i)\,.$$

Proof. Uniqueness is clear, and existence follows from Lemma 1.4.11 after we have taken care of putting together values of f in same classes of $G_F Z_{\mathbf{A}} \backslash G_{\mathbf{A}} / K'$ and throwing out zero terms. □

4.1.2 Definition. With the notation of the preceding proposition we define

$$\mathcal{U}_{\Phi,K'}([g]) = \{([g],[g_i],m_i)\}_{i=1,\dots,r}\,.$$

The classes $[g_i]$ are called the *Φ-neighbours of $[g]$ (relative to K').*

The *graph $\mathcal{G}_{\Phi,K'}$ of Φ (relative to K')* consists of vertices

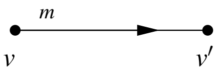
$$\mathrm{Vert}\,\mathcal{G}_{\Phi,K'} \;=\; G_F Z_{\mathbf{A}} \backslash G_{\mathbf{A}} / K'$$

and oriented weighted edges

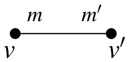
$$\text{Edge}\,\mathcal{G}_{\Phi,K'} = \bigcup_{v\in\text{Vert}\,\mathcal{G}_{\Phi,K'}} \mathcal{U}_{\Phi,K'}(v)\,.$$

4.1.3 Remark. The usual notation for an edge in a graph with weighted edges consists of pairs that code the origin and the terminus, and an additional function on the set of edges that gives the weight. For our purposes, it is more convenient to replace the set of edges by the graph of the weight function and to call the resulting tripels that consist of origin, terminus and the weight the edges of $\mathcal{G}_{\Phi,K'}$.

4.1.4 We make the following drawing conventions to illustrate the graph of a Hecke operator: vertices are represented by labelled dots, and an edge (v,v',m) together with its origin v and its terminus v' is drawn as



If there is precisely one edge from v to v' and precisely one from v' to v, which we call the inverse edge, we draw



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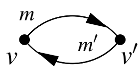
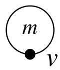


, and


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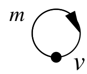

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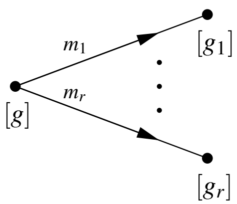
4.1.5 By the very definition of the graph of Φ, we have for $f \in \mathcal{A}^{K'}$ and $[g] \in G_F Z_\mathbf{A} \backslash G_\mathbf{A} / K'$ that

$$\Phi(f)(g) = \sum_{\substack{([g],[g'],m')\\ \in\text{Edge}\,\mathcal{G}_{\Phi,K'}}} m' f(g')\,.$$

Hence one can read off the effect of a Hecke operator on the value of an automorphic function from the illustration of the graph:



4.1.6 We collect some first properties: Since $\mathcal{H} = \bigcup \mathcal{H}_{K'}$, with K' running over all compact opens in $G_\mathbf{A}$, the notion of the graph of a Hecke operator applies to any $\Phi \in \mathcal{H}$.

The set of vertices of the graph of a Hecke operator $\Phi \in \mathcal{H}_{K'}$ only depends on K', and only the edges depend on the particular chosen Φ. There is at most one edge for each two vertices and each direction, and the weight of an edge is always non-zero. Each vertex is connected with only finitely many other vertices.

The algebra structure of $\mathcal{H}_{K'}$ has the following implications for the structure of the set of edges. We define the empty sum as 0.

4.1.7 Proposition. *For the zero element $0 \in \mathcal{H}_{K'}$, the multiplicative unit $1 \in \mathcal{H}_{K'}$, and arbitrary $\Phi_1, \Phi_2 \in \mathcal{H}_{K'}$, $r \in \mathbf{C}^\times$ we obtain that*

$$\begin{aligned}
\text{Edge}\,\mathcal{G}_{0,K'} &= \emptyset\,,\\
\text{Edge}\,\mathcal{G}_{1,K'} &= \big\{(v,v,1)\big\}_{v\in\text{Vert}\,\mathcal{G}_{1,K'}}\,,\\
\text{Edge}\,\mathcal{G}_{\Phi_1+\Phi_2,K'} &= \Big\{(v,v',m)\,\Big|\, m = \sum_{(v,v',m')\in\text{Edge}\,\mathcal{G}_{\Phi_1,K'}} m' + \sum_{(v,v',m'')\in\text{Edge}\,\mathcal{G}_{\Phi_2,K'}} m'' \neq 0\Big\}\,,\\
\text{Edge}\,\mathcal{G}_{r\Phi_1,K'} &= \big\{(v,v',rm)\,\big|\,(v,v',m)\in\text{Edge}\,\mathcal{G}_{\Phi_1,K'}\big\}\,,\ \textit{and}\\
\text{Edge}\,\mathcal{G}_{\Phi_1*\Phi_2,K'} &= \Big\{(v,v',m)\,\Big|\, m = \sum_{\substack{(v,v'',m')\in\text{Edge}\,\mathcal{G}_{\Phi_1,K'}\\ \textit{and}\\ (v'',v',m'')\in\text{Edge}\,\mathcal{G}_{\Phi_2,K'}}} m'\cdot m'' \neq 0\Big\}.
\end{aligned}$$

If $K'' < K'$ and $\Phi \in \mathcal{H}_{K'}$, then also $\Phi \in \mathcal{H}_{K''}$. This implies that we have a canonical map $P : \mathcal{G}_{\Phi,K''} \to \mathcal{G}_{\Phi,K'}$, which is given by

$$\text{Vert}\,\mathcal{G}_{\Phi,K''} = G_F Z_{\mathbf{A}} \backslash G_{\mathbf{A}} / K'' \quad \xrightarrow{P} \quad G_F Z_{\mathbf{A}} \backslash G_{\mathbf{A}} / K' = \text{Vert}\,\mathcal{G}_{\Phi,K'}$$

and

$$\begin{array}{ccc}
\text{Edge}\,\mathcal{G}_{\Phi,K''} & \xrightarrow{P} & \text{Edge}\,\mathcal{G}_{\Phi,K'}\,.\\
(v,v',m') & \longmapsto & (P(v),P(v'),m')
\end{array}$$

4.1.8 One can also collect the data of $\mathcal{G}_{\Phi,K'}$ in an infinite-dimensional matrix $M_{\Phi,K'}$, which we call *the matrix associated to* $\mathcal{G}_{\Phi,K'}$, by putting $(M_{\Phi,K'})_{v',v} = m$ if $(v,v',m) \in$ Edge $\mathcal{G}_{\Phi,K'}$, and $(M_{\Phi,K'})_{v',v} = 0$ otherwise. Thus each row and each column has only finitely many non-vanishing entries.

The above proposition implies:

$$\begin{aligned}
M_{0,K'} &= 0, && \text{the zero matrix,}\\
M_{1,K'} &= 1, && \text{the identity matrix,}\\
M_{\Phi_1+\Phi_2,K'} &= M_{\Phi_1,K'} + M_{\Phi_2,K'}\,,\\
M_{r\Phi_1,K'} &= rM_{\Phi_1,K'}\,, && \text{and}\\
M_{\Phi_1*\Phi_2,K'} &= M_{\Phi_2,K'} M_{\Phi_1,K'}\,.
\end{aligned}$$

Thus, we may regard $\mathcal{H}_{K'}$ as a subalgebra of the algebra of $\mathbf{C}$-linear maps

$$\bigoplus_{G_F Z_{\mathbf{A}} \backslash G_{\mathbf{A}} / K'} \mathbf{C} \longrightarrow \bigoplus_{G_F Z_{\mathbf{A}} \backslash G_{\mathbf{A}} / K'} \mathbf{C}\,.$$

4.2 Unramified Hecke operators

From now on we will restrict ourselves to unramified automorphic forms and unramified Hecke operators. Recall from Lemma 1.4.15 that the Hecke operators Φ_x together with elements that act as 1 on $\mathcal{A}^K$ generate $\mathcal{H}_K$ as a complex algebra. By Proposition 4.1.7 it is thus enough to know the graphs of generators to determine all graphs of unramified Hecke operators. We use the shorthand notation $\mathcal{G}_x$ for the graph $\mathcal{G}_{\Phi_x,K}$, and $\mathcal{U}_x(v)$ for the Φ_x-neighbours $\mathcal{U}_{\Phi_x,K}(v)$ of v.

4.2.1 We introduce the *"lower x convention"* that says that a lower index x on an algebraic group defined over the adeles of F will consist of only the component at x of the adelic points, for example, $G_x = G_{F_x}$, $Z_x = Z_{F_x}$, etc. Analogously, we have $K_x = G_{\mathcal{O}_x}$.

The *"upper x convention"* means that a upper index x on an algebraic group defined over the adeles of F will consist of all components except for the one at x. In particular, we first define $\mathbf{A}^x = \prod'_{y\neq x} F_y$, the restricted product relative to $\mathcal{O}^x = \prod_{y\neq x}\mathcal{O}_y$ over all places y that do not equal x. Then examples for groups with upper x are $G^x = G_{\mathbf{A}^x}$, $Z^x = Z_{\mathbf{A}^x}$, etc. Put $K^x = G_{\mathcal{O}^x}$.

4.2.2 For the standard Borel subgroup $B < G$, we have the local and the global form of the *Iwasawa decomposition*, respectively:

$$G_x = B_x K_x \qquad \text{and} \qquad G_{\mathbf{A}} = B_{\mathbf{A}} K.$$

Recall from paragraph 1.1.2 that the uniformisers $\pi_x \in F$ are considered as ideles embedded via $F^\times \subset F_x^\times \subset \mathbf{A}^\times$. Also, we embed κ_x via $\kappa_x \subset F_x \subset \mathbf{A}$, thus an element $b \in \kappa_x$ will be considered as the adele whose component at x is b and whose other components are 0. Let $\mathbf{P}^1$ be the projective line. Define for $w \in \mathbf{P}^1(\kappa_x)$,

$$\xi_w = \begin{pmatrix} \pi_x & b \\ & 1 \end{pmatrix} \quad \text{if } w = [1:b] \quad \text{and} \quad \xi_w = \begin{pmatrix} 1 & \\ & \pi_x \end{pmatrix} \quad \text{if } w = [0:1].$$

Note that $\xi_w \in G_{\mathbf{A}}$ depends on x as w is an element of $\mathbf{P}^1(\kappa_x)$.

4.2.3 Lemma.

$$K\begin{pmatrix} \pi_x & \\ & 1 \end{pmatrix}K = \coprod_{w\in\mathbf{P}^1(\kappa_x)} \xi_w K\,.$$

Proof. It is clear that $K\left(\begin{smallmatrix} \pi_x & \\ & 1 \end{smallmatrix}\right)K$ is a disjoint union of cosets of the form ξK for certain $\xi \in G_{\mathbf{A}}$. The question can be solved componentwise at each place y. If $y \neq x$, then $K_y\left(\begin{smallmatrix} 1 & \\ & 1 \end{smallmatrix}\right)K_y = K_y$ as desired.

If $y = x$, then by the Iwasawa decomposition, ξ_x can be chosen to be upper triangular. Since

$$\left|\det\left(k\begin{pmatrix} \pi_x & \\ & 1 \end{pmatrix}k'\right)\right|_x = |\pi_x|_x$$

for $k, k' \in K_x$ and all entries of ξ_x have to lie in $\mathcal{O}_x$, the only possible cosets are the ones occurring in the lemma. On the other hand they indeed occur since

$$\begin{pmatrix} 1 & b \\ & 1 \end{pmatrix}\begin{pmatrix} \pi_x & \\ & 1 \end{pmatrix}\begin{pmatrix} 1 & \\ & 1 \end{pmatrix} = \begin{pmatrix} \pi_x & b \\ & 1 \end{pmatrix}$$

and

$$\begin{pmatrix} & 1 \\ 1 & \end{pmatrix}\begin{pmatrix} \pi_x & \\ & 1 \end{pmatrix}\begin{pmatrix} & 1 \\ 1 & \end{pmatrix} = \begin{pmatrix} 1 & \\ & \pi_x \end{pmatrix}. \qquad \square$$

Since we normalise the operators Φ_x with the factor $(\operatorname{vol} K)^{-1}$, the weights of edges in $\mathcal{G}_x$ are positive integers. We shall also refer to them as the *multiplicity* of a Φ_x-neighbour. The above lemma implies the following.

4.2.4 Proposition. *The Φ_x-neighbours of $[g]$ are the classes $[g\xi_w]$ with ξ_w as in the previous lemma, and the multiplicity of an edge from $[g]$ to $[g']$ equals the number of $w \in \mathbf{P}^1(\kappa_x)$ such that $[g\xi_w] = [g']$. The multiplicities of the edges originating in $[g]$ sum up to $\# \mathbf{P}^1(\kappa_x) = q_x + 1$.*

4.3 Examples for rational function fields

This section contains first examples of graphs of Hecke operators for a rational function field, which can be calculated by elementary matrix manipulations. It serves to give an impression of what the graphs of Hecke operators look like, but is not needed for the subsequent theory. Hence, we do not show all calculations that led to the pictures as presented. The reader will find more figures in section 7.3.

Let F be $\mathbf{F}_q(T)$, the function field of the projective line over $\mathbf{F}_q$, which has $q+1$ rational points and trivial class group. Fix a place x of degree 1.

4.3.1 Using strong approximation for SL_2 (cf. Proposition 4.4.11, where J is trivial in this case), we see that the map obtained by adding the identity matrix e at all places $y \neq x$,

$$\begin{array}{ccc} \Gamma \backslash G_x / Z_x K_x & \longrightarrow & G_F \backslash G_\mathbf{A} / K Z_\mathbf{A}\,, \\ [g_x] & \longmapsto & [(g_x, e)] \end{array}$$

is a bijection.

We define an empty sum as 0. Recall the notation:

- $\mathcal{O}_F^x = \bigcap_{y\neq x}(\mathcal{O}_y \cap F)$ is the collection of all elements in F of the form $\sum_{i=m}^{0} b_i \pi_x^i$ with $b_i \in \mathbf{F}_q$ for $i = m, \dots, 0$ for some integer m.
- $K_x = \mathrm{GL}_2(\mathcal{O}_x)$, where $\mathcal{O}_x$ is the collection of all power series $\sum_{i\geq 0} b_i \pi_x^i$ with $b_i \in \mathbf{F}_q$ for $i \geq 0$.
- $\Gamma = G_F \cap K^x = \mathrm{GL}_2(\mathcal{O}_F^x)$ (cf. Remark 4.4.9).

4.3.2 For better readability, we write π for the uniformiser π_x at x and g for a matrix in G_x. We say $g \sim g'$ if they represent the same class $[g] = [g']$ in $\Gamma \backslash G_x / Z_x K_x$, and indicate by subscripts to '$\sim$' how to alter one representative to another. The following changes of the representative g of a class $[g] \in \Gamma \backslash G_x / Z_x K_x$ provide an algorithm to determine a standard representative for the class of any matrix $g \in G_x$:

(i) By the Iwasawa decomposition, cf. paragraph 4.2.2, every class in $\Gamma \backslash G_x / Z_x K_x$ is represented by an upper triangular matrix, and

$$\begin{pmatrix} a & b \\ & d \end{pmatrix} \underset{/Z_x}{\sim} \begin{pmatrix} a & b \\ & d \end{pmatrix} \begin{pmatrix} d^{-1} & \\ & d^{-1} \end{pmatrix} = \begin{pmatrix} a/d & b/d \\ & 1 \end{pmatrix}.$$

(ii) Write $a/d = r\pi^n$ for some integer n and $r \in \mathcal{O}_x^\times$, then with $b' = b/d$, we have

$$\begin{pmatrix} r\pi^n & b' \\ & 1 \end{pmatrix} \underset{/K_x}{\sim} \begin{pmatrix} r\pi^n & b' \\ & 1 \end{pmatrix} \begin{pmatrix} r^{-1} & \\ & 1 \end{pmatrix} = \begin{pmatrix} \pi^n & b' \\ & 1 \end{pmatrix}.$$

(iii) If $b' = \sum_{i \geq m} b_i \pi^i$ for some integer m and coefficients $b_i \in \mathbf{F}_q$ for $i \geq m$, then

$$\begin{pmatrix} \pi^n & \sum_{i \geq m} b_i \pi^i \\ & 1 \end{pmatrix} \underset{/K_x}{\sim} \begin{pmatrix} \pi^n & \sum_{i \geq m} b_i \pi^i \\ & 1 \end{pmatrix} \begin{pmatrix} 1 & -\pi^{-n}(\sum_{i \geq n} b_i \pi^i) \\ & 1 \end{pmatrix}$$
$$= \begin{pmatrix} \pi^n & b_m \pi + \ldots + b_{n-1}\pi^{n-1} \\ & 1 \end{pmatrix}.$$

(iv) One can further perform the following step:

$$\begin{pmatrix} \pi^n & b_m \pi^m + \ldots + b_{n-1}\pi^{n-1} \\ & 1 \end{pmatrix}$$
$$\underset{\Gamma\backslash}{\sim} \begin{pmatrix} 1 & -(b_m \pi^m + \ldots + b_0 \pi^0) \\ & 1 \end{pmatrix} \begin{pmatrix} \pi^n & b_m \pi^m + \ldots + b_{n-1}\pi^{n-1} \\ & 1 \end{pmatrix}$$
$$= \begin{pmatrix} \pi^n & b_1 \pi + \ldots + b_{n-1}\pi^{n-1} \\ & 1 \end{pmatrix}.$$

(v) If $b = b_1 \pi + \ldots + b_{n-1}\pi^{n-1} \neq 0$, then $b = s\pi^k$ with $1 \leq k \leq n-1$, $s \in O_x^\times$ and

$$\begin{pmatrix} \pi^n & s\pi^k \\ & 1 \end{pmatrix} \underset{\Gamma\backslash\,/Z_x K_x}{\sim} \begin{pmatrix} & 1 \\ 1 & \end{pmatrix} \begin{pmatrix} \pi^n & s\pi^k \\ & 1 \end{pmatrix} \begin{pmatrix} s^{-1}\pi^{-k} & \\ & s^{-1}\pi^{-k} \end{pmatrix} \begin{pmatrix} & -s^2 \\ s\pi^{n-k} & 1 \end{pmatrix}$$
$$= \begin{pmatrix} \pi^{n-2k} & s^{-1}\pi^{-k} \\ & 1 \end{pmatrix}.$$

(vi) The last trick is

$$\begin{pmatrix} \pi^n & \\ & 1 \end{pmatrix} \underset{\Gamma\backslash\,/Z_x K_x}{\sim} \begin{pmatrix} & 1 \\ 1 & \end{pmatrix} \begin{pmatrix} \pi^n & \\ & 1 \end{pmatrix} \begin{pmatrix} \pi^{-n} & \\ & \pi^{-n} \end{pmatrix} \begin{pmatrix} & 1 \\ 1 & \end{pmatrix} = \begin{pmatrix} \pi^{-n} & \\ & 1 \end{pmatrix}.$$

Executing these steps (possibly (iii)–(v) several times) will finally lead to a matrix of the form

$$p_n = \begin{pmatrix} \pi^{-n} & \\ & 1 \end{pmatrix}$$

for some $n \geq 0$, and $p_n \sim p_m$ for positive integers n, m if and only if $n = m$, as can be proven by geometric methods (Example 5.4.11 and [60, Example 2.4.1]) or by more tedious elementary methods. We denote the classes $[p_n]$ by c_{nx} (with nx considered as divisor, cf. 5.2.2) and derive:

4.3.3 Proposition. *For* $\Phi \in \mathcal{H}_K$, $\quad$ Vert $\mathcal{G}_{\Phi,K} = \{c_{nx}\}_{n\geq 0}$.

4.3.4 Example (Graph of 0 **and** 1**).** Following Proposition 4.1.7, the graphs for the zero element 0 and the identity 1 in $\mathcal{H}_K$ are given in Figures 4.1 and 4.2, respectively.

Figure 4.1: The graph of the zero element in $\mathcal{H}_K$

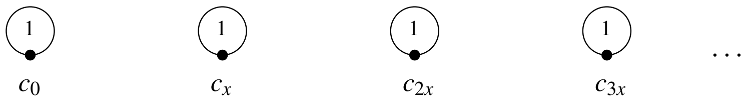


Figure 4.2: The graph of the identity in $\mathcal{H}_K$

4.3.5 Example (Graph of Φ_x**).** Let ξ_w be as in Lemma 4.2.3. We are only concerned with the x-component of ξ_w, which we shall also denote by the symbol ξ_w in this example. Proposition 4.2.4 describes the edges, and the reduction steps (i)–(vi) in paragraph 4.3.2 describe how to find the standard representative p_i for the class of $p_j\xi_w$:

- For $i \geq 0$ and $w = [0:1]$,

$$p_i\xi_{[0:1]} = \begin{pmatrix} \pi^{-i} & \\ & 1 \end{pmatrix}\begin{pmatrix} 1 & \\ & \pi \end{pmatrix} \underset{(i)}{\sim} \begin{pmatrix} \pi^{-(i+1)} & \\ & 1 \end{pmatrix} = p_{i+1}\ .$$

- For $i = 0$ and $w = [1:b_0]$ with $b_0 \in \mathbf{F}_q$,

$$p_0\xi_w = \begin{pmatrix} \pi & b_0 \\ & 1 \end{pmatrix} \underset{(iv)}{\sim} \begin{pmatrix} \pi & \\ & 1 \end{pmatrix} \underset{(vi)}{\sim} p_1\ .$$

- For $i \geq 1$ and $w = [1:b_0]$ with $b_0 \in \mathbf{F}_q$,

$$p_i\xi_w = \begin{pmatrix} \pi^{-i} & \\ & 1 \end{pmatrix}\begin{pmatrix} \pi & b_0 \\ & 1 \end{pmatrix} \underset{(iv)}{\sim} \begin{pmatrix} \pi^{-(i-1)} & \\ & 1 \end{pmatrix} = p_{i-1}\ .$$

We conclude that $c_0 = c_{0x}$ is connected to $c_x = c_{1x}$ with multiplicity $q+1$, and for positive n, c_{nx} is connected to $c_{(n-1)x}$ with multiplicity q and to $c_{(n+1)x}$ with multiplicity 1 . Thus $\mathcal{G}_x$ can be illustrated as in Figure 4.3.

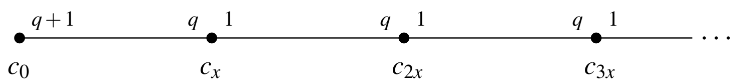


Figure 4.3: The graph of Φ_x

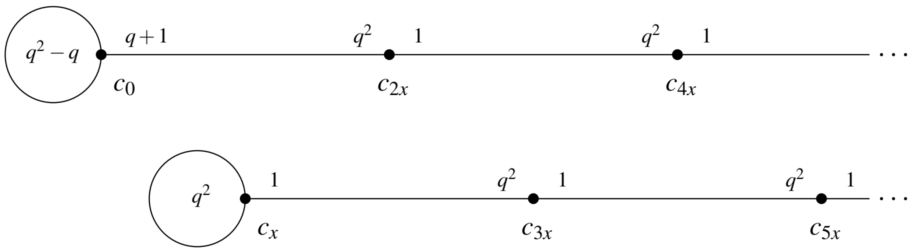


Figure 4.4: The graph of Φ_y for a place y of degree 2

4.3.6 Example (Graph of Φ_y for $y \neq x$). If we want to determine the edges of $\mathcal{G}_y$ for a place y of degree d that differs from x, we have to find the standard representative p_j for elements

$$p_i \begin{pmatrix} \pi_y & b \\ & 1 \end{pmatrix} \quad \text{with } b \in \kappa_y, \text{ and} \quad p_i \begin{pmatrix} 1 & \\ & \pi_y \end{pmatrix}.$$

As F has class number 1, we can assume that $\pi_y \in F$ has nontrivial valuation in y and x only. Let $\gamma \in G_F$ denote the inverse of one of the matrices $\left(\begin{smallmatrix} \pi_y & b \\ & 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} 1 & \\ & \pi_y \end{smallmatrix}\right)$. For all places $z \neq x, y$, the canonical embedding $G_F \to G_z$ sends γ to a matrix $\gamma_z \in K_z$ since $v_z(\pi_y) = 0$ by assumption. Thus multiplying with $\gamma \in G_F$ from the left, which operates diagonally on the components of all places, and multiplying componentwise with $\gamma_z^{-1} \in K_z$ from the right for all $z \neq x, y$, gives an element that is nontrivial only in x (also compare with [23, Lemma 3.7]). The matrices that we obtain in this way are:

$$\begin{pmatrix} \pi_x^d & b_0 + \cdots + b_{d-1}\pi_x^{d-1} \\ & 1 \end{pmatrix} p_i \quad \text{with } b_i \in \kappa_x \text{ for } i = 0, \ldots, d-1, \text{ and} \quad \begin{pmatrix} 1 & \\ & \pi_x^d \end{pmatrix} p_i \,.$$

The reduction steps (i)–(vi) of paragraph 4.3.2 tell us which classes are represented, and we are able to determine the edges similarly to the previous example. Thus we obtain that $\mathcal{G}_y$ only depends on the degree of y. Note that if y is of degree 1, then $\mathcal{G}_y$ equals $\mathcal{G}_x$. Figures 4.4, 4.5, 4.10, and 4.13 show the graphs for degrees 2, 3, 4 and 5, respectively.

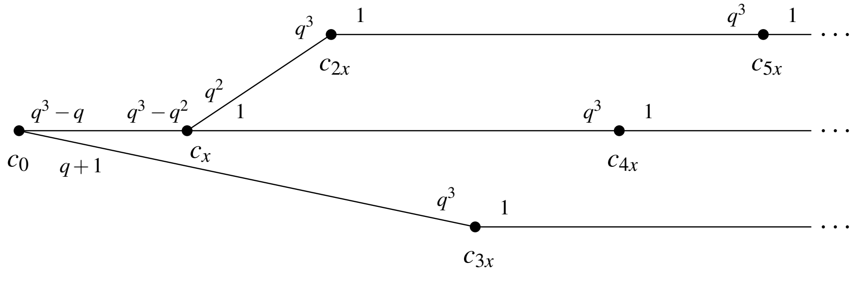


Figure 4.5: The graph of Φ_y for a place y of degree 3

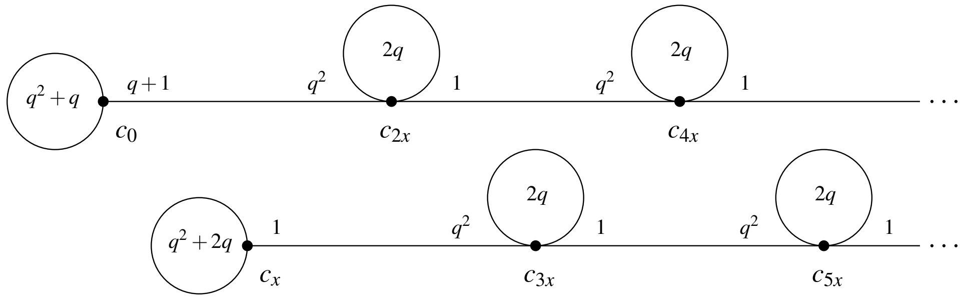


Figure 4.6: The graph of Φ_x^2

4.3.7 Example (The graph of powers of Φ_x). It is interesting to compare the graph of Φ_y with $\deg y = d$ with the graph of Φ_x^d. The latter graph is easily deduced from $\mathcal{G}_x$ by means of Proposition 4.1.7. Namely, a vertex v' is a Φ_x^d-neighbour of a vertex v in $\mathcal{G}_{\Phi_x^d,K}$ if there is a path of length d from v to v' in $\mathcal{G}_x$, i.e. a sequence $(v_0, v_1, \ldots, v_d)$ of vertices in $\mathcal{G}_x$ with $v_0 = v$ and $v_d = v'$ such that for all $i = 1, \ldots, d$, there is an edge (v_{i-1}, v_i, m_i) in $\mathcal{G}_x$. The weight of an edge from v to v' in the graph of $\mathcal{G}_x^d$ is obtained by taking the sum of the products $m_1 \cdot \ldots \cdot m_d$ over all paths of length d from v to v' in $\mathcal{G}_x$.

Figure 4.6 and 4.7 show the graphs of Φ_x^2 and Φ_x^3, respectively, and we see that for $\deg y = 2$, we have $\Phi_x^2 = \Phi_y + 2q \cdot 1$, and for $\deg y = 3$, we have $\Phi_x^3 = \Phi_y + 3q \cdot \Phi_x$.

4.3.8 Remark. In these first examples, we saw graphs that contain a finite subgraph that is irregular such that the complement follows a regular pattern that repeats periodically. This behaviour is common to all graphs of Hecke operators, and when we illustrate a graph, we will always picture the irregular part and at least one complete period. The geometric methods in the next chapter will give an explanation for this periodical behaviour of graphs of Hecke operators relative to K.

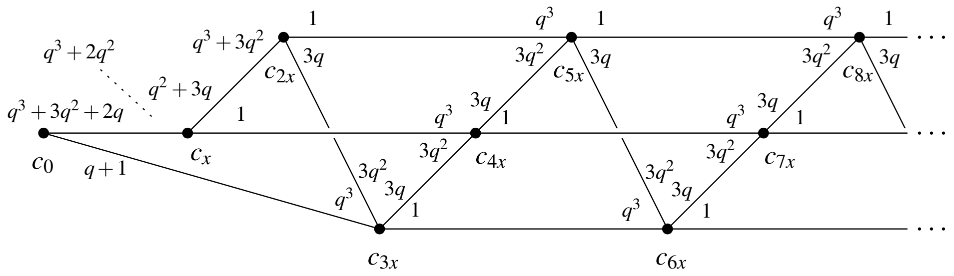


Figure 4.7: The graph of Φ_x^3

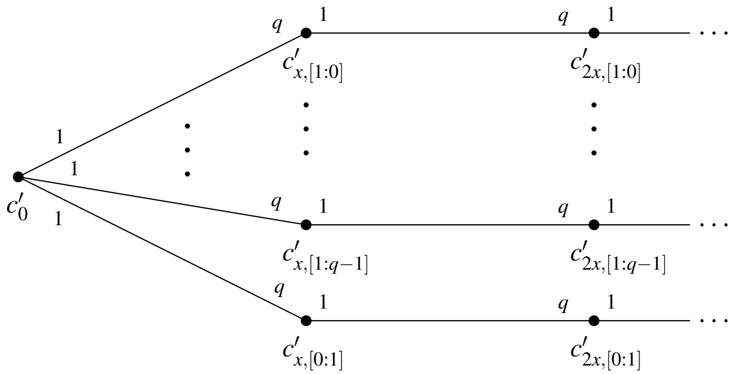


Figure 4.8: Graph of $\Phi'_{y,e}$ as defined in Example 4.3.9

4.3.9 Example (The graphs of two ramified Hecke operators). It is also possible to determine examples for Hecke operators in $\mathcal{H}_{K'}$ by elementary matrix manipulations, when $K' < K$ is a subgroup of finite index. We will show two examples, which are illustrated in Figures 4.8 and 4.9. We omit the calculation, but only point out why the crucial differences between the two graphs occur.

For $K' = \{k \in K \mid k_x \equiv \left(\begin{smallmatrix}1 & \\ & 1\end{smallmatrix}\right) \pmod{\pi_x}\}$, the fibres of the projection

$$P : G_F \backslash G_{\mathbf{A}} / Z_{\mathbf{A}} K' \longrightarrow G_F \backslash G_{\mathbf{A}} / Z_{\mathbf{A}} K$$

are given by $P^{-1}(c_0) = \{[p_0]\}$ and for positive n, by $P^{-1}(c_{nx}) = \{[p_{nx}\vartheta_w]\}_{w \in \mathbf{P}^1(\kappa_x)}$ with $\vartheta_{[1:c]} = \left(\begin{smallmatrix}1 & c \\ & 1\end{smallmatrix}\right)$ and $\vartheta_{[0:1]} = \left(\begin{smallmatrix} & 1 \\ 1 & \end{smallmatrix}\right)$. The union of these fibres equals the set of vertices of a Hecke operator in $\mathcal{H}_{K'}$. We shall denote the vertices by $c'_0 = [p_0]$ and $c'_{nx,w} = [p_{nx}\vartheta_w]$ for $n \geq 1$ and $w \in \mathbf{P}^1(\kappa_x)$. Note that $G_{\mathbf{F}_q} = G_{\kappa_x}$ acts on $\mathbf{P}^1(\kappa_x)$ from the right, so if $\gamma \in G_{\mathbf{F}_q}$, then $w \mapsto w\gamma$ permutes the elements of $\mathbf{P}^1(\kappa_x)$.

The first Hecke operator $\Phi'_{y,\gamma} \in \mathcal{H}_{K'}$ that we consider is $(\operatorname{vol} K / \operatorname{vol} K')$ times the characteristic function of $K'\left(\begin{smallmatrix}\pi_y & \\ & 1\end{smallmatrix}\right)\gamma K'$, where y is a degree one place different to x and $\gamma \in G_{\mathbf{A}}$ is a matrix whose only nontrivial component is $\gamma_x \in G_{\mathbf{F}_q}$. (The factor $(\operatorname{vol} K / \operatorname{vol} K')$ is only included to obtain integer weights). Since $K'\left(\begin{smallmatrix}\pi_y & \\ & 1\end{smallmatrix}\right)\gamma K' \subset K\left(\begin{smallmatrix}\pi_y & \\ & 1\end{smallmatrix}\right)\gamma K$, the graph of $\Phi'_{y,\gamma}$ relative to K' can have an edge from v to w only if $\mathcal{G}_y$ has an edge from $P(v)$ to $P(w)$. Because $K'_y = K_y$, we argue as for K that $K'\left(\begin{smallmatrix}\pi_y & \\ & 1\end{smallmatrix}\right)\gamma K' = \coprod_{w \in \mathbf{P}^1(\kappa_y)} \xi_w \gamma K'$. Applying the same methods as in Example 4.3.6, one obtains that

$$\mathcal{U}_{\Phi'_{y,\gamma},K'}(c'_0) \;=\; \{(c'_0, c'_{x,w}, 1)\}_{w \in \mathbf{P}^1(\kappa_x)}$$

and for every $n \geq 1$ and $w \in \mathbf{P}^1(\kappa_x)$ that

$$\mathcal{U}_{\Phi'_{y,\gamma},K'}(c'_{nx,w}) \;=\; \{(c'_0, c'_{(n+1)x,w\gamma}, 1), (c'_0, c'_{(n-1)x,w\gamma}, q)\}\,.$$

For the case that γ equals the identity matrix e, the graph is illustrated in Figure 4.8. Note that for general γ, an edge does not necessarily have an inverse edge since $w\gamma^2$ does not have to equal w.

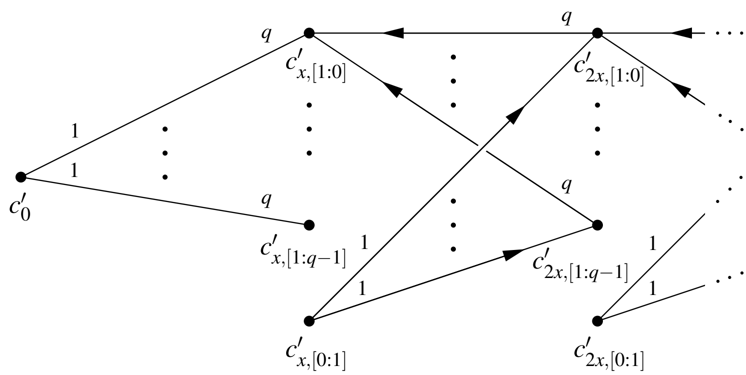


Figure 4.9: Graph of Φ'_x as defined in Example 4.3.9

The second Hecke operator $\Phi'_x \in \mathcal{H}_{K'}$ is $(\operatorname{vol} K / \operatorname{vol} K')$ times the characteristic function of $K'\begin{pmatrix}\pi_x & \\ & 1\end{pmatrix}K'$. This case behaves differently, since K'_x and K_x are not equal: We have $K'\begin{pmatrix}\pi_x & \\ & 1\end{pmatrix}K' = \coprod_{b\in\kappa_x}\begin{pmatrix}\pi_x & b\pi_x \\ & 1\end{pmatrix}K'$. This allows us to compute the edges as illustrated in Figure 4.9. Note that for $n \geq 1$, the vertices of the form $c'_{nx,[1:0]}$ and $c'_{nx,[0:1]}$ behave particularly.

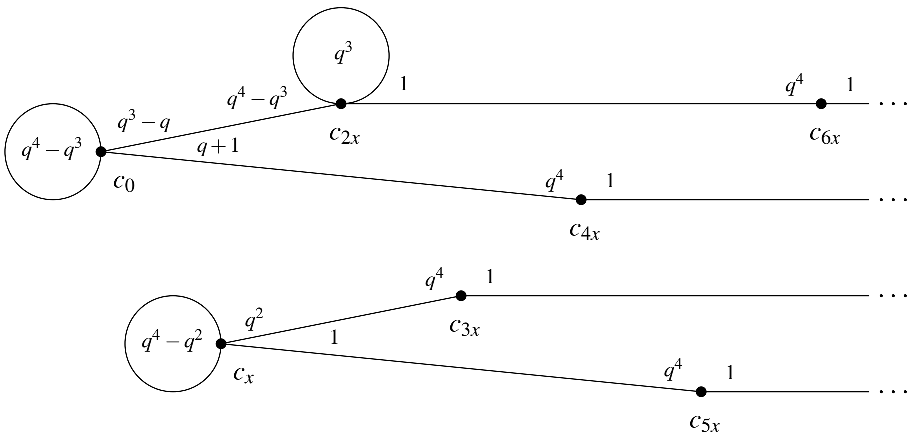


Figure 4.10: The graph of Φ_y for a place y of degree 4

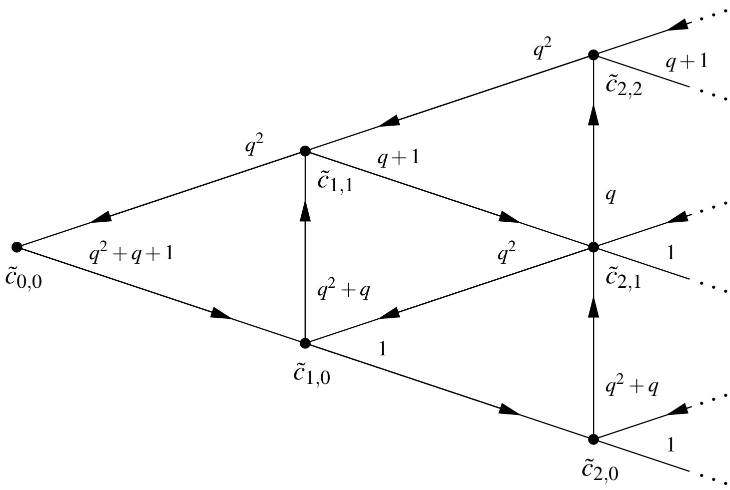


Figure 4.11: Graph of $\Phi_{x,1}$ for GL_3

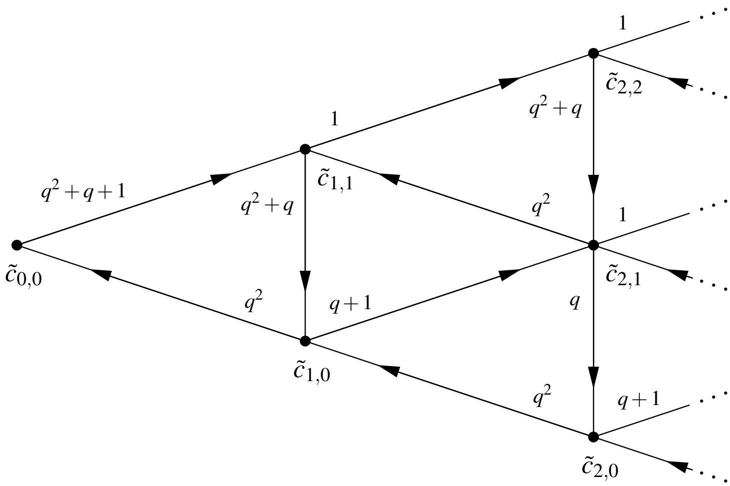


Figure 4.12: Graph of $\Phi_{x,2}$ for GL_3

4.3.10 Example (Two graphs for GL_3**).** There is a generalisation of the notion of the graph of a Hecke operator to other algebraic groups. We carry out two examples for $\widetilde{G} = \mathrm{GL}_3$. The notions of automorphic forms and Hecke operators as given in Chapter 1 transfer literally to GL_3. Let $\widetilde{K} = \widetilde{G}_{\mathcal{O}_\mathbf{A}}$ be the standard maximal compact subgroup of $\widetilde{G}_\mathbf{A}$, $\widetilde{Z} < \widetilde{G}$ the centre and $\widetilde{\mathcal{H}}$ the Hecke algebra. The elements of $\widetilde{G}_F \backslash \widetilde{G}_\mathbf{A} / \widetilde{Z}_\mathbf{A} \widetilde{K}$ are the double cosets

$$\tilde{c}_{i,j} = \widetilde{G}_F \begin{pmatrix} \pi^{-i} & & \\ & \pi^{-j} & \\ & & 1 \end{pmatrix} \widetilde{Z}_\mathbf{A} \widetilde{K}$$

for $i \geq j \geq 0$. The role of Φ_x is played by the two elements

$$\widetilde{\Phi}_{x,1} = \mathrm{char}_{\widetilde{K}\begin{pmatrix} \pi & & \\ & 1 & \\ & & 1 \end{pmatrix}\widetilde{K}} \qquad \text{and} \qquad \widetilde{\Phi}_{x,2} = \mathrm{char}_{\widetilde{K}\begin{pmatrix} \pi & & \\ & \pi & \\ & & 1 \end{pmatrix}\widetilde{K}}$$

Proposition 4.2.4 generalises to GL_3 for certain $\tilde{\xi}_w$, but the index w runs over the Grassmannian $\mathbf{Gr}_{1,3}(\kappa_x)$ for $\widetilde{\Phi}_{x,1}$ and over $\mathbf{Gr}_{2,3}(\kappa_x)$ for $\widetilde{\Phi}_{x,2}$. The reduction steps (i)–(vi) of paragraph 4.3.2 also generalise, allowing us to calculate the graphs of $\widetilde{\Phi}_{x,1}$ and $\widetilde{\Phi}_{x,2}$ in the same way as we did for Φ_x in Example 4.3.5. The result is shown in Figures 4.11 and 4.12. It is interesting to remark that the duality between $\mathbf{Gr}_{1,3}$ and $\mathbf{Gr}_{2,3}$ is reflected in the graphs.

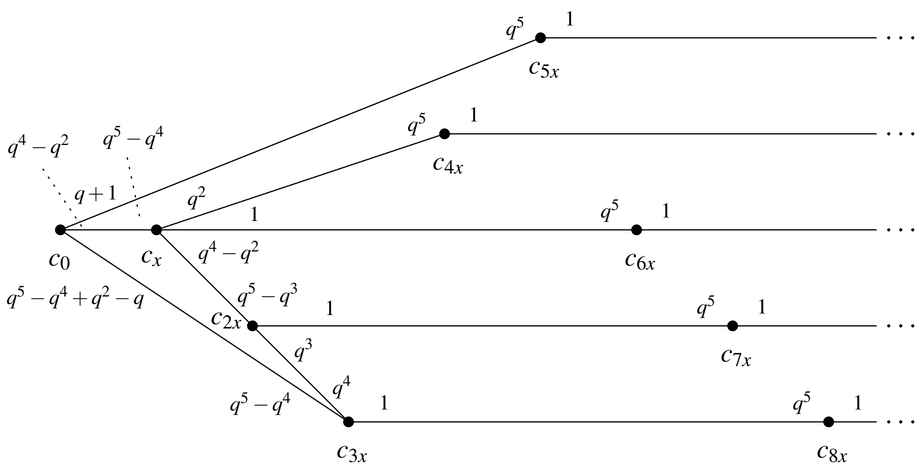


Figure 4.13: The graph of Φ_y for a place y of degree 5

4.4 Connection with Bruhat-Tits trees

Let x be a place. In this section we define maps from so-called Bruhat-Tits trees to $\mathcal{G}_x$. This will enable us to determine the components of $\mathcal{G}_x$. We let ξ_w be as in Proposition 4.2.3.

4.4.1 Definition. The *Bruhat-Tits tree* $\mathcal{T}_x$ for F_x has vertices

$$\mathrm{Vert}\,\mathcal{T}_x \;=\; G_x\,/\,K_x Z_x$$

and edges

$$\mathrm{Edge}\,\mathcal{T}_x \;=\; \{\, ([g],[g']) \mid \exists w \in \mathbf{P}^1(\kappa_x),\; g \equiv g'\xi_w \pmod{K_x Z_x} \,\}\,.$$

4.4.2 For each $h \in G_{\mathbf{A}}$, we define a map

$$\Psi_{x,h} : \mathcal{T}_x \longrightarrow \mathcal{G}_x$$

by

$$\begin{array}{ccc} \mathrm{Vert}\,\mathcal{T}_x = G_x\,/\,K_x Z_x & \longrightarrow & G_F \backslash G_{\mathbf{A}}\,/\,K Z_{\mathbf{A}} = \mathrm{Vert}\,\mathcal{G}_x \\ {[g]} & \longmapsto & [hg] \end{array}$$

and

$$\begin{array}{ccc} \mathrm{Edge}\,\mathcal{T}_x & \longrightarrow & \mathrm{Edge}\,\mathcal{G}_x \\ ([g],[g']) & \longmapsto & ([hg],[hg'],m) \end{array}$$

with m being the number of vertices $[g'']$ that are adjacent to $[g]$ in $\mathcal{T}_x$ such that $\Psi_{x,h}([g'']) = \Psi_{x,h}([g'])$.

By Proposition 4.2.4 and the definition of a Bruhat-Tits tree, $\Psi_{x,h}$ is well-defined and *locally surjective*, i.e. it is locally surjective as map between the associated simplicial complexes of $\mathcal{T}_x$ and $\mathcal{G}_x$ with suppressed weights.

To explain this in more detail: The associated simplicial complex gives the notion of a component. Two vertices lie in the same *component*, if there is a sequence of vertices beginning with the one and ending with the other vertex in question such that each two consecutive vertices in this sequence are connected by an edge. Edges lie in the component of their origin. A map is locally surjective if for each vertex or edge in the image of that map, every other vertex and edge of the corresponding component also lies in the image.

Since Bruhat-Tits trees are indeed trees ([60, II.1, Thm. 1]), hence in particular connected, the image of each $\Psi_{x,h}$ is precisely one component of $\mathcal{G}_x$.

4.4.3 Proposition. *If* $(v,v',m) \in \mathrm{Edge}\,\mathcal{G}_x$*, then there is a* $m' \in \mathbf{C}^\times$ *such that* $(v',v,m') \in \mathrm{Edge}\,\mathcal{G}_x$.

Proof. Let $(v,v',m) \in \mathrm{Edge}\,\mathcal{G}_x$, and let $h \in G_{\mathbf{A}}$ represent $v = [h]$. Since $\Psi_{x,h}$ is locally surjective and $\Psi_{x,h}([e]) = v$ if e is the identity matrix, it is enough to show that for every edge $(1,w)$, there is also the edge $(w,1) \in \mathrm{Edge}\,\mathcal{T}_x$.

But this follows from

$$\xi_{[1:b]}\xi_{[0:1]} = \begin{pmatrix} \pi_x & b \\ & 1 \end{pmatrix}\begin{pmatrix} 1 & \\ & \pi_x \end{pmatrix} = \begin{pmatrix} \pi_x & b\pi_x \\ & \pi_x \end{pmatrix} \equiv \begin{pmatrix} 1 & \\ & 1 \end{pmatrix} \pmod{K_x Z_x}$$

and

$$\xi_{[0:1]}\xi_{[1:0]} = \begin{pmatrix} 1 & \\ & \pi_x \end{pmatrix}\begin{pmatrix} \pi_x & \\ & 1 \end{pmatrix} = \begin{pmatrix} \pi_x & \\ & \pi_x \end{pmatrix} \equiv \begin{pmatrix} 1 & \\ & 1 \end{pmatrix} \pmod{K_x Z_x} .$$

□

4.4.4 Remark. This symmetry of edges is a property that is special to unramified Hecke operators for $G = GL_2$. In case of ramification, the symmetry is broken, cf. Example 4.3.9. For other algebraic groups, even unramified Hecke operators occur that have edges without an inverse edge, cf. Example 4.3.10.

4.4.5 Theorem ([39, Thm. E.2.1]). SL_2 *has the strong approximation property, i.e. for every place* x, $\mathrm{SL}_2 F$ *is a dense subset of* $\mathrm{SL}_2 \mathbf{A}^x$ *with respect to the adelic topology.*

This theorem was first proven by Martin Kneser ([33], 1965) for number fields and extended independently by Gopal Prasad ([52], 1977) and Gregory Margulis ([46], 1977) to global fields.

4.4.6 Lemma.

$$G_F \backslash G^x / K^x \xrightarrow{\det} F^\times \backslash (\mathbf{A}^x)^\times / (\mathcal{O}^x)^\times$$

is bijective.

Proof. For surjectivity, we observe that for each $a \in (\mathbf{A}^x)^\times$,

$$\det \begin{pmatrix} a & \\ & 1 \end{pmatrix} = a .$$

Fix an arbitrary $a \in (\mathbf{A}^x)^\times$. For injectivity, we have to show that each $g \in G^x$ with $\det g = a$ represents the same class as $\left(\begin{smallmatrix} a & \\ & 1 \end{smallmatrix}\right)$ in $G_F \backslash G^x / K^x$. Since

$$\det \left(\begin{pmatrix} a & \\ & 1 \end{pmatrix} g^{-1} \right) = 1 ,$$

$\left(\begin{smallmatrix} a & \\ & 1 \end{smallmatrix}\right) g^{-1} \in \mathrm{SL}_2 \mathbf{A}^x$. We put $D_y = |v_y(a)|$ for all $y \neq x$, and choose K' to be the collection of all elements $k' \in \mathrm{SL}_2 \mathbf{A}^x$ such that for all $y \in |X|$, we have $k'_y \equiv \left(\begin{smallmatrix} 1 & \\ & 1 \end{smallmatrix}\right) \pmod{\mathfrak{m}_y^{D_y} \mathcal{O}_y}$, which is an open subgroup of $\mathrm{SL}_2 \mathbf{A}^x$. For all $k' \in K'$, we have $k = \left(\begin{smallmatrix} a^{-1} & \\ & 1 \end{smallmatrix}\right) k' \left(\begin{smallmatrix} a & \\ & 1 \end{smallmatrix}\right) \in K^x$.

By the strong approximation property, there is a $\gamma \in \mathrm{SL}_2 F \cap \left(K' \left(\begin{smallmatrix} a & \\ & 1 \end{smallmatrix}\right) g^{-1}\right) \subset G_F$. Thus, we can find a $k' \in K'$ such that

$$k' \begin{pmatrix} a & \\ & 1 \end{pmatrix} g^{-1} = \gamma .$$

With $k \in K^x$ as above this gives

$$\begin{pmatrix} a & \\ & 1 \end{pmatrix} k = \gamma g \,.$$

In other words, $[g] = \left[\begin{pmatrix} a & \\ & 1\end{pmatrix}\right]$ in $G_F \backslash G^x / K^x$. □

4.4.7 Let x be a place of degree d. Since $\det : G^x \to (\mathbf{A}^x)^\times$ is a group homomorphism, and $F^\times \backslash (\mathbf{A}^x)^\times / (\mathcal{O}^x)^\times$ is a group, the bijection of the lemma defines a group structure on $G_F \backslash G^x / K^x$ which coincides with the quotient product structure inherited from G^x, even though neither $G_F \backslash G^x$ nor G^x / K^x is a group.

The quotient group $F^\times \backslash (\mathbf{A}^x)^\times / (\mathcal{O}^x)^\times$ is nothing else but the class group of the *integers $\mathcal{O}_F^x = \bigcap_{y \neq x} (\mathcal{O}_y \cap F)$ coprime to x*. Thus we have isomorphisms of groups

$$G_F \backslash G^x / K^x \;\simeq\; F^\times \backslash (\mathbf{A}^x)^\times / (\mathcal{O}^x)^\times \;\simeq\; \mathrm{Cl}\, \mathcal{O}_F^x \;\simeq\; \mathrm{Cl}^0 F \times \mathbf{Z}/d\mathbf{Z} \,.$$

Let $S \subset G^x$ be a set of representatives for $G_F \backslash G^x / K^x$. Then Lemma 4.4.6 implies that for every $g \in G_\mathbf{A}$, which can be written as $g = g^x g_x$ with $g^x \in G^x$ and $g_x \in G_x$, there are $s \in S$, $\gamma \in G_F$ and $k \in K^x$ such that $g = \gamma s k \tilde{g}_x$ such that $\gamma s k$ equals g in all components $z \neq x$ and $\tilde{g}_x = \gamma^{-1} g_x$. The condition $[\det s] = [\det g^x]$ as cosets in $G_F \backslash G^x / K^x$ implies that s is uniquely determined by g^x. Observe that

$$G_\mathbf{A} / K Z_x \;=\; \big(G^x / K^x\big) \times \big(G_x / K_x Z_x\big) \;=\; \big(G^x / K^x\big) \times \mathrm{Vert}\, \mathcal{T}_x \,,$$

and define $\Gamma_s = G_F \cap s K^x s^{-1}$. Then we obtain the following, also cf. [53, (2.1.3)].

4.4.8 Proposition. *The decomposition $g = \gamma s k \tilde{g}_x$ induces a bijective map*

$$\begin{array}{ccc} G_F \backslash G_\mathbf{A} / K Z_x & \longrightarrow & \coprod_{s \in S} \Gamma_s \backslash \mathrm{Vert}\, \mathcal{T}_x \,. \\ [g] & \longmapsto & (s, [\tilde{g}_x]) \end{array}$$

Its inverse is obtained by putting together the components $s \in G^x$ and $\tilde{g}_x \in G_x$. □

4.4.9 Remark. On the right hand side of the bijection in Proposition 4.4.8, we have a finite union of quotients of the form $\Gamma_s \backslash \mathrm{Vert}\, \mathcal{T}_x$. If s is the identity element e, then $\Gamma = \Gamma_e = G_{\mathcal{O}_F^x}$ is an arithmetic group of the form that Serre considers in [60, II.2.3]. For general s, we are not aware of any results about $\Gamma_s \backslash \mathrm{Vert}\, \mathcal{T}_x$ in the literature.

4.4.10 So far, we have only divided out the action of the x-component Z_x of the centre. We still have to consider the action of Z^x. If we restrict the determinant map to the centre and write $J = \{z \in Z_F \backslash Z^x / Z_{\mathcal{O}^x} \mid |\det z| = 1\}$, then we have an exact sequence of abelian groups

$$1 \;\to\; J \;\to\; Z_F \backslash Z^x / Z_{\mathcal{O}^x} \;\xrightarrow{\det}\; \mathrm{Cl}\, \mathcal{O}_F^x \;\to\; \mathrm{Cl}\, \mathcal{O}_F^x / 2\, \mathrm{Cl}\, \mathcal{O}_F^x \;\to\; 0 \,.$$

Let S be as in paragraph 4.4.7. The action of Z^x on S factors through $2\,\mathrm{Cl}\, \mathcal{O}_F^x$ and the action of Z^x on $\Gamma_s \backslash \mathrm{Vert}\, \mathcal{T}_x$ factors through J for each $s \in S$. If we let $S' \subset G^x$ be a set of representatives for $\mathrm{Cl}\, \mathcal{O}_F^x / 2\,\mathrm{Cl}\, \mathcal{O}_F^x$, and $h_2 = \#(\mathrm{Cl}\, F)[2]$ the cardinality of the 2-torsion, then we obtain:

4.4.11 Proposition. *The decomposition $g = \gamma s k \tilde{g}_x$ induces a bijective map*

$$G_F \backslash G_{\mathbf{A}} / K Z_{\mathbf{A}} \longrightarrow \coprod_{s \in S'} J\, \Gamma_s \backslash \operatorname{Vert} \mathcal{T}_x \,.$$

The inverse maps an element $(s, [\tilde{g}_x])$ to the class of the adelic matrix with components $s \in G^x$ and $\tilde{g}_x \in G_x$. The number of components of $\mathcal{G}_x$ equals

$$\#\big(\operatorname{Cl} \mathcal{O}_F^x / 2 \operatorname{Cl} \mathcal{O}_F^x\big) = \#(\operatorname{Cl} O_F^x)[2] = \begin{cases} h_2 & \textit{if } \deg x \textit{ is odd,} \\ 2h_2 & \textit{if } \deg x \textit{ is even.} \end{cases}$$

Proof. Everything follows from Proposition 4.4.8 and paragraph 4.4.10 except for the two equalities. Regarding the former, observe that both dividing out the squares and taking 2-torsion commutes with products, so by the structure theorem of finite abelian groups, we can reduce the proof to groups of the form $\mathbf{Z}/\tilde{p}^m\mathbf{Z}$ with $\tilde{p}$ prime. If $\tilde{p} \neq 2$, then every element is a square and there is no 2-torsion, hence the equality holds. If $\tilde{p} = 2$, then $\mathbf{Z}/\tilde{p}^m\mathbf{Z}$ modulo squares has one nontrivial class, and there is exactly one nontrivial element in $\mathbf{Z}/\tilde{p}^m\mathbf{Z}$ that is 2-torsion.

Regarding the latter equality, recall that $\operatorname{Cl} \mathcal{O}_F^x \simeq \operatorname{Cl}^0 F \times \mathbf{Z}/d\mathbf{Z}$, where $d = \deg x$. As above, $\mathbf{Z}/d\mathbf{Z}$ modulo squares has a nontrivial class if and only if d is even, and in this case there is only one such class. □

4.5 A vertex labelling

Let $\mathcal{Q}_{\mathbf{A}} = \langle a^2 \,|\, a \in \mathbf{A}^\times \rangle$ be the subgroup of squares. We look once more at the determinant map

$$\operatorname{Vert} \mathcal{G}_x = G_F \backslash G_{\mathbf{A}} / K Z_{\mathbf{A}} \xrightarrow{\det} F^\times \backslash \mathbf{A}^\times / \mathcal{O}_{\mathbf{A}}^\times \mathcal{Q}_{\mathbf{A}} \simeq \operatorname{Cl} F / 2 \operatorname{Cl} F \,.$$

This map assigns to every vertex in $\mathcal{G}_x$ a label in $\operatorname{Cl} F / 2\operatorname{Cl} F$. Let h_2 be as in Proposition 4.4.11. Observe that $\operatorname{Cl} F / 2 \operatorname{Cl} F$ has $2h_2$ elements, since the elements of even degree in $\operatorname{Cl} F$ are precisely the inverse image of $\operatorname{Cl}^0 F / 2 \operatorname{Cl}^0 F$, whose order is h_2.

4.5.1 Proposition. *If the prime divisor x is a square in the divisor class group then all vertices in the same component of $\mathcal{G}_x$ have the same label, and there are $2h_2$ components, each of which has a different label. Otherwise, the vertices of each component have one of two labels that differ by x in $\operatorname{Cl} F / 2 \operatorname{Cl} F$, and two adjacent vertices have different labels, so each connected component is bipartite.*

Proof. First of all, observe that each label is realised, since if we represent a label by some idele a, then the vertex represented by $\begin{pmatrix} a & \\ & 1 \end{pmatrix}$ has this label.

Let $\mathcal{Q}_x = \langle b^2 \mid b \in F_x^\times \rangle$ and $\operatorname{Cl} F_x = F_x^\times / \mathcal{O}_x^\times$, a group isomorphic to $\mathbf{Z}$. For the Bruhat-Tits tree $\mathcal{T}_x$, the determinant map

$$\operatorname{Vert} \mathcal{T}_x = G_x / K_x Z_x \xrightarrow{\det} F_x^\times / \mathcal{O}_x^\times \mathcal{Q}_x \simeq \operatorname{Cl} F_x / 2 \operatorname{Cl} F_x \simeq \mathbf{Z}/2\mathbf{Z}$$

defines a labelling of the vertices, and the two classes of $F_x^\times / \mathcal{O}_x^\times \mathcal{Q}_x$ are represented by 1 and π_x. Two adjacent vertices have the different labels since for $g \in G_x$ and ξ_w as in Definition 4.4.1, $\det(g\xi_w) = \pi_x \det g$ represents a class different from $\det g$ in Vert $\mathcal{T}_x$.

Define for $a \in \mathbf{A}^\times$ a map $\psi_{x,a} : F_x^\times / \mathcal{O}_x^\times \mathcal{Q}_x \to F^\times \backslash \mathbf{A}^\times / \mathcal{O}_\mathbf{A}^\times \mathcal{Q}_\mathbf{A}$ by $\psi_{x,a}([b]) = [ab]$, where b is viewed as the idele concentrated in x. For every $h \in G_\mathbf{A}$ we obtain a commutative diagram

$$\begin{array}{ccccccc}
\mathrm{Vert}\,\mathcal{T}_x & = & G_x / K_x Z_x & \xrightarrow{\Psi_{x,h}} & G_F \backslash G_\mathbf{A} / K Z_\mathbf{A} & = & \mathrm{Vert}\,\mathcal{G}_x \\
\downarrow & & \downarrow \scriptstyle\det & & \downarrow \scriptstyle\det & & \downarrow \\
\mathrm{Cl}\, F_x / 2\,\mathrm{Cl}\, F_x & \simeq & F_x^\times / \mathcal{O}_x^\times \mathcal{Q}_x & \xrightarrow{\psi_{x,\det h}} & F^\times \backslash \mathbf{A}^\times / \mathcal{O}_\mathbf{A}^\times \mathcal{Q}_\mathbf{A} & \simeq & \mathrm{Cl}\, F / 2\,\mathrm{Cl}\, F\,.
\end{array}$$

This means that vertices with equal labels map to vertices with equal labels.

Each component of $\mathcal{G}_x$ lies in the image of a suitable $\Psi_{x,h}$, thus has at most two labels. On the other hand, the two labels of $\mathcal{T}_x$ map to $\psi_{x,\det h}([1]) = [a]$ and $\psi_{x,\det h}([\pi_x]) = [a\pi_x]$. The divisor classes of $[a]$ and $[a\pi_x]$ differ by the class of the prime divisor x, and are equal if and only if x is a square in the divisor class group. If so, according to Proposition 4.4.11, there must be $2h_2$ components so that the $2h_2$ labels are spread over all components. If x is not a square then by the local surjectivity of $\Psi_{x,h}$ on edges two adjacent vertices of $\mathcal{G}_x$ also have different labels. $\square$

Geometry of Hecke operators

A global field of positive characteristic can be interpreted as the function field of a curve over a finite field. This provides the theory of automorphic forms over global function fields with a geometrical meaning. The domain of an unramified automorphic form translates to isomorphism classes of projective line bundles over the curve and the action of a Hecke operator can be described by certain exact sequences of sheaves on the curve. This approach allows us to apply methods from algebraic geometry, which lead to a complete description of the graph of a Hecke operator up to a finite subgraph.

5.1 Geometric description of unramified Hecke operators

Let $\mathcal{G}_x$ be as in Section 4.2. We will give a brief introduction to the geometric concepts needed for a description of $\mathcal{G}_x$.

5.1.1 For each global function field F with constants $\mathbf{F}_q$, there is, up to isomorphism, precisely one geometrically irreducible smooth projective curve X over $\mathbf{F}_q$ whose function field is isomorphic to F. One can construct X as follows.

The topological space X^{top} of X consists of all places x of F and a generic point η, where the nontrivial closed sets are finite unions of places. Then we find back $|X|$, which we defined in 1.1.2 as the set of closed points of X^{top}. Define the stalks of the structure sheaf $\mathcal{O}_X$ and their embedding into the generic stalk by

$$\mathcal{O}_{X,x} \;:=\; \mathcal{O}_x \cap F \;\hookrightarrow\; F \;=:\; \mathcal{O}_{X,\eta}\,.$$

Then for an open set $U \subset X^{\mathrm{top}}$,

$$\mathcal{O}_X(U) \;:=\; \bigcap_{x\in U} \mathcal{O}_{X,x} \;=\; \bigcap_{x\in U} (\mathcal{O}_x \cap F) \;\subset\; F\,.$$

Let ω_X be the canonical bundle and let

$$g_X \;= \dim_{\mathbf{F}_q} \Gamma(X,\omega_X) \;=\; \dim_{\mathbf{F}_q} H^1(X,\mathcal{O}_X)$$

be the genus of the curve, which equals the genus g_F of F as defined in paragraph 1.1.5.

We sometimes interchange F and X in our notation, e.g. we write $\mathrm{Cl}\,X$ for $\mathrm{Cl}\,F$, h_X for h_F, or $\mathcal{O}_{F,x}$ for $\mathcal{O}_{X,x}$ etc.

5.1.2 Remark. The usage of the letter $\mathcal{O}$ for both the structure sheaf of X and its stalks as well as for the rings of integers of F, F_x and $\mathbf{A}$ may cause confusion if not read carefully, but different indices avoid ambiguity. There are various relationships between these objects, e.g. $\mathcal{O}_x$ is the completion of $\mathcal{O}_{X,x}$.

5.1.3 We shall consider vector bundles on X to be embedded in the category of sheaves ([28, Ex. II.5.18]). We denote by $\mathrm{Bun}_n X$ the set of isomorphism classes of *rank n bundles* over X and by $\mathrm{Pic}\, X$ the *Picard group*, i.e. the isomorphism classes of *line bundles* together with the tensor product, which turns it into an abelian group. For $\mathcal{L}_1, \mathcal{L}_2 \in \mathrm{Pic}\, X$, we use the shorthand notation $\mathcal{L}_1\mathcal{L}_2$ for $\mathcal{L}_1 \otimes \mathcal{L}_2$. There is a natural action

$$\begin{array}{ccc} \mathrm{Bun}_n X \times \mathrm{Pic}\, X & \longrightarrow & \mathrm{Bun}_n X\,. \\ (\mathcal{M}, \mathcal{L}) & \longmapsto & \mathcal{M} \otimes \mathcal{L} \end{array}$$

Let $\mathbf{P}\mathrm{Bun}_n X$ be the orbit set $\mathrm{Bun}_n X \,/\, \mathrm{Pic}\, X$, which is nothing else but the set of isomorphism classes of $\mathbf{P}^{n-1}$-bundles over X ([28, Ex. II.7.10]). Accordingly we will call elements of $\mathbf{P}\mathrm{Bun}_n X$ *projective space bundles*, or in the case $n = 2$, *projective line bundles*. If we regard the total space of a projective line bundle as a scheme, then we obtain nothing else but a ruled surface, cf. [28, Prop. V.2.2]. Thus $\mathbf{P}\mathrm{Bun}_2 X$ may also be seen as the set of isomorphism classes of ruled surfaces over X.

If two vector bundles $\mathcal{M}_1$ and $\mathcal{M}_2$ are in the same orbit of the action of $\mathrm{Pic}\, X$, we write

$$\mathcal{M}_1 \sim \mathcal{M}_2\,,$$

and say that $\mathcal{M}_1$ and $\mathcal{M}_2$ are *projectively equivalent*. By $[\mathcal{M}] \in \mathbf{P}\mathrm{Bun}_n X$, we mean the class that is represented by the rank n bundle $\mathcal{M}$.

The *determinant* map ([28, Ex. II.6.11])

$$\begin{array}{cccc} \det: & \mathrm{Bun}_n X & \longrightarrow & \mathrm{Pic}\, X \\ & \mathcal{M} & \longmapsto & (n\text{-th exterior power of } \mathcal{M}) \end{array}$$

is multiplicative in exact sequences, i.e. if there is an exact sequence of vector bundles

$$0 \longrightarrow \mathcal{M}' \longrightarrow \mathcal{M} \longrightarrow \mathcal{M}'' \longrightarrow 0\ ,$$

then $\det \mathcal{M} \;=\; \det \mathcal{M}' \otimes \det \mathcal{M}''$.

Taking the *associated line bundle*

$$\begin{array}{ccc} \mathrm{Cl}\, F & \longrightarrow & \mathrm{Pic}\, X \\ [D] & \longmapsto & \mathcal{L}_D \end{array}$$

is an isomorphism of abelian groups ([28, Prop. II.6.13]), which allows us to define the *degree of a vector bundle* by $\deg \mathcal{M} = \deg D$ when $\det \mathcal{M} \simeq \mathcal{L}_D$. If $\mathcal{F}$ is a torsion sheaf, i.e. a coherent sheaf whose stalk at η is zero ([28, Ex. II.6.12]), then one defines its degree by $\deg \mathcal{F} = \sum_{x\in|X|} \dim_{\mathbf{F}_q}(\mathcal{F}_x)$.

The degree is additive in exact sequences of vector bundles, i.e. that for an exact sequence as above, $\deg \mathcal{M} = \deg \mathcal{M}' + \deg \mathcal{M}''$. Additivity holds also if one replaces $\mathcal{M}''$ by a torsion sheaf, see [28, Ex. II.6.10-6.12].

5.1.4 Remark. Note that if $D = x$ is a prime divisor, the notation for the associated line bundle $\mathcal{L}_x$ coincides with the notation for the stalk of $\mathcal{L}$ at x. In order to avoid confusion, we will reserve the notation $\mathcal{L}_x$ strictly for the associated line bundle. In case we have to consider the stalk of a line bundle, we will use a symbol different from $\mathcal{L}$ for the line bundle.

5.1.5 The bijection

$$\begin{array}{ccccccc} F^\times \backslash \mathbf{A}^\times / \mathcal{O}_\mathbf{A}^\times & = & \operatorname{Cl} F & \overset{1:1}{\longleftrightarrow} & \operatorname{Pic} X & = & \operatorname{Bun}_1 X\,, \\ [a] & & & \longmapsto & & & \mathcal{L}_a \end{array}$$

where $\mathcal{L}_a = \mathcal{L}_D$ if D is the divisor determined by a, generalises to all vector bundles as follows, cf. [20, Lemma 3.1] and [22, 2.1].

A rank n bundle $\mathcal{M}$ can be described by choosing bases

$$\mathcal{M}_\eta \overset{\sim}{\longrightarrow} \mathcal{O}_{X,\eta}^n = F^n \qquad \text{and} \qquad \mathcal{M}_x \overset{\sim}{\longrightarrow} \mathcal{O}_{F,x}^n = (\mathcal{O}_x \cap F)^n$$

for all stalks. This gives a diagram

$$\begin{array}{ccc} \mathcal{O}_x^n & \overset{}{\underset{g_x^{-1}}{\hookrightarrow}} & F_x^n \\ \uparrow & & \uparrow \\ \mathcal{O}_{F,x}^n & \underset{g_x^{-1}}{\hookrightarrow} & F^n \\ \sim\uparrow & & \sim\uparrow \\ \mathcal{M}_x & \hookrightarrow & \mathcal{M}_\eta \end{array}$$

for every closed point x, where the matrix $g_x \in \operatorname{GL}_n F$ is determined by the constraint that its inverse describes the unique linear map such that diagram commutes. By the nature of a vector bundle, $g_x \in \operatorname{GL}_n \mathcal{O}_{F,x} \subset \operatorname{GL}_n \mathcal{O}_x$ for almost all places x. In this way, $\mathcal{M}$ defines a class $[g] = [(g_x)] \in \operatorname{GL}_n F \backslash \operatorname{GL}_n \mathbf{A} / \operatorname{GL}_n \mathcal{O}_\mathbf{A}$.

To see that this assignment is well-defined on isomorphism classes of vector bundles, take a vector bundle $\mathcal{M}'$ that is isomorphic $\mathcal{M}$ and suppose that choices of bases for its stalks defines an element $g' = (g'_x) \in \operatorname{GL}_n \mathbf{A}$. An isomorphism $\mathcal{M}' \to \mathcal{M}$ induces isomorphisms of the stalks

$$\mathcal{M}'_\eta \overset{\sim}{\longrightarrow} \mathcal{M}_\eta \qquad \text{and} \qquad \mathcal{M}'_x \overset{\sim}{\longrightarrow} \mathcal{M}_x$$

for all x. Altogether, this fits into a larger diagram for every x:

$$\begin{array}{ccccccc} \mathcal{O}_x^n & \underset{k_x^{-1}}{\overset{\sim}{\longrightarrow}} & \mathcal{O}_x^n & \underset{g_x^{-1}}{\hookrightarrow} & F_x^n & \underset{\gamma}{\overset{\sim}{\longleftarrow}} & F_x^n \\ \uparrow & & \uparrow & & \uparrow & & \uparrow \\ \mathcal{O}_{F,x}^n & \underset{k_x^{-1}}{\overset{\sim}{\longrightarrow}} & \mathcal{O}_{F,x}^n & \underset{g_x^{-1}}{\hookrightarrow} & F^n & \underset{\gamma}{\overset{\sim}{\longleftarrow}} & F^n \\ \sim\uparrow & & \sim\uparrow & & \sim\uparrow & & \sim\uparrow \\ \mathcal{M}'_x & \overset{\sim}{\longrightarrow} & \mathcal{M}_x & \hookrightarrow & \mathcal{M}_\eta & \overset{\sim}{\longleftarrow} & \mathcal{M}'_\eta\,. \end{array}$$

Here, again, the matrices $k_x \in \mathrm{GL}_n\,\mathcal{O}_{F,x} \subset \mathrm{GL}_n\,\mathcal{O}_x$ and $\gamma \in \mathrm{GL}_n\,F$ are uniquely determined by the constrains of commutativity. Thus we see that $(g'_x)^{-1} = k_x^{-1} g_x^{-1} \gamma^{-1}$ for all x, or equivalently, if we put $k = (k_x) \in \mathrm{GL}_n(\mathcal{O}_\mathbf{A})$, that $g' = \gamma g k$, and thus $[g'] = [g]$ as classes in $\mathrm{GL}_n\,F \setminus \mathrm{GL}_n\,\mathbf{A} / \mathrm{GL}_n\,\mathcal{O}_\mathbf{A}$.

Since the inclusion $F \subset F_x$ is dense for every place x, and $\mathrm{GL}_n\,\mathcal{O}_\mathbf{A}$ is open in $\mathrm{GL}_n\,\mathbf{A}$, every class in $\mathrm{GL}_n\,F \setminus \mathrm{GL}_n\,\mathbf{A} / \mathrm{GL}_n\,\mathcal{O}_\mathbf{A}$ is represented by a $g = (g_x) \in \mathrm{GL}_n\,\mathbf{A}$ such that $g_x \in \mathrm{GL}_n\,F$ for all places x. This means that the above construction can be reversed. We obtain:

5.1.6 Lemma. *For every $n \geq 1$, there is a bijection*

$$\begin{array}{ccc} \mathrm{GL}_n\,F \setminus \mathrm{GL}_n\,\mathbf{A} / \mathrm{GL}_n\,\mathcal{O}_\mathbf{A} & \overset{1:1}{\longleftrightarrow} & \mathrm{Bun}_n\,X \\ [g] & \longmapsto & \mathcal{M}_g \end{array}$$

such that $\mathcal{M}_g \otimes \mathcal{L}_a = \mathcal{M}_{ag}$ for $a \in \mathbf{A}^\times$, and $\deg \mathcal{M}_g = \deg(\det g)$. □

5.1.7 Lemma. *If $Z_n\,\mathbf{A}$ denotes the centre of* $\mathrm{GL}_n\,\mathbf{A}$, *then there is a bijection*

$$\mathrm{GL}_n\,F\,Z_n\,\mathbf{A} \setminus \mathrm{GL}_n\,\mathbf{A} / \mathrm{GL}_n\,\mathcal{O}_\mathbf{A} \overset{1:1}{\longleftrightarrow} \mathbf{P}\mathrm{Bun}_n\,X$$

for every $n \geq 1$. □

5.1.8 The last lemma identifies the set of vertices of $\mathscr{G}_x$ with the geometric object $\mathbf{P}\mathrm{Bun}_2\,X$. The next task is to describe edges of $\mathscr{G}_x$ in geometric terms.

We say that two exact sequences of sheaves

$$0 \to \mathcal{F}_1 \to \mathcal{F} \to \mathcal{F}'_1 \to 0 \qquad \text{and} \qquad 0 \to \mathcal{F}_2 \to \mathcal{F} \to \mathcal{F}'_2 \to 0\,,$$

are *isomorphic with fixed $\mathcal{F}$* if there are isomorphisms $\mathcal{F}_1 \to \mathcal{F}_2$ and $\mathcal{F}'_1 \to \mathcal{F}'_2$ such that

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{F}_1 & \longrightarrow & \mathcal{F} & \longrightarrow & \mathcal{F}'_1 & \longrightarrow & 0 \\ & & \downarrow \simeq & & \| & & \downarrow \simeq & & \\ 0 & \longrightarrow & \mathcal{F}_2 & \longrightarrow & \mathcal{F} & \longrightarrow & \mathcal{F}'_2 & \longrightarrow & 0 \end{array}$$

commutes.

Let $\mathcal{K}_x$ be the torsion sheaf that is supported at x and has stalk κ_x at x, where κ_x is the residue field at x. Fix a representative $\mathcal{M}$ of $[\mathcal{M}] \in \mathbf{P}\mathrm{Bun}_2\,X$. Then we define $m_x([\mathcal{M}],[\mathcal{M}'])$ as the number of isomorphism classes of exact sequences

$$0 \longrightarrow \mathcal{M}'' \longrightarrow \mathcal{M} \longrightarrow \mathcal{K}_x \longrightarrow 0\ ,$$

with fixed $\mathcal{M}$ and with $\mathcal{M}''$ representing $[\mathcal{M}']$. This number is independent of the choice of the representative $\mathcal{M}$ because for another choice, which would be a vector bundle of the form $\mathcal{M} \otimes \mathcal{L}$ for some $\mathcal{L} \in \mathrm{Pic}\,X$, we have the bijection

$$\begin{array}{ccc} \left\{ \begin{array}{c} \text{isomorphism classes} \\ 0 \to \mathcal{M}'' \to \mathcal{M} \to \mathcal{K}_x \to 0 \\ \text{with fixed } \mathcal{M} \end{array} \right\} & \longrightarrow & \left\{ \begin{array}{c} \text{isomorphism classes} \\ 0 \to \mathcal{M}''' \to \mathcal{M} \otimes \mathcal{L} \to \mathcal{K}_x \to 0 \\ \text{with fixed } \mathcal{M} \end{array} \right\}. \\ \\ (0 \to \mathcal{M}'' \to \mathcal{M} \to \mathcal{K}_x \to 0) & \longmapsto & (0 \to \mathcal{M}'' \otimes \mathcal{L} \to \mathcal{M} \otimes \mathcal{L} \to \mathcal{K}_x \to 0) \end{array}$$

5.1.9 Definition. Let x be a place. For a projective line bundle $[\mathcal{M}] \in \mathbf{P}\mathrm{Bun}_2\, X$ we define

$$\mathcal{U}_x([\mathcal{M}]) \;=\; \big\{([\mathcal{M}],[\mathcal{M}'],m) \;\big|\; m = m_x([\mathcal{M}],[\mathcal{M}']) \neq 0\big\}\ ,$$

and call the occurring $[\mathcal{M}']$ the *Φ_x-neighbours of* $[\mathcal{M}]$, and $m_x([\mathcal{M}],[\mathcal{M}'])$ their *multiplicity*.

5.1.10 We shall show that this concept of neighbours is the same as the one defined for classes in $G_F Z_{\mathbf{A}} \backslash G_{\mathbf{A}} / K$ in Definition 4.1.2. In Proposition 4.2.4, we determined the Φ_x-neighbours of a class $[g] \in G_F Z_{\mathbf{A}} \backslash G_{\mathbf{A}} / K$ to be of the form $[g\xi_w]$ for a $w \in \mathbf{P}^1(\kappa_x)$. Fix a basis $(\mathcal{M}_g)_y \xrightarrow{\sim} \mathcal{O}^2_{X,y}$ for each $y \in |X|$. Note that by the definition of ξ_w in paragraph 4.2.2, multiplying an element of $\mathcal{O}^2_{X,y}$ with the component $(\xi_w)_y$ from the right yields an element of $\mathcal{O}^2_{X,y}$. Thus we obtain an exact sequence of $\mathbf{F}_q$-modules

$$0 \longrightarrow \prod_{y\in|X|} \mathcal{O}^2_{X,y} \xrightarrow{\ \xi_w\ } \prod_{y\in|X|} \mathcal{O}^2_{X,y} \longrightarrow \kappa_x \longrightarrow 0\ ,$$

and by the correspondence explained in paragraph 5.1.5 an exact sequence of sheaves

$$0 \longrightarrow \mathcal{M}_{g\xi_w} \longrightarrow \mathcal{M}_g \longrightarrow \mathcal{K}_x \longrightarrow 0\ .$$

This maps $w \in \mathbf{P}^1(\kappa_x)$ to the isomorphism class of $\big(0 \to \mathcal{M}_{g\xi_w} \to \mathcal{M}_g \to \mathcal{K}_x \to 0\big)$ with fixed $\mathcal{M}_g$.

On the other hand, as we have chosen a basis for the stalk at x, each isomorphism class of sequences $\big(0 \to \mathcal{M}' \to \mathcal{M} \to \mathcal{K}_x \to 0\big)$ with fixed $\mathcal{M}$ defines an element in $\mathbf{P}\big(\mathcal{O}^2_{X,x} / (\mathfrak{m}_x \mathcal{O}_{X,x})^2\big) = \mathbf{P}^1(\kappa_x)$, which gives back w.

We have proven the following.

5.1.11 Lemma. *For every $x \in |X|$, the map*

$$\begin{array}{ccc} \mathcal{U}_x([g]) & \longrightarrow & \mathcal{U}_x([\mathcal{M}_g]) \\ ([g],[g'],m) & \longmapsto & ([\mathcal{M}_g],[\mathcal{M}_{g'}],m) \end{array}$$

is a well-defined bijection. □

Lemmas 5.1.6 and 5.1.11 imply:

5.1.12 Proposition. *Let $x \in |X|$. The graph $\mathcal{G}_x$ of Φ_x is described in geometric terms as:*

$$\begin{aligned} \mathrm{Vert}\,\mathcal{G}_x &= \mathbf{P}\mathrm{Bun}_2\, X \quad \textit{and} \\ \mathrm{Edge}\,\mathcal{G}_x &= \coprod_{[\mathcal{M}]\in\mathbf{P}\mathrm{Bun}_2 X} \mathcal{U}_x([\mathcal{M}])\,. \quad \square \end{aligned}$$

5.1.13 Remark. This interpretation shows that the graphs that we consider are a global version of the graphs of Serre ([60, Chapter II.2]). We are looking at all rank 2 bundles on X modulo the action of the Picard group of X while Serre considers rank 2 bundles that trivialise outside a given place x modulo line bundles that trivialise outside x. As already remarked in 4.4.9, we obtain a projection of the graph of Serre to the component of the trivial class c_0.

Serre describes his graphs as quotients of Bruhat-Tits trees by the action of the group $\Gamma = G_{\mathcal{O}_F^\times}$ (cf. Remark 4.4.9) on both vertices and edges. This leads in general to multiple edges between vertices in the quotient graph, see e.g. [60, 2.4.2c]. This does not happen with graphs of Hecke operators: there is at most one edge with given origin and terminus.

Relative to the action of Γ on Serre's graphs, one can define the weight of an edge as the order of the stabiliser of its origin in the stabiliser of the edge. The projection from Serre's graphs to graphs of Hecke operators identifies all the different edges between two vertices, adding up their weights to obtain the weight of the image edge.

5.2 Geometric classification of vertices

Our aim is to show that the set of isomorphism classes of projective line bundles over X can be separated into subspaces corresponding to certain quotients of the the divisor class group of F, the divisor class group of $\mathbf{F}_{q^2}F$ and geometrically indecomposable projective line bundles.

5.2.1 We denote *the dual vector bundle* of $\mathcal{M}$ by $\mathcal{M}^\vee$. For a line bundle $\mathcal{L}$,

$$\mathcal{L} \otimes \mathcal{L}^\vee \simeq \mathcal{O}_X ,$$

thus the dual line bundle $\mathcal{L}^\vee$ represents the multiplicative inverse $\mathcal{L}^{-1}$ ([28, Prop. II.6.12]).

For two vector bundles $\mathcal{M}_1$ and $\mathcal{M}_2$ over X, the $\mathbf{F}_q$-vector space of sheaf morphisms

$$\mathrm{Hom}(\mathcal{M}_1, \mathcal{M}_2) \simeq \Gamma(X, \mathcal{M}_1^\vee \otimes \mathcal{M}_2)$$

is finite-dimensional.

We call a vector bundle $\mathcal{M}$ *indecomposable* if for every decomposition

$$\mathcal{M} = \mathcal{M}_1 \oplus \mathcal{M}_2$$

into two subbundles $\mathcal{M}_1$ and $\mathcal{M}_2$, one factor is trivial and the other is isomorphic to $\mathcal{M}$. The *Krull-Schmidt theorem* holds for the category of vector bundles over X, i.e. every vector bundle $\mathcal{M}$ on X defined over $\mathbf{F}_q$ has, up to permutation of factors, a unique decomposition into a direct sum of indecomposable subbundles, see [4, Thm. 2].

An extension of scalars $\mathbf{F}_{q^i}F/F$, or geometrically, $p : X' = X \otimes \mathbf{F}_{q^i} \to X$, defines the inverse image or the *constant extension* of vector bundles

$$\begin{array}{rccc} p^* : & \mathrm{Bun}_n X & \longrightarrow & \mathrm{Bun}_n X' . \\ & \mathcal{M} & \longmapsto & p^*\mathcal{M} \end{array}$$

The isomorphism classes of rank n bundles that after extension of constants to $\mathbf{F}_{q^i}$ become isomorphic to $p^*\mathcal{M}$ are classified by $H^1\big(\mathrm{Gal}(\mathbf{F}_{q^i}/\mathbf{F}_q), \mathrm{Aut}(\mathcal{M} \otimes \mathbf{F}_{q^i})\big)$, cf. [1, Section 1]. The algebraic group $\mathrm{Aut}(\mathcal{M} \otimes \mathbf{F}_{q^i})$ is an open subvariety of the connected algebraic group

$\mathrm{End}(\mathcal{M}\otimes \mathbf{F}_{q^i})$, and thus it is itself a connected algebraic group. As a consequence of Lang's theorem ([36, Cor. to Thm. 1]), we have $H^1\big(\mathrm{Gal}(\mathbf{F}_{q^i}/\mathbf{F}_q),\mathrm{Aut}(\mathcal{M}\otimes \mathbf{F}_{q^i})\big)=1$.

We deduce that p^* is injective. In particular, one can consider the constant extension to the geometric curve $\overline{X}=X\otimes\overline{\mathbf{F}}_q$ over an algebraic closure $\overline{\mathbf{F}}_q$ of $\mathbf{F}_q$. Then two vector bundles are isomorphic if and only if they are geometrically isomorphic, i.e. that their constant extensions to $\overline{X}$ are isomorphic. We can therefore think of $\mathrm{Bun}_n X$ as a subset of $\mathrm{Bun}_n X'$ and $\mathrm{Bun}_n \overline{X}$. Although we will point out at many places that $\mathrm{Pic}\, X$ is mapped to $\mathrm{Pic}\, X'$ via p^*, we will consider $\mathrm{Cl}\, X$ as a subgroup of $\mathrm{Cl}\, X'$ and omit p^* from the notation.

On the other hand, $p: X'\to X$ defines the direct image or the *trace* of vector bundles

$$\begin{array}{cccc} p_*: & \mathrm{Bun}_n X' & \longrightarrow & \mathrm{Bun}_{ni} X\ , \\ & \mathcal{M} & \longmapsto & p_*\mathcal{M} \end{array}$$

and we have that for $\mathcal{M}\in\mathrm{Bun}_n X$

$$p_*p^*\mathcal{M} \ \simeq\ \mathcal{M}^i\ .$$

There is a natural action of $\mathrm{Gal}(\mathbf{F}_{q^i}/\mathbf{F}_q)=\mathrm{Gal}(\mathbf{F}_{q^i}F/F)$

$$\begin{array}{ccc} \mathrm{Gal}(\mathbf{F}_{q^i}/\mathbf{F}_q)\times \mathrm{Bun}_n X' & \longrightarrow & \mathrm{Bun}_n X' \\ (\ \tau\quad ,\quad \mathcal{M}\ ) & \longmapsto & \mathcal{M}^\tau \end{array}$$

where $\mathcal{M}^\tau$ denotes the vector bundle with stalks $\mathcal{M}^\tau_x=\mathcal{M}_{\tau^{-1}(x)}$. Then for $\mathcal{M}\in\mathrm{Bun}_n X'$,

$$p^*p_*\mathcal{M}\ \simeq \bigoplus_{\tau\in\mathrm{Gal}(\mathbf{F}_{q^i}/\mathbf{F}_q)} \mathcal{M}^\tau\ .$$

The right hand side of the equation is a decomposition of $p^*(p_*\mathcal{M})$ over X'. This is a decomposition over X only if the factors are defined over X. This shows that if $\mathcal{M}$ is not defined over X, the notion of an indecomposable vector bundle is not stable under constant extension. We call a vector bundle *geometrically indecomposable* if its extension to $\overline{X}$ is indecomposable. In [1, Thm. 1.8], it is shown that every indecomposable vector bundle over X is the trace of an geometrically indecomposable bundle over some constant extension X' of X.

There are certain compatibilities of constant extension and trace with tensor products. Namely, for a vector bundle $\mathcal{M}$ and a line bundle $\mathcal{L}$ over X, we have

$$p^*(\mathcal{M}\otimes\mathcal{L})\ \simeq\ p^*\mathcal{M}\otimes p^*\mathcal{L}\ ,$$

and for a vector bundle $\mathcal{M}'$ over X',

$$p_*\mathcal{M}'\otimes\mathcal{L}\ \simeq\ p_*(\mathcal{M}'\otimes p^*\mathcal{L})\ .$$

Thus p^* induces a map denoted by the same symbol

$$\begin{array}{cccc} p^*: & \mathbf{P}\mathrm{Bun}_n X & \longrightarrow & \mathbf{P}\mathrm{Bun}_n X'\ , \\ & [\mathcal{M}] & \longmapsto & [p^*\mathcal{M}] \end{array}$$

and p_* induces

$$\begin{array}{cccc} p_*: & \mathrm{Bun}_n X'/\,p^*\,\mathrm{Pic}\, X & \longrightarrow & \mathbf{P}\mathrm{Bun}_{ni} X\ . \\ & [\mathcal{M}] & \longmapsto & [p_*\mathcal{M}] \end{array}$$

5.2.2 We look at the situation for $n=2$ and $i=2$. Let σ be the nontrivial automorphism of $\mathbf{F}_{q^2}/\mathbf{F}_q$. The set $\mathbf{P}\mathrm{Bun}_2 X$ is the disjoint union of the set of classes of decomposable rank 2 bundles, i.e. rank 2 bundles that are isomorphic to the direct sum of two line bundles, and the set of classes of indecomposable bundles. We denote these sets by $\mathbf{P}\mathrm{Bun}_2^{\mathrm{dec}} X$ and $\mathbf{P}\mathrm{Bun}_2^{\mathrm{indec}} X$, respectively. Let $\mathbf{P}\mathrm{Bun}_2^{\mathrm{gi}} X \subset \mathbf{P}\mathrm{Bun}_2^{\mathrm{indec}} X$ be the subset of classes of geometrically indecomposable bundles. Since the rank is 2, the complement $\mathbf{P}\mathrm{Bun}_2^{\mathrm{tr}} X = \mathbf{P}\mathrm{Bun}_2^{\mathrm{indec}} X - \mathbf{P}\mathrm{Bun}_2^{\mathrm{gi}} X$ consists of classes of traces of line bundles that are defined over the quadratic extension $X' = X \otimes \mathbf{F}_{q^2}$. Thus, we have a disjoint union

$$\mathbf{P}\mathrm{Bun}_2 X \;=\; \mathbf{P}\mathrm{Bun}_2^{\mathrm{dec}} X \;\amalg\; \mathbf{P}\mathrm{Bun}_2^{\mathrm{tr}} X \;\amalg\; \mathbf{P}\mathrm{Bun}_2^{\mathrm{gi}} X \;.$$

One has to be aware of the fact that there are traces of line bundles $\mathcal{L}$ over X' that decompose over X; more precisely, $p_*\mathcal{L}$ decomposes if and only if $\mathcal{L} \in p^* \operatorname{Pic} X$, and then $p_*\mathcal{L} \sim \mathcal{O}_X \oplus \mathcal{O}_X$.

For $[D] \in \operatorname{Cl} X$, define

$$c_D \;=\; [\mathcal{L}_D \oplus \mathcal{O}_X] \;\in\; \mathbf{P}\mathrm{Bun}_2^{\mathrm{dec}} X \;,$$

and for a $[D] \in \operatorname{Cl} X'$, define

$$t_D \;=\; [p_*\mathcal{L}_D] \;\in\; \mathbf{P}\mathrm{Bun}_2^{\mathrm{tr}} X \cup \{c_0\} \;.$$

Note that σ acts on $\operatorname{Cl} X'$ in a way compatible with the identification $\operatorname{Cl} X' \simeq \operatorname{Pic} X'$. Since $p^* p_*(\mathcal{L}) \simeq \mathcal{L} \oplus \mathcal{L}^\sigma \simeq p^* p_*(\mathcal{L}^\sigma)$ for $\mathcal{L} \in \operatorname{Pic} X'$, and isomorphism classes of vector bundles are stable under constant extensions, we have $t_D = t_{\sigma D}$.

We derive the following characterisations of $\mathbf{P}\mathrm{Bun}_2^{\mathrm{dec}} X$ and $\mathbf{P}\mathrm{Bun}_2^{\mathrm{tr}} X$:

5.2.3 Proposition.

$$\begin{array}{ccc} \operatorname{Cl} X & \longrightarrow & \mathbf{P}\mathrm{Bun}_2^{\mathrm{dec}} X \\ [D] & \longmapsto & c_D \end{array}$$

is surjective with fibres of the form $\{[D],[-D]\}$.

Proof. Let $\mathcal{M}$ decompose into $\mathcal{L}_1 \oplus \mathcal{L}_2$. Then

$$\mathcal{M} \;\simeq\; \mathcal{L}_1 \oplus \mathcal{L}_2 \;\sim\; \big(\mathcal{L}_1 \oplus \mathcal{L}_2\big) \otimes \mathcal{L}_2^{-1} \;\simeq\; \mathcal{L}_1\mathcal{L}_2^{-1} \oplus \mathcal{O}_X \;,$$

thus surjectivity follows. Let $\mathcal{L}_{D'} \oplus \mathcal{O}_X$ represent the same projective line bundle as $\mathcal{L}_D \oplus \mathcal{O}_X$, then there is a line bundle $\mathcal{L}_0$ such that

$$\mathcal{L}_D \oplus \mathcal{O}_X \;\simeq\; \big(\mathcal{L}_{D'} \oplus \mathcal{O}_X\big) \otimes \mathcal{L}_0 \;,$$

and thus either $\mathcal{L}_0 \simeq \mathcal{O}_X$ and $\mathcal{L}_D \simeq \mathcal{L}_{D'}$ or $\mathcal{L}_0 \simeq \mathcal{L}_D$ and $\mathcal{L}_{D'} \otimes \mathcal{L}_D \simeq \mathcal{O}_X$. Hence $[D']$ either equals $[D]$ or $[-D]$. □

5.2.4 Proposition.

$$\begin{array}{ccc} \operatorname{Cl} X' / \operatorname{Cl} X & \longrightarrow & \mathbf{P}\mathrm{Bun}_2^{\mathrm{tr}} X \cup \{c_0\} \\ [D] & \longmapsto & t_D \end{array}$$

is surjective with fibres of the form $\{[D],[-D]\}$.

Proof. Surjectivity is clear. Assume that $[D_1],[D_2] \in \mathrm{Cl}\, X'$ have the same image, then there is a $\mathcal{L}_0 \in \mathrm{Pic}\, X$ such that

$$p_*\mathcal{L}_1 \simeq p_*\mathcal{L}_2 \otimes \mathcal{L}_0 ,$$

where we briefly wrote $\mathcal{L}_i$ for $\mathcal{L}_{D_i}$. Then in $\mathbf{P}\mathrm{Bun}_2\, X'$, we see that

$$\begin{aligned} \mathcal{L}_1 \oplus \mathcal{L}_1^\sigma &\simeq p^* p_* \mathcal{L}_1 \\ &\simeq p^* p_* \mathcal{L}_2 \otimes p^* \mathcal{L}_0 \\ &\simeq (\mathcal{L}_2 \otimes p^*\mathcal{L}_0) \oplus (\mathcal{L}_2^\sigma \otimes p^*\mathcal{L}_0) , \end{aligned}$$

thus either $\mathcal{L}_1 \simeq \mathcal{L}_2 \otimes p^*\mathcal{L}_0$, which implies that D_1 and D_2 represent the same class in $\mathrm{Cl}\, X' / \mathrm{Cl}\, X$, or $\mathcal{L}_1 \simeq \mathcal{L}_2^\sigma \otimes p^*\mathcal{L}_0$, which means that D_1 represents the same class as σD_2. But in $\mathrm{Cl}\, X' / \mathrm{Cl}\, X$,

$$[\sigma D_2] = [\underbrace{\sigma D_2 + D_2}_{\in \mathrm{Cl}\, X} - D_2] = [-D_2] . \quad \square$$

5.2.5 Lemma. *The constant extension restricts to an injective map*

$$p^* : \mathbf{P}\mathrm{Bun}_2^{\mathrm{dec}}\, X \amalg \mathbf{P}\mathrm{Bun}_2^{\mathrm{tr}}\, X \hookrightarrow \mathbf{P}\mathrm{Bun}_2^{\mathrm{dec}}\, X' .$$

Proof. Since $p^* p_*(\mathcal{L}) \simeq \mathcal{L} \oplus \mathcal{L}^\sigma$ for a line bundle $\mathcal{L}$ over X', it is clear that the image is contained in $\mathbf{P}\mathrm{Bun}_2^{\mathrm{dec}}\, X'$. The images of $\mathbf{P}\mathrm{Bun}_2^{\mathrm{dec}}\, X$ and $\mathbf{P}\mathrm{Bun}_2^{\mathrm{tr}}\, X$ are disjoint since elements of the image of the latter set decompose into line bundles over X' that are not defined over X. If we denote taking the inverse elements by inv, then by Proposition 5.2.3, p^* is injective restricted to $\mathbf{P}\mathrm{Bun}_2^{\mathrm{dec}}\, X$ because $(\mathrm{Cl}\, X/\mathrm{inv}) \to (\mathrm{Cl}\, X'/\mathrm{inv})$ is. Regarding $\mathbf{P}\mathrm{Bun}_2^{\mathrm{tr}}\, X$, observe that

$$\begin{aligned} p^*(t_D) &= p^* p_*(\mathcal{L}_D) \\ &\simeq \mathcal{L}_D \oplus \mathcal{L}_{\sigma D} \\ &\sim \mathcal{L}_{D-\sigma D} \oplus \mathcal{O}_X \\ &= c_{D-\sigma D} , \end{aligned}$$

where by Proposition 5.2.4, D represents an element in $\big(\mathrm{Cl}\, X'/\mathrm{Cl}\, X\big)/\mathrm{inv}$, and by Proposition 5.2.3, $D - \sigma D$ represents an element in $\mathrm{Cl}\, X/\mathrm{inv}$. If there are $[D_1],[D_2] \in \mathrm{Cl}\, X'$ such that $(D_1 - \sigma D_1) = \pm(D_2 - \sigma D_2)$, then we have $D_1 \mp D_2 = \sigma(D_1 \mp D_2)$, and consequently $[D_1 \mp D_2] \in \mathrm{Cl}\, X$. $\square$

5.2.6 Remark. The constant extension also restricts to a map

$$p^* : \mathbf{P}\mathrm{Bun}_2^{\mathrm{gi}}\, X \longrightarrow \mathbf{P}\mathrm{Bun}_2^{\mathrm{gi}}\, X' .$$

But this restriction is in general not injective in contrast to the previous result. For a counterexample to injectivity, consider Remark 7.1.7.

5.3 Reduction theory for rank 2 bundles

This section introduces reduction theory for rank 2 bundles, i.e. the investigation of vector bundles by looking at proper subbundles.

5.3.1 Vector bundles do not form a full subcategory of the category of sheaves, to wit, if $\mathcal{M}_1$ and $\mathcal{M}_2$ are vector bundles and $\mathcal{M}_1 \to \mathcal{M}_2$ is a morphism of sheaves, then the cokernel may have nontrivial torsion, which does not occur for a morphism of vector bundles. Thus by a *line subbundle* $\mathcal{L} \to \mathcal{M}$ of a vector bundle $\mathcal{M}$, we mean an injective morphism of sheaves such that the cokernel $\mathcal{M}/\mathcal{L}$ is again a vector bundle.

But every locally free subsheaf $\mathcal{L} \to \mathcal{M}$ of rank 1 extends to a uniquely determined line subbundle $\overline{\mathcal{L}} \to \mathcal{M}$, viz. $\overline{\mathcal{L}}$ is determined by the constraint $\mathcal{L} \subset \overline{\mathcal{L}}$ ([60, p. 100]). On the other hand, every rank 2 bundle has a line subbundle ([28, Corollary V.2.7]).

Two line subbundles $\mathcal{L} \to \mathcal{M}$ and $\mathcal{L}' \to \mathcal{M}$ are said to be the same if their image coincides, or in other words, if there is an isomorphism $\mathcal{L} \simeq \mathcal{L}'$ that commutes with the inclusions into $\mathcal{M}$.

For a line subbundle $\mathcal{L} \to \mathcal{M}$ of a rank 2 bundle $\mathcal{M}$, we define

$$\delta(\mathcal{L},\mathcal{M}) \ := \ \deg\mathcal{L} - \deg(\mathcal{M}/\mathcal{L}) \ = \ 2\deg\mathcal{L} - \deg\mathcal{M}$$

and

$$\delta(\mathcal{M}) \ := \sup_{\substack{\mathcal{L}\to\mathcal{M} \\ \text{line subbundle}}} \delta(\mathcal{L},\mathcal{M}) \ .$$

If $\delta(\mathcal{M}) = \delta(\mathcal{L},\mathcal{M})$, then we call $\mathcal{L}$ a *line subbundle of maximal degree*, or briefly, a *maximal subbundle*. Since $\delta(\mathcal{L}\otimes\mathcal{L}',\mathcal{M}\otimes\mathcal{L}') = \delta(\mathcal{L},\mathcal{M})$ for a line bundle $\mathcal{L}'$, $\delta(\mathcal{M})$ is a well-defined invariant on $\mathbf{P}\mathrm{Bun}_2 X$, and we put $\delta([\mathcal{M}]) = \delta(\mathcal{M})$.

Let g_X be the genus of X. Then the Riemann-Roch theorem and Serre duality imply:

5.3.2 Proposition ([60, II.2.2, Prop. 6 and 7]). *For every rank* 2 *bundle* $\mathcal{M}$*,*

$$-2g_X \le \delta(\mathcal{M}) < \infty \ .$$

If $\mathcal{L} \to \mathcal{M}$ *is a line subbundle with* $\delta(\mathcal{L},\mathcal{M}) > 2g_X - 2$*, then* $\mathcal{M} \simeq \mathcal{L} \oplus \mathcal{M}/\mathcal{L}$*.*

5.3.3 Every extension of a line bundle $\mathcal{L}'$ by a line bundle $\mathcal{L}$, i.e. every exact sequence of the form

$$0 \longrightarrow \mathcal{L} \longrightarrow \mathcal{M} \longrightarrow \mathcal{L}' \longrightarrow 0 \ ,$$

determines a rank 2 bundle $\mathcal{M} \in \mathrm{Bun}_2 X$. This defines for all $\mathcal{L},\mathcal{L}' \in \mathrm{Pic}\, X$ a map

$$\mathrm{Ext}^1(\mathcal{L},\mathcal{L}') \ \longrightarrow \ \mathrm{Bun}_2 X \ ,$$

which maps the zero element to $\mathcal{L} \oplus \mathcal{L}'$. Remark that since decomposable bundles may have line subbundles that differ from its given two factors, nontrivial elements can give rise to decomposable bundles.

The units $\mathbf{F}_q^\times$ operate by multiplication on the $\mathbf{F}_q$-vector space

$$\mathrm{Ext}^1(\mathcal{L},\mathcal{L}') \underset{\substack{\text{Serre} \\ \text{duality}}}{\simeq} \mathrm{Hom}(\mathcal{L},\mathcal{L}'\omega_X^\vee) \ .$$

The multiplication of a morphism $\mathcal{L} \to \mathcal{L}'\omega_X^\vee$ by an $a \in \mathbf{F}_q^\times$ is nothing else but multiplying the stalk $(\mathcal{L})_\eta$ by a^{-1} and all stalks $(\mathcal{L}'\omega_X^\vee)_x$ at closed points x by a, which induces automorphisms on both $\mathcal{L}$ and $\mathcal{L}'\omega_X^\vee$, respectively. Thus, two elements of $\operatorname{Ext}^1(\mathcal{L},\mathcal{L}')$ that are $\mathbf{F}_q^\times$-multiples of each other define the same bundle $\mathcal{M} \in \operatorname{Bun}_2 X$. We get a well-defined map

$$\mathbf{P}\operatorname{Ext}^1(\mathcal{L},\mathcal{L}') \longrightarrow \operatorname{Bun}_2 X$$

where the projective space $\mathbf{P}\operatorname{Ext}^1(\mathcal{L},\mathcal{L}')$ is defined as the empty set when $\operatorname{Ext}^1(\mathcal{L},\mathcal{L}')$ is trivial. If we further project to $\mathbf{P}\operatorname{Bun}_2 X$, we can reformulate the above properties of the invariant δ as follows.

5.3.4 Proposition. *The map*

$$\coprod_{-2g_X \leq \deg \mathcal{L} \leq 2g_X - 2} \mathbf{P}\operatorname{Ext}^1(\mathcal{L},\mathcal{O}_X) \longrightarrow \mathbf{P}\operatorname{Bun}_2 X$$

meets every element of $\mathbf{P}\operatorname{Bun}_2^{\text{indec}} X$*, and the fibre of any* $[\mathcal{M}] \in \mathbf{P}\operatorname{Bun}_2 X$ *is of the form*

$$\left\{ 0 \to \mathcal{L} \to \mathcal{M} \to \mathcal{O}_X \to 0 \;\middle|\; \substack{\delta(\mathcal{L},\mathcal{M}) \geq -2g_X \\ \text{and } \mathcal{M} \not\simeq \mathcal{L}\oplus\mathcal{O}_X} \right\}.$$

Proof. We know that every $[\mathcal{M}] \in \mathbf{P}\operatorname{Bun}_2 X$ has a reduction

$$0 \longrightarrow \mathcal{L} \longrightarrow \mathcal{M} \longrightarrow \mathcal{L}' \longrightarrow 0$$

with $\delta(\mathcal{L},\mathcal{M}) \geq -2g_X$, where we may assume that $\mathcal{L}' = \mathcal{O}_X$ by replacing $\mathcal{M}$ with $\mathcal{M} \otimes (\mathcal{L}')^{-1}$, hence $\delta(\mathcal{L},\mathcal{M}) = \deg \mathcal{L}$. If $\deg \mathcal{L} > 2g_X - 2$, then $\mathcal{M}$ decomposes, so $\operatorname{Ext}^1(\mathcal{L},\mathcal{O}_X)$ is trivial and $\mathbf{P}\operatorname{Ext}^1(\mathcal{L},\mathcal{O}_X)$ is the empty set. This explains the form of the fibres and that $\mathbf{P}\operatorname{Bun}_2^{\text{indec}} X$ is contained in the image. □

5.3.5 Corollary. *There are only finitely many isomorphism classes of indecomposable projective line bundles.*

Proof. This is clear since $\coprod_{-2g_X \leq \deg \mathcal{L} \leq 2g_X - 2} \mathbf{P}\operatorname{Ext}^1(\mathcal{L},\mathcal{O}_X)$ is a finite union of finite sets. □

5.3.6 Lemma. *If* $\mathcal{L} \to \mathcal{M}$ *is a maximal subbundle, then for every line subbundle* $\mathcal{L}' \to \mathcal{M}$ *that is different from* $\mathcal{L} \to \mathcal{M}$*,*

$$\delta(\mathcal{L}',\mathcal{M}) \leq -\delta(\mathcal{L},\mathcal{M}) .$$

Equality holds if and only if $\mathcal{M} \simeq \mathcal{L} \oplus \mathcal{L}'$*, i.e.* $\mathcal{M}$ *decomposes and* $\mathcal{L}'$ *is a complement of* $\mathcal{L}$ *in* $\mathcal{M}$*.*

Proof. Compare with [56, Lemma 3.1.1.]. Since $\mathcal{L}' \to \mathcal{M}$ is different from $\mathcal{L} \to \mathcal{M}$, there is no inclusion $\mathcal{L}' \to \mathcal{L}$ that commutes with the inclusions into $\mathcal{M}$. Hence the composed morphism $\mathcal{L}' \to \mathcal{M} \to \mathcal{M}/\mathcal{L}$ must be injective, and $\deg \mathcal{L}' \leq \deg \mathcal{M}/\mathcal{L} = \deg \mathcal{M} - \deg \mathcal{L}$. This implies that $\delta(\mathcal{L}',\mathcal{M}) = 2\deg \mathcal{L}' - \deg \mathcal{M} \leq \deg \mathcal{M} - 2\deg \mathcal{L} = -\delta(\mathcal{L},\mathcal{M})$. Equality holds if and only if $\mathcal{L}' \to \mathcal{M}/\mathcal{L}'$ is an isomorphism, but its inverse then defines a section $\mathcal{M}/\mathcal{L} \simeq \mathcal{L}' \to \mathcal{M}$. □

5.3.7 Proposition.

(i) *A rank 2 bundle $\mathcal{M}$ has at most one line subbundle $\mathcal{L} \to \mathcal{M}$ such that $\delta(\mathcal{L}, \mathcal{M}) \geq 1$.*

(ii) *If $\mathcal{L} \to \mathcal{M}$ is a line subbundle with $\delta(\mathcal{L}, \mathcal{M}) \geq 0$, then $\delta(\mathcal{M}) = \delta(\mathcal{L}, \mathcal{M})$.*

(iii) *If $\delta(\mathcal{M}) = 0$, we distinguish three cases.*

   (1) *$\mathcal{M}$ has only one maximal line bundle: this happens if and only if $\mathcal{M}$ is indecomposable.*

   (2) *$\mathcal{M}$ has exactly two maximal subbundles $\mathcal{L}_1 \to \mathcal{M}$ and $\mathcal{L}_2 \to \mathcal{M}$: this happens if and only if $\mathcal{L}_1 \not\simeq \mathcal{L}_2$ and $\mathcal{M} \simeq \mathcal{L}_1 \oplus \mathcal{L}_2$.*

   (3) *$\mathcal{M}$ has exactly $q+1$ maximal subbundles: this happens if and only if all maximal subbundles are of the same isomorphism type $\mathcal{L}$ and $\mathcal{M} \simeq \mathcal{L} \oplus \mathcal{L}$.*

(iv) $\delta(c_D) = |\deg D|$.

(v) *$\delta(\mathcal{M})$ is invariant under extension of constants for $[\mathcal{M}] \in \mathbf{P}\mathrm{Bun}_2^{\mathrm{dec}} X$.*

Proof. Everything follows from preceding lemmas, except for the fact that $\mathcal{L} \oplus \mathcal{L}$ has precisely $q+1$ maximal subbundles in part (iii3), which needs some explanation.

First observe that by tensoring with $\mathcal{L}^{-1}$, we reduce the question to searching the maximal subbundles of $\mathcal{O}_X \oplus \mathcal{O}_X$. This bundle has canonical bases at every stalk, which induce the canonical inclusions $\mathcal{O}_{X,x}^2 \hookrightarrow \mathcal{O}_{X,\eta}^2$ of the stalks at closed points x into the stalk at the generic point η. This allows us to choose for any line subbundle $\mathcal{F} \to \mathcal{O}_X \oplus \mathcal{O}_X$ a trivialisation with trivial coordinate changes. Thus for every open subset over which $\mathcal{F}$ trivialises, we obtain the same 1-dimensional F-subspace $\mathcal{F}_\eta \subset \mathcal{O}_{X,\eta}^2 = F^2$. On the other hand, every 1-dimensional subspace $\mathcal{F}_\eta \subset \mathcal{O}_{X,\eta}^2$ gives back the line subbundle by the inclusion of stalks $\mathcal{F}_x = \mathcal{F}_\eta \cap \mathcal{O}_{X,x}^2 \hookrightarrow \mathcal{F}_\eta$. We see that for every place x, $\deg_x \mathcal{F} \geq 0$, and only the lines in $\mathcal{O}_{X,\eta}^2 = F^2$ that are generated by an element in $\mathbf{F}_q^2 \subset F^2$ define line subbundles $\mathcal{F} \to \mathcal{O}_X \oplus \mathcal{O}_X$ with $\deg_x \mathcal{F} = 0$ for every place x. But there are $q+1 = \#\mathbf{P}^1(\mathbf{F}_q)$ different such line subbundles. □

5.3.8 Proposition. *Let $p : X' = X \otimes \mathbf{F}_{q^2} \to X$ and $\mathcal{L} \in \operatorname{Pic} X'$, then $\delta(p_*\mathcal{L})$ is an even non-positive integer. It equals 0 if and only if $\mathcal{L} \in p^* \operatorname{Pic} X$.*

Proof. Over X', we have $p^* p_* \mathcal{L} \simeq \mathcal{L} \oplus \mathcal{L}^\sigma$, and $\deg \mathcal{L} = \deg \mathcal{L}^\sigma$, thus by the previous paragraph, a maximal subbundle of $p_*\mathcal{L}$ has at most the same degree as $\mathcal{L}$, or, equivalently, $\delta(p_*\mathcal{L}) \leq 0$. A maximal subbundle has the same degree as $\mathcal{L}$ if and only if it is isomorphic to $\mathcal{L}$ or $\mathcal{L}^\sigma$ which can only be the case when $\mathcal{L}$ already is defined over X. Finally, by the very definition of $\delta(\mathcal{M})$ for rank 2 bundles $\mathcal{M}$, it follows that

$$\delta(\mathcal{M}) \equiv \deg \mathcal{M} \pmod 2,$$

and $\deg(p_*\mathcal{L}) = 2 \deg \mathcal{L}$ is even. □

5.3.9 Remark. We see that for $[\mathcal{M}] \in \mathbf{P}\mathrm{Bun}_2^{\mathrm{tr}} X$, the invariant $\delta(\mathcal{M})$ must get larger if we extend constants to $\mathbf{F}_{q^2}$, because $p^*(\mathcal{M})$ decomposes over X'. This stays in contrast to the result for classes in $\mathbf{P}\mathrm{Bun}_2^{\mathrm{tr}} X$ (Proposition 5.3.7 (v)).

5.4 Geometric classification of edges

This section will define certain subgraphs of $\mathscr{G}_x$ for a place x, namely, the cusp of a divisor class modulo x, which is an infinite subgraph of a simple nature, and the nucleus, which is a finite subgraph that depends heavily on the arithmetic of F. Finally, $\mathscr{G}_x$ can be described as the union of the nucleus with a finite number of cusps.

5.4.1 We use reduction theory to investigate sequences of the form

$$0 \longrightarrow \mathcal{M}' \longrightarrow \mathcal{M} \longrightarrow \mathcal{K}_x \longrightarrow 0 \ ,$$

which occur in the definition of $\mathcal{U}_x([\mathcal{M}])$. By paragraph 5.1.3, $\deg \mathcal{M}' = \deg \mathcal{M} - d_x$ when $d_x = \dim_{\mathbf{F}_q} \kappa_x$ is the degree of x.

If $\mathcal{L} \to \mathcal{M}$ is a line subbundle, then we say that it lifts to $\mathcal{M}'$ if there exists a morphism $\mathcal{L} \to \mathcal{M}'$ such that the diagram

$$\begin{array}{ccc} & & \mathcal{L} \\ & \swarrow & \downarrow \\ \mathcal{M}' & \longrightarrow & \mathcal{M} \end{array}$$

commutes. In this case, $\mathcal{L} \to \mathcal{M}'$ is indeed a subbundle since otherwise it would extend nontrivially to a subbundle $\overline{\mathcal{L}} \to \mathcal{M}' \subset \mathcal{M}$ and would contradict the hypothesis that $\mathcal{L}$ is a subbundle of $\mathcal{M}$. By exactness of the above sequence, a line subbundle $\mathcal{L} \to \mathcal{M}$ lifts to $\mathcal{M}'$ if and only if the image of $\mathcal{L}$ in $\mathcal{K}_x$ is 0.

Let $\mathcal{J}_x \subset \mathcal{O}_X$ be the kernel of $\mathcal{O}_X \to \mathcal{K}_x$. This is also a line bundle, since $\mathcal{K}_x$ is a torsion sheaf. For every line bundle $\mathcal{L}$, we may think of $\mathcal{L}\mathcal{J}_x$ as a subsheaf of $\mathcal{L}$. In $\operatorname{Pic} X$, the line bundle $\mathcal{J}_x$ represents the inverse of $\mathcal{L}_x$,the line bundle associated to the divisor x. In particular, $\deg \mathcal{J}_x = \deg \mathcal{L}_x^{-1} = -d_x$.

If $\mathcal{L} \to \mathcal{M}$ does not lift to a subbundle of $\mathcal{M}'$, we have that $\mathcal{L}\mathcal{J}_x \subset \mathcal{L} \to \mathcal{M}$ lifts to a subbundle of $\mathcal{M}'$:

$$\begin{array}{ccc} \mathcal{J}_x\mathcal{L} & \subset & \mathcal{L} \\ \downarrow & & \downarrow \\ \mathcal{M}' & \longrightarrow & \mathcal{M} \ . \end{array}$$

Note that every subbundle $\mathcal{L} \to \mathcal{M}'$ is a locally free subsheaf of $\mathcal{L} \to \mathcal{M}$ and thus extends to a subbundle $\overline{\mathcal{L}} \to \mathcal{M}$. If thus $\mathcal{L} \to \mathcal{M}$ is a maximal subbundle that lifts to a subbundle $\mathcal{L} \to \mathcal{M}'$, then $\mathcal{L} \to \mathcal{M}'$ is a maximal subbundle. If, however, $\mathcal{L} \to \mathcal{M}$ is a maximal subbundle that does not lift to a subbundle $\mathcal{L} \to \mathcal{M}'$, then $\mathcal{L}\mathcal{J}_x \to \mathcal{M}'$ is a subbundle, which is not necessarily maximal. These considerations imply that

$$\begin{array}{lllll} \delta(\mathcal{M}') \leq 2\deg \mathcal{L} - \deg \mathcal{M}' & = & 2\deg \mathcal{L} - (\deg \mathcal{M} - d_x) & = & \delta(\mathcal{M}) + d_x \quad \text{and} \\ \delta(\mathcal{M}') \geq 2\deg \mathcal{J}_x\mathcal{L} - \deg \mathcal{M}' & = & 2\deg \mathcal{L} - 2d_x - (\deg \mathcal{M} - d_x) & = & \delta(\mathcal{M}) - d_x \ . \end{array}$$

Since $\delta(\mathcal{M}') \equiv \deg \mathcal{M}' = \deg \mathcal{M} - d_x \pmod 2$, we derive:

5.4.2 Lemma. *If* $0 \to \mathcal{M}' \to \mathcal{M} \to \mathcal{K}_x \to 0$ *is exact, then*

$$\delta(\mathcal{M}') \in \left\{ \delta(\mathcal{M}) - d_x, \ \delta(\mathcal{M}) - d_x + 2, \ldots, \delta(\mathcal{M}) + d_x \right\} . \quad \square$$

5.4.3 Every line subbundle $\mathcal{L} \to \mathcal{M}$ defines a line $\mathcal{L}/\mathcal{L}\mathcal{J}_x$ in $\mathbf{P}^1\big(\mathcal{M}/(\mathcal{M}\otimes\mathcal{J}_x)\big)$. By the bijection of paragraph 5.1.10,

$$\begin{array}{ccc} \left\{ \begin{array}{c}\text{isomorphism classes of exact}\\ 0\to\mathcal{M}'\to\mathcal{M}\to\mathcal{K}_x\to 0\\ \text{with fixed } \mathcal{M}\end{array}\right\} & \overset{1:1}{\longrightarrow} & \mathbf{P}^1\big(\mathcal{M}\,/\,(\mathcal{M}\otimes\mathcal{J}_x)\big)\ , \\ \big(0\to\mathcal{M}'\to\mathcal{M}\to\mathcal{K}_x\to 0\big) & \longmapsto & \mathcal{M}'/(\mathcal{M}\otimes\mathcal{J}_x) \end{array}$$

there is an unique

$$0 \longrightarrow \mathcal{M}' \longrightarrow \mathcal{M} \longrightarrow \mathcal{K}_x \longrightarrow 0\ ,$$

up to isomorphism with fixed $\mathcal{M}$, such that $\mathcal{L}\to\mathcal{M}$ lifts to $\mathcal{L}\to\mathcal{M}'$. We call this the *sequence associated to* $\mathcal{L}\to\mathcal{M}$ *relative to* Φ_x, or for short the *associated sequence*, and $[\mathcal{M}']$ the *associated* Φ_x*-neighbour*. It follows that $\delta(\mathcal{M}') \geq \delta(\mathcal{L},\mathcal{M}) + d_x$.

We summarise this as follows.

5.4.4 Lemma. *If $\mathcal{L}\to\mathcal{M}$ is a maximal subbundle, then the associated Φ_x-neighbour $[\mathcal{M}']$ has $\delta(\mathcal{M}') = \delta(\mathcal{M}) + d_x$, and*

$$\sum_{\substack{([\mathcal{M}],[\mathcal{M}'],m)\in\mathcal{U}_x([\mathcal{M}])\\ \delta(\mathcal{M}')=\delta(\mathcal{M})+d_x}} m \quad = \quad \#\left\{ \overline{\mathcal{L}} \in \mathbf{P}^1\big(\mathcal{M}\,/\,(\mathcal{M}\otimes\mathcal{J}_x)\big) \;\middle|\; \begin{array}{l}\exists\mathcal{L}\to\mathcal{M} \textit{ maximal subbundle}\\ \textit{with } \mathcal{L}\equiv\overline{\mathcal{L}} \pmod{\mathcal{M}\otimes\mathcal{J}_x}\end{array}\right\}. \qquad \square$$

5.4.5 Definition. Let x be a place. Define the number

$$m_X \;=\; \max\{2g_X - 2, 0\}\ ,$$

and let the divisor D represent a class $[D] \in \mathrm{Cl}\,\mathcal{O}_X^x = \mathrm{Cl}\,X\,/\langle x\rangle$.

We define the *cusp* $\mathcal{C}_x(D)$ *(of D in $\mathcal{G}_x$)* as the full subgraph of $\mathcal{G}_x$ with vertices

$$\mathrm{Vert}\,\mathcal{C}_x(D) \;=\; \left\{ c_{D'} \,\middle|\, [D'] \equiv [D] \pmod{\langle x\rangle}, \text{ and } \deg D' > m_X \right\},$$

and the *nucleus* $\mathcal{N}_x$ *(of $\mathcal{G}_x$)* as the full subgraph of $\mathcal{G}_x$ with vertices

$$\mathrm{Vert}\,\mathcal{N}_x \;=\; \left\{ [\mathcal{M}] \in \mathbf{P}\mathrm{Bun}_2\,X \,\middle|\, \delta(\mathcal{M}) \leq m_X + d_x \right\}.$$

5.4.6 Theorem. *Let x be a place and $[D] \in \mathrm{Cl}\,X$ be a divisor of non-negative degree. The Φ_x-neighbours v of c_D with $\delta(v) = \deg D + d_x$ are given by the following list:*

$$\begin{array}{rcll} (c_0, c_x, q+1) & \in & \mathcal{U}_x(c_0), & \\ (c_D, c_{D+x}, 2) & \in & \mathcal{U}_x(c_D) & \textit{if } [D] \in (\mathrm{Cl}^0\,X)[2] - \{0\}, \\ (c_D, c_{D+x}, 1), (c_D, c_{-D+x}, 1) & \in & \mathcal{U}_x(c_D) & \textit{if } [D] \in \mathrm{Cl}^0\,X - (\mathrm{Cl}^0\,X)[2], \textit{ and} \\ (c_D, c_{D+x}, 1) & \in & \mathcal{U}_x(c_D) & \textit{if } \deg D \textit{ is positive.} \end{array}$$

For all Φ_x-neighbours v of c_D not occurring in this list, $\delta(v) < \delta(c_D) + d_x$. If furthermore $\deg D > d_x$, then $\delta(v) = \deg D - d_x$, and if $\deg D > m_X + d_x$, then

$$\mathcal{U}_x(c_D) \;=\; \{(c_D, c_{D-x}, q_x), (c_D, c_{D+x}, 1)\}\,.$$

Proof. By Lemma 5.4.4, the Φ_x-neighbours v of c_D with $\delta(v) = \delta(c_D) + d_x$ counted with multiplicity correspond to the maximal subbundles of a rank 2 bundle $\mathcal{M}$ that represents c_D. Since $\delta(\mathcal{M}) = \delta(c_D) \geq 0$, the list of all Φ_x-neighbours v of c_D with $\delta(v) = \deg D + d_x = \delta(c_D) + d_x$ follows from the different cases in Proposition (5.3.7) (i) and (iii). Be aware that $c_D = c_{-D}$ by Proposition 5.2.3; hence it makes a difference whether or not D is 2-torsion.

For the latter statements, write $\mathcal{M} = \mathcal{L}_D \oplus \mathcal{O}_X$ and let $\mathcal{M}'$ be a subsheaf of $\mathcal{M}$ with cokernel $\mathcal{K}_x$ such that $\delta(\mathcal{M}') < \delta(\mathcal{M}) + d_x$. Then $\mathcal{L}_D \to \mathcal{M}$ does not lift to $\mathcal{M}'$, but $\mathcal{L}_D \mathcal{J}_x \to \mathcal{M}'$ is a line subbundle and

$$\mathcal{M}'/\mathcal{L}_D\mathcal{J}_x \;\simeq\; (\det \mathcal{M}')(\mathcal{L}_D\mathcal{J}_x)^\vee \;\simeq\; (\det \mathcal{M})\mathcal{J}_x(\mathcal{L}_D\mathcal{J}_x)^\vee \;\simeq\; \mathcal{L}_D\mathcal{J}_x(\mathcal{L}_D\mathcal{J}_x)^\vee \;\simeq\; \mathcal{O}_X\,.$$

If $\deg D > d_x$, then

$$\delta(\mathcal{L}_D\mathcal{J}_x, \mathcal{M}') \;=\; \deg \mathcal{L}_D\mathcal{J}_x - \deg \mathcal{O}_X \;=\; \deg D - d_x \;>\; 0\,.$$

Proposition 5.3.7 (i) implies that $\mathcal{L}_D \to \mathcal{M}$ is the unique maximal subbundle of $\mathcal{M}'$ and thus $\delta(\mathcal{M}') = \delta(\mathcal{M}) - d_x$.

If $\delta(\mathcal{M}) > m_X + d_x$, then $\delta(\mathcal{M}') > m_X \geq 2g_X - 2$, hence $\mathcal{M}'$ decomposes and represents c_{D-x}. Since the multiplicities of all Φ_x-neighbours of a vertex sum up to $q_x + 1$, this proves the last part of our assertions. □

5.4.7 Applying the proposition to the vertices of the cusp $\mathcal{C}_x(D)$ determines all edges that lie in the cusp. If $m_X < \deg D \leq m_X + d_x$, the cusp can be illustrated as in Figure 5.1. Note that a cusp is an infinite graph. It has a regular pattern that repeats periodically. In diagrams we draw the pattern and indicate its periodic continuation with dots.

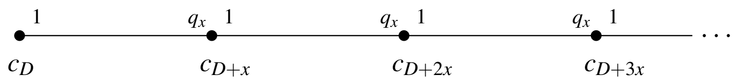


Figure 5.1: A cusp

5.4.8 Remark. Note that the notation c_D for vertices in $\mathbf{P}\mathrm{Bun}_2^{\mathrm{dec}} X$ coincides with the notation for the vertices in the examples of section 4.3.

We summarise the theory so far in the following theorem that describes the general structure of $\mathcal{G}_x$.

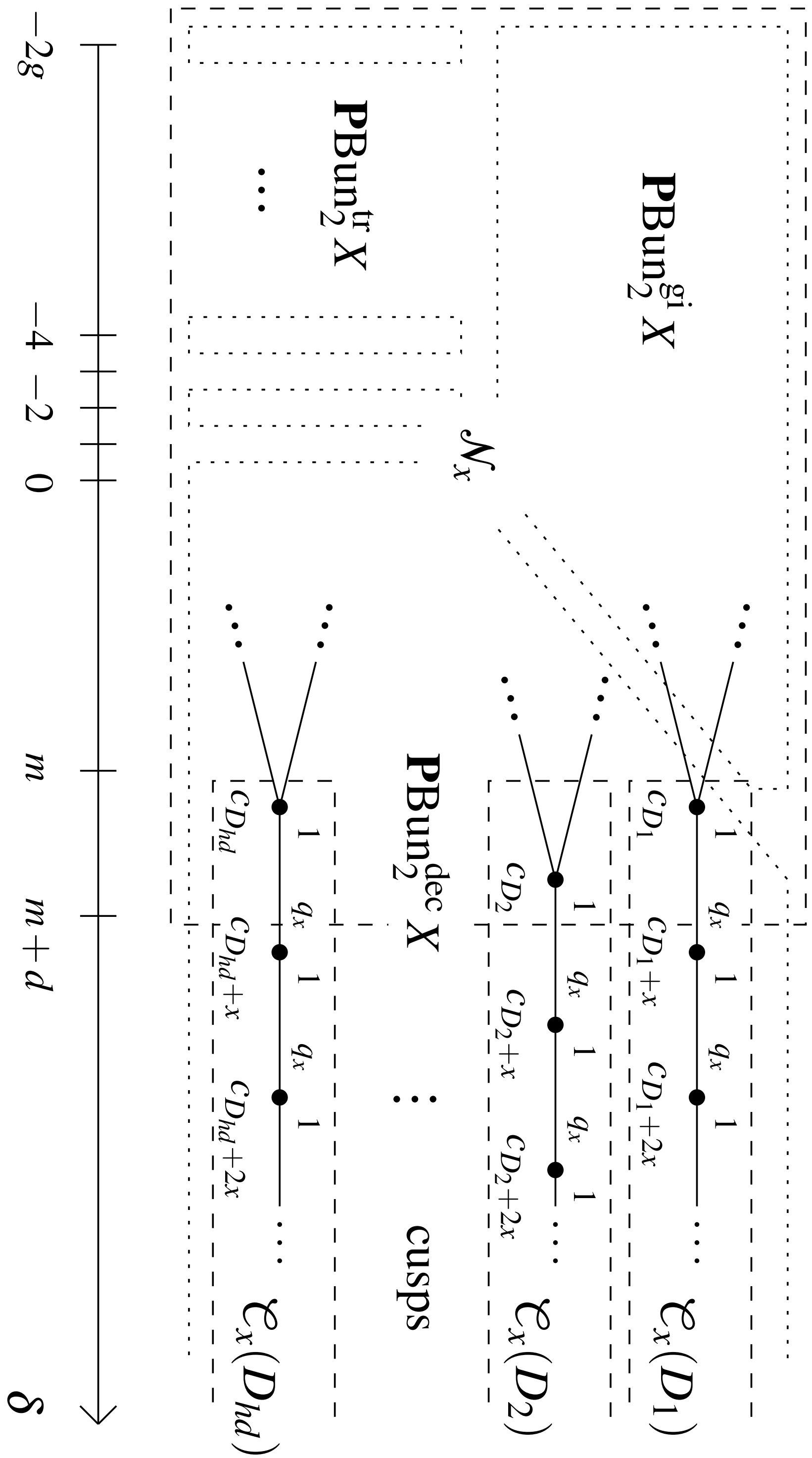


Figure 5.2: General structure of $\mathscr{G}_x$

5.4.9 Theorem. *Let x be a place of degree d_x and h_X the class number, then:*

(i) *$\mathcal{G}_x$ has $h_X d_x$ cusps and*

$$\mathcal{G}_x \;=\; \mathcal{N}_x \cup \coprod_{[D]\in \mathrm{Cl}\,\mathcal{O}_F^x} \mathcal{C}_x(D)\,,$$

where $\mathrm{Vert}\,\mathcal{N}_x \cap \mathrm{Vert}\,\mathcal{C}_x(D) = \{c_D\}$ if D represents $[D]$ and $m_X < \deg D \le m_X + d_x$. The union of the edges is disjoint.

(ii) *$\mathcal{N}_x$ is finite and has $\#\big(\mathrm{Cl}\,\mathcal{O}_F^x\,/\,2\,\mathrm{Cl}\,\mathcal{O}_F^x\big)$ components. Each vertex of $\mathcal{N}_x$ is at distance $\le (2g_X + m_X + d_x)/d_x$ from some cusp. The associated CW-complexes of $\mathcal{N}_x$ and $\mathcal{G}_x$ are homotopy equivalent.*

(iii) *If $[D] \in \mathrm{Cl}\,\mathcal{O}_F^x$, then $\mathrm{Vert}\,\mathcal{C}_x(D) \subset \mathbf{P}\mathrm{Bun}_2^{\mathrm{dec}}\,X$. Furthermore*

$$\begin{aligned}
\mathbf{P}\mathrm{Bun}_2^{\mathrm{dec}}\,X \;&\subset\; \{v \in \mathrm{Vert}\,\mathcal{G}_x \mid \delta(v) \ge 0\}\,,\\
\mathbf{P}\mathrm{Bun}_2^{\mathrm{gi}}\,X \;&\subset\; \{v \in \mathrm{Vert}\,\mathcal{G}_x \mid \delta(v) \le 2g-2\} \text{ and}\\
\mathbf{P}\mathrm{Bun}_2^{\mathrm{tr}}\,X \;&\subset\; \{v \in \mathrm{Vert}\,\mathcal{G}_x \mid \delta(v) < 0 \text{ and even}\}\,.
\end{aligned}$$

Proof. The number of cusps is $\#\mathrm{Cl}\,\mathcal{O}_X^x = \#(\mathrm{Cl}\,X\,/\,\langle x\rangle) = \#\mathrm{Cl}^0\,X \cdot \#(\mathbf{Z}/d_x\mathbf{Z}) = h_X d_x$. That the vertices of cusps are disjoint and only intersect in the given point with the nucleus, is clear by definition. Regarding the edges, recall from paragraph 4.4.2 that if there is an edge from v to w in $\mathcal{G}_x$, then there is also an edge from w to v. But Theorem 5.4.6 implies that each vertex of a cusp that does not lie in the nucleus only connects to a vertex of the same cusp, hence every edge of $\mathcal{G}_x$ either lies in a cusp or in the nucleus, and we have proven (i).

The nucleus is finite since $\mathbf{P}\mathrm{Bun}_2^{\mathrm{indec}}\,X$ is finite by Corollary 5.3.5 and the intersection $\mathbf{P}\mathrm{Bun}_2^{\mathrm{dec}}\,X \cap \mathrm{Vert}\,\mathcal{N}_x$ is finite by the definition of the nucleus and Proposition 5.2.3. Since the cusps are contractible as CW-complexes, $\mathcal{N}_x$ and $\mathcal{G}_x$ have the same homotopy type. Therefore the number of components is $\#\big(\mathrm{Cl}\,\mathcal{O}_F^x\,/\,2\mathcal{O}_F^x\big)$ by Proposition 4.4.11. By Lemma 5.4.4, every vertex v has a Φ_x-neighbour w with $\delta(w) = \delta(v) + d_x$, thus the upper bound for the distance of vertices in the nucleus to one of the cusps. This proves (ii).

The four statements of Part (iii) follow from the definition of a cusp, Proposition 5.3.7 (iv), Proposition 5.3.2 and Proposition 5.3.8, respectively. □

5.4.10 (Remark on Figure 5.2) Define $h = h_X$, $m = m_X$ and $d = d_x$. Further let $D_1, \dots, D_{hd}$ be representatives for $\mathrm{Cl}\,\mathcal{O}_F^x$ with $m < \deg D_i \le m + d$ for $i = 1\dots, hd$. The cusps $\mathcal{C}_x(D_i)$, $i = 1, \dots, hd$, can be seen in Figure 5.2 as the regions in the dotted squares that are open to the right. The nucleus $\mathcal{N}_x$ is contained in the dotted rectangle to the left. Since we have no further information about the nucleus, we leave the area in the rectangle open.

The δ-line on the bottom of the picture indicates the value $\delta(v)$ for the vertices v in the graph that lie vertically above $\delta(v)$.

The dotted lines refer to the vertices, which are elements of either $\mathbf{P}\mathrm{Bun}_2^{\mathrm{gi}}\,X$, $\mathbf{P}\mathrm{Bun}_2^{\mathrm{tr}}\,X$, or $\mathbf{P}\mathrm{Bun}_2^{\mathrm{dec}}\,X$. These lines are drawn with reference to the δ-line to reflect part (iii) of the theorem.

5.4.11 Example (The projective line). Let X be the projective line over $\mathbf{F}_q$. Then $g_X = 0$, $h_X = 1$ and X has a closed point x of degree 1. This means that

$$\mathbf{P}\mathrm{Bun}_2^{\mathrm{dec}} X \;=\; \{c_{nx}\}_{n\geq 0}\,.$$

Since an indecomposable bundle $\mathcal{M}$ must satisfy both $\delta(\mathcal{M}) \geq 0$ and $\delta(\mathcal{M}) \leq -2$ which is impossible, all projective line bundles decompose. Theorem 5.4.6 together with the fact that the weights around each vertex sum to $q+1$ in the graph of Φ_x determines $\mathcal{G}_x$ completely, as illustrated in Figure 5.3, and we recover the result from Example 4.3.5.

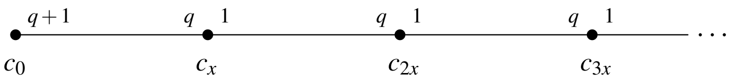


Figure 5.3: The graph of Φ_x for a degree one place x of a rational function field

We conclude this section with two useful lemmas. Recall that $\mathcal{L}_x$ denotes the line bundle associated to the divisor class $[x] \in \mathrm{Cl}\, X$.

5.4.12 Lemma. *Consider an exact sequence*

$$0 \longrightarrow \mathcal{L} \oplus \mathcal{L}' \longrightarrow \mathcal{M} \longrightarrow \mathcal{K}_x \longrightarrow 0\;\;.$$

If $\mathcal{L} \to \mathcal{M}$ is not a subbundle, then $\mathcal{M} \simeq \mathcal{L}\mathcal{L}_x \oplus \mathcal{L}'$.

Proof. Because $\mathcal{L} \to \mathcal{M}$ is not a subbundle, it extends to a subsheaf $\mathcal{L}\mathcal{L}_x \to \mathcal{M}$, and consequently we obtain a short exact sequence

$$0 \longrightarrow \mathcal{L}\mathcal{L}_x \oplus \mathcal{L}' \longrightarrow \mathcal{M} \longrightarrow \mathcal{F} \longrightarrow 0$$

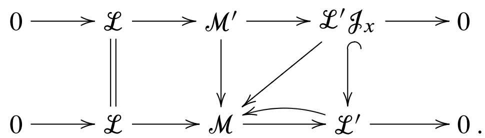
with some torsion sheaf $\mathcal{F}$. But $\deg \mathcal{F} = \deg \mathcal{M} - (\deg \mathcal{L} + \deg \mathcal{L}_x + \deg \mathcal{L}') = 0$, thus $\mathcal{F}$ is the zero sheaf, and $\mathcal{L}\mathcal{L}_x \oplus \mathcal{L}' \to \mathcal{M}$ an isomorphism. □

5.4.13 Lemma. *Let $\mathcal{L} \to \mathcal{M}$ be a line subbundle and*

$$0 \longrightarrow \mathcal{M}' \longrightarrow \mathcal{M} \longrightarrow \mathcal{K}_x \longrightarrow 0$$

the associated sequence. Let $\mathcal{L}' = \mathcal{M}/\mathcal{L}$. If $\mathcal{M} \simeq \mathcal{L} \oplus \mathcal{L}'$, then $\mathcal{M}' \simeq \mathcal{L} \oplus \mathcal{L}'\mathcal{J}_x$.

Proof. Note that $\mathcal{M}'/\mathcal{L} \simeq (\det \mathcal{M})\mathcal{J}_x\mathcal{L}^\vee \simeq \mathcal{L}'\mathcal{J}_x$. The hypothesis can be illustrated by:



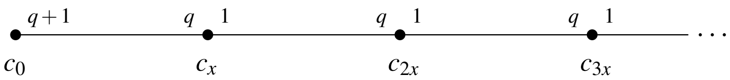
Since the composition $\mathcal{L}'\mathcal{J}_x \to \mathcal{M} \to \mathcal{K}_x$ is zero, $\mathcal{L}'\mathcal{J}_x \to \mathcal{M}$ lifts to $\mathcal{L}'\mathcal{J}_x \to \mathcal{M}'$, and the upper sequence also splits. □

5.5 Automorphic forms on graphs

Since we know the structure of $\mathcal{G}_x$ for places x in general, we are able to describe a strategy to discover unramified automorphic forms as functions on Vert $\mathcal{G}_x$ by solving eigenvalue equations for Φ_x. To have a notation that suits our translation of automorphic as functions on graphs, we put $f([g]) := f(g)$ for $[g] \in \operatorname{Vert} \mathcal{G}_x$.

5.5.1 Since all functions on Vert $\mathcal{G}_x$ are smooth, K-finite and $G_F Z_{\mathbf{A}}$-invariant, only the condition of moderate growth (paragraph 1.3.3) needs some consideration. It is easily seen to be equivalent with the existence of constants C and N such that for every divisor D of positive degree, one has $f(c_D) \leq C q^{N \deg D}$. This means that the growth behaviour on cusps should be at most polynomial in $q^{\deg D}$.

5.5.2 Example (Eigenfunctions on the projective line). We begin with calculating eigenfunctions in the easiest case. Let X be the projective line over $\mathbf{F}_q$ and x a place of degree 1. Note that $h_X = 1$. Recall that $\mathcal{G}_x$ looks like:



We investigate the space $\mathcal{A}(\Phi_x, \lambda)^K$, i.e. we search for functions on Vert $\mathcal{G}_x$ that satisfy the eigenvalue equation $\Phi_x(f) = \lambda f$. Evaluating this equation at the vertices yields

$$\lambda f(c_0) \;=\; \Phi_x(f)(c_0) \;=\; (q+1)\, f(c_x)$$

$$\text{and for } i \geq 1, \qquad \lambda f(c_{ix}) \;=\; \Phi_x(f)(c_{ix}) \;=\; f(c_{(i+1)x}) + q\, f(c_{(i-1)x}),$$

or equivalently that

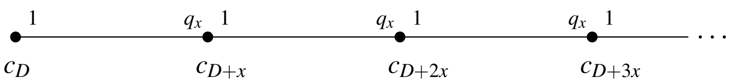
$$f(c_x) \;=\; \lambda\,(q+1)^{-1} f(c_0)$$

$$\text{and for } i \geq 1, \qquad f(c_{(i+1)x}) \;=\; \lambda f(c_{ix}) - q\, f(c_{(i-1)x}).$$

which determines all values $f(c_{ix})$ for $i \geq 1$ if $f(c_0)$ and λ are given. Thus $\mathcal{A}(\Phi_x, \lambda)^K$ is 1-dimensional for any λ. From Theorem 3.6.2 together with Lemmas 3.7.2 and 3.7.3, we know that there is precisely one set $\{\chi, \chi^{-1}\}$ such that $\tilde{E}(\,\cdot\,, \chi)$ is an eigenfunction of Φ_x with eigenvalue λ, so $\tilde{E}(\,\cdot\,, \chi)$ spans $\mathcal{A}(\Phi_x, \lambda)^K$, and—up to a constant multiple—its values at the vertices can be calculated by the equations we have just found.

Since these calculations hold for arbitrary $\lambda \in \mathbf{C}$, we have proven that there are no unramified cusp forms for the projective line.

5.5.3 Let X be any projective smooth irreducible curve over $\mathbf{F}_q$ and let x be a place of degree d_x. Choose a divisor D with $m_X < \deg D \leq m_X + d_x$ and consider the cusp $\mathcal{C}_x(D)$:



Let $f \in \mathcal{A}(\Phi_x, \lambda)^K$, then we obtain from evaluating the eigenvalue equation $\Phi_x f = \lambda f$ for every $i \geq 1$,

$$f(c_{D+(i+1)x}) \;=\; \lambda f(c_{D+ix}) - q\, f(c_{D+(i-1)x})\,.$$

Thus the restriction of f to Vert $\mathcal{C}_x(D)$ is determined by the eigenvalue λ, once its values at c_D and c_{D+x} are given, but there is no further restriction on $f(c_D)$ and $f(c_{D+x})$.

This consideration justifies that we only have to evaluate the eigenvalue equation at vertices of the nucleus to determine the eigenfunctions of Φ_x. If f vanishes at two consecutive vertices of a cusp, then it vanishes at all vertices of the cusp. We conclude that f restricted to the vertices of a cusp has compact support if and only if $f(c_D) = f(c_{D+x}) = 0$.

5.5.4 We proceed with $\widetilde{\mathcal{E}}(\Phi_x,\lambda)^K$. We know from Theorem 3.6.2 together with Lemmas 3.7.2 and 3.7.3 that this space is $h_X d_x$-dimensional, and all nontrivial elements are functions that have non-compact support. Even if we restrict the domain of these functions to the vertices of the cusps, they span an $h_X d_x$-dimensional space since otherwise, $\widetilde{\mathcal{E}}(\Phi_x,\lambda)^K$ would contain a Φ_x-eigenfunctions that vanishes at all vertices on the cusps and thus would be a cusp form, which contradicts the decomposition in Theorem 3.6.2.

Applying the results from paragraph 5.5.3, we see that the functions in $\widetilde{\mathcal{E}}(\Phi_x,\lambda)^K$ are determined by their values in the $2h_X d_x$ divisor classes of degrees $m_X+1,\dots,m_X+2d_x$. Evaluating the eigenvalue equation at all vertices of the nucleus defines an $h_X d_x$-dimensional subspace of the functions on these divisor classes.

In paragraph 3.7.17, we defined a finite set S of places such that for any $\chi \in \Xi_0$, the span of $\tilde{E}(\,\cdot\,,\chi)$ equals the intersection of all $\widetilde{\mathcal{E}}(\Phi_x,\lambda_x(\chi))^K$ with $x \in S$. Thus we can determine $\tilde{E}(\,\cdot\,,\chi)$ up to a constant multiple by evaluating the eigenvalue equations $\Phi_x f = \lambda_x(\chi) f$ at all vertices of the nuclei $\mathcal{N}_x$ for all $x \in S$ if there are no nontrivial cusp forms that have the same eigenvalues $\lambda_x(\chi)$ for Φ_x for all $x \in S$.

5.5.5 Residues of Eisenstein series fit perfectly in the picture of the completed Eisenstein part, but on the graph, they can be described in a particularly simple way: Theorem 2.4.2 states that if $\chi = \omega|\,|^{\pm 1/2} \in \Xi_0$ with $\omega^2 = 1$, then the residue $\tilde{E}(\,\cdot\,,\chi)$ is nothing else but a constant multiple of $\omega \circ \det$.

In Section 4.5 we defined a labelling of the vertices by taking the determinant

$$\operatorname{Vert}\mathcal{G}_x \;=\; G_F\backslash G_{\mathbf{A}}/KZ_{\mathbf{A}} \xrightarrow{\det} F^\times\backslash\mathbf{A}^\times/\mathcal{O}_{\mathbf{A}}^\times\mathcal{Q}_{\mathbf{A}} \;\simeq\; \operatorname{Cl}F/2\operatorname{Cl}F\,.$$

Since $\omega^2 = 1$, this character factors through $\operatorname{Cl}F/2\operatorname{Cl}F$, and $\omega\circ\det$ is thus a function that assigns to the vertices of $\mathcal{G}_x$ the values ± 1 depending on their label. The residue $\tilde{E}(\,\cdot\,,\chi)$ is a Φ_x-eigenfunction with eigenvalue (q_x+1) if the values of adjacent vertices have the same sign, and the eigenvalue is $-(q_x+1)$ if the values of adjacent vertices have opposite signs.

5.5.6 The $\mathcal{H}_K$-eigenfunctions that are cusp forms are characterised as those $\mathcal{H}_K$-eigenfunctions with compact support. More precisely, from paragraph 5.5.3, it follows that the support of a cusp form is contained in the set of vertices $v \in \mathcal{G}_x$ with $\delta(v) \le m_X$, a finite set. In particular the space of unramified cusp forms $\mathcal{A}_0^K$ is finite dimensional. By the multiplicity one theorem (3.5.3), $\mathcal{A}_0^K$ has a basis of $\mathcal{H}_K$-eigenfunctions, which are unique up to constant multiple. Since $\mathcal{A}_0^K$ is finite dimensional, there is a finite set $S \subset |X|$ such that the Hecke operators Φ_x for $x \in S$ can distinguish these basis vectors.

5.5.7 We give a reformulation in terms of the matrix associated to $\mathcal{G}_x$ as defined in paragraph 4.1.8. Let M_x be the matrix associated to Φ_x. This infinite dimensional matrix is characterised by the property that $\Phi_x f = M_x f$ for every $f \in \mathcal{A}^K$, where $M_x f$ is defined by identifying an unramified automorphic form with the infinite dimensional vector consisting of its values at all vertices of $\mathcal{G}_x$. An unramified automorphic form f is thus a Φ_x-eigenfunction with eigenvalue λ if and only if it lies in the kernel of $M_x(\lambda) = M_x - \lambda \cdot 1$.

Define the submatrix $(M'_x(\lambda))_{v,v'}$, where the row index ranges over $v \in \operatorname{Vert} \mathcal{N}_x$, i.e. over the vertices v with $\delta(v) \leq m_X + d_x$, and the column index ranges over all $v' \in V$, where $V \subset \operatorname{Vert} \mathcal{G}_x$ is subset of vertices v' with $\delta(v) \leq m_X + 2d_x$.

Let A denote the space of functions on V. Then $M'_x(\lambda)$ can be seen as the restriction of $M_x(\lambda)$ to A, where we delete all rows of $M_x(\lambda)$ that have entries outside V. The restriction map $\mathcal{A}^K \to A$ induces a bijection of the kernel of $M_x(\lambda)$ with the kernel of $M'_x(\lambda)$ because a function that satisfies the eigenvalue equation at all $v \in V$ determines a unique Φ_x-eigenfunction on $\operatorname{Vert} \mathcal{G}_x$, see paragraph 5.5.3.

Since there are $h_X d_x$ vertices $v \in \mathcal{G}_x$ with $m_X + d_x < \delta(v) \leq m_X + 2d_x$, the kernel of $M'_x(\lambda)$ has at least dimension $h_X d_x$ independent of the value of λ. On the other hand, there are only finitely many values for λ such that the kernel of $M'_x(\lambda)$ has a larger dimension.

We know from paragraph 5.5.4 that $h_X d_x$ linearly independent functions of the completed Eisenstein part lie in the kernel, which are characterised by the property that they do not vanish on all vertices v with $m_X < \delta(v) \leq m_X + 2d_x$. This means that we can sort out the cusp form by looking at the functions in the kernel of $M'_x(\lambda)$ that vanish on all vertices v with $m_X < \delta(v) \leq m_X + 2d_x$.

5.5.8 Remark. When we want to determine the cusp forms that are $\mathcal{H}_K$-eigenfunctions, we look for the solutions of a system of linear equations with integer coefficients, which contain the eigenvalues for Hecke operators as parameters. The cusp forms occur if these eigenvalues satisfy certain algebraic relations, which occur as vanishing condition on determinants of submatrices of $M_x(\lambda)$ as considered in the previous paragraph. This means that the eigenvalues are algebraic numbers. Moreover, the degree of the defining algebraic relations is bounded by the number of vertices v with $\delta(v) \leq m_X$, since only for these vertices, the eigenvalue equation can contain a non-zero multiple of the eigenvalue.

5.5.9 Example (Derivatives of Eisenstein series). We will show in the example of the projective line X over $\mathbf{F}_q$ and a place x of degree 1 how to determine the derivatives of Eisenstein series (or residues) as functions on $\operatorname{Vert} \mathcal{G}_x = \{c_{ix}\}_{i \geq 0}$.

Let $\chi \in \Xi_0$ with $\chi^2 \neq 1$, then we know from Theorem 3.6.2 that

$$\Phi_x(\tilde{E}^{(1)}(c_{ix}, \chi)) \;=\; \lambda_x(\chi)\, \tilde{E}^{(1)}(c_{ix}, \chi) + \ln q_x \, \lambda_x^-(\chi)\, \tilde{E}(c_{ix}, \chi)$$

for all $i \geq 0$. For better readability, put $f = \tilde{E}(\,\cdot\,, \chi)$, $f' = \tilde{E}^{(1)}(\,\cdot\,, \chi)$, $\lambda = \lambda_x(\chi)$ and $\lambda^- = \ln q_x \, \lambda_x^-(\chi)$. Calculating $\Phi_x(\tilde{E}^{(1)}(c_{ix}, \chi))$ for every $i \geq 0$ gives:

$$f'(c_x) \;=\; (q+1)^{-1} \big(\lambda\, f'(c_0) + \lambda^-\, f(c_0)\big)$$

$$\text{and for } i \geq 1, \qquad f'(c_{(i+1)x}) \;=\; \lambda\, f'(c_{ix}) - q\, f'(c_{(i-1)x}) + \lambda^-\, f(c_{ix}).$$

Since we have already determined $f = \tilde{E}(\,\cdot\,, \chi)$ (up to a constant multiple), this system of equations determines $f' = \tilde{E}^{(1)}(\,\cdot\,, \chi)$ up to a constant multiple.

Note that in case $\chi^2 = 1$, one obtains the same results, but the constants look different.

5.5.10 Remark. This strategy generalises to a way of determining all functions in $\widetilde{\mathcal{E}}^K$. Thus we are able to determine the values of each unramified admissible automorphic form by solving a finite system of linear equations.

5.5.11 Example (An automorphic form that is not admissible). Let X be the projective line and x a place of degree 1. Define $f \in \mathcal{A}^K$ by $f(c_0) = 1$ and $f(c_{ix}) = 0$ for all $i \geq 1$.

Then $\Phi_x(f)$ is the function with $\Phi_x(f)(c_0) = 0$, $\Phi_x(f)(c_x) = q + 1$ and trivial on all c_{ix} with $i \geq 2$. One shows by induction on n that the support $\Phi_x^n(f)$ is contained in $\{c_{ix}\}_{i=0,\dots,n}$ and that $\Phi_x^n(f)(c_{nx}) = q + 1$.

Thus the set of functions $\{\Phi_x^n(f)\}_{n\geq 0}$ spans an infinite dimensional space in $\mathcal{A}^K$, and we see that f is not admissible.

Note that the space of unramified automorphic forms with compact support is invariant under the action of $\mathcal{H}_K$. The subspace of admissible functions is precisely $\mathcal{A}_0^K$. Therefore, every unramified automorphic form with compact support that is not an $\mathcal{H}_K$-eigenfunction is not admissible.

The theory of toroidal automorphic forms

In this chapter, the theories of the previous chapters combine into the main results of this thesis about the space of unramified toroidal automorphic forms. The first step is to prove their finite dimensionality, which implies that they are contained in the direct sum of the Eisenstein, residual and cuspidal part. This chapter shows that the Eisenstein part admits a $(g_F - 1)h_F + 1$-dimensional subspace of unramified toroidal automorphic forms and the residual part contains no nontrivial toroidal automorphic form at all. A translation of a result of Waldspurger from number fields to global function fields would clarify the question of the existence of toroidal cusp forms. Finally, the question of unitarizability and the connection with the Riemann hypothesis are discussed.

6.1 Finite dimensionality

In this section, we will give a finite upper bound for the dimension of the space $\mathcal{A}^K_{\mathrm{tor}}(E)$ of unramified toroidal automorphic forms of the quadratic constant field extension $E = \mathbf{F}_{q^2} F$ over F. In particular, this shows that the space $\mathcal{A}^K_{\mathrm{tor}}$ of unramified toroidal automorphic forms is finite dimensional.

6.1.1 Let $p : X' \to X$ be the map of curves that corresponds to the field extension E/F. If f is an unramified automorphic form and $\mathcal{M} = \mathcal{M}_g \in \mathrm{Bun}_2 X$ for $g \in G_{\mathbf{A}}$ (Lemma 5.1.6), then we write $f([\mathcal{M}]) = f(g)$, where $[\mathcal{M}] \in \mathbf{P}\mathrm{Bun}_2 X$ is the class represented by $\mathcal{M}$.

Let T be a torus corresponding to the inclusion

$$\Theta_E :\ E^\times \simeq \mathrm{Aut}_E(E) \hookrightarrow \mathrm{Aut}_F(E) \stackrel{\sim}{\longrightarrow} G_F$$

of the units of E given by a basis of E/F that is contained in $\mathbf{F}_{q^2}$, cf. paragraph 1.5.2. Recall from paragraph 1.5.12 the definition

$$f_T(g) = \int\limits_{T_F Z_{\mathbf{A}} \backslash T_{\mathbf{A}}} f(tg)\, dt \ .$$

6.1.2 Theorem. *If T is as above and $c_T = \mathrm{vol}(T_F Z_{\mathbf{A}} \backslash T_{\mathbf{A}}) / \#\big(\mathrm{Pic}\, X' / p^*(\mathrm{Pic}\, X)\big)$, then for all $f \in \mathcal{A}^K$,*

$$f_T(e) \quad = \quad c_T \quad \cdot \sum_{[\mathcal{L}] \in \mathrm{Pic}\, X' / p^*(\mathrm{Pic}\, X)} f([p_* \mathcal{L}]) \ .$$

Proof. To avoid confusion, we write $\mathbf{A}_F = \mathbf{A}$. We introduce the following notation. For an $x \in |X|$ that is inert in E/F, we define $\mathcal{O}_{E,x} := \mathcal{O}_{E,y}$, where y is the unique place that lies over x. For an $x \in |X|$ that is split in E/F, we define $\mathcal{O}_{E,x} := \mathcal{O}_{E,y_1} \oplus \mathcal{O}_{E,y_2}$, where y_1 and y_2 are the two places that lie over x. Note that there is no place that ramifies. Let $\mathcal{O}_{E_x}$ denote the completion of $\mathcal{O}_{E,x}$. Then $\mathcal{O}_{E_x}$ is a free module of rank 2 over $\mathcal{O}_{F_x} = \mathcal{O}_x$ for every $x \in |X|$.

The basis of E over F that defines T is contained in $\mathbf{F}_{q^2}$. It is thus a basis of $\mathcal{O}_{E_x}$ over $\mathcal{O}_{F_x}$ for every $x \in |X|$. This shows at once that $\Theta_E^{-1}(K) = \mathcal{O}_{\mathbf{A}_E}^\times$ and that the diagram

$$\begin{array}{ccc} E^\times \backslash \mathbf{A}_E^\times / \mathcal{O}_{\mathbf{A}_E}^\times & \xrightarrow{1:1} & \operatorname{Pic} X' \\ \downarrow \Theta_E & & \downarrow p_* \\ G_F \backslash G_{\mathbf{A}_F} / K & \xrightarrow{1:1} & \operatorname{Bun}_2 X \end{array}$$

commutes, where the horizontal arrows are the bijections as described in paragraph 5.1.5.

The action of $\mathbf{A}_F$ on $E^\times \backslash \mathbf{A}_E^\times / \mathcal{O}_{\mathbf{A}_E}^\times$ and $G_F \backslash G_{\mathbf{A}_F} / K$ by scalar multiplication is compatible with the action of $\operatorname{Pic} X$ on $\operatorname{Pic} X'$ and $\operatorname{Bun}_2 X$ by tensoring in the sence that all maps in the above diagram become equivariant if we identify $\operatorname{Pic} X$ with $F^\times \backslash \mathbf{A}_F^\times / \mathcal{O}_{\mathbf{A}_F}^\times$, cf. Lemma 5.1.6. Taking orbits under these compatible actions yields the commutative diagram

$$\begin{array}{ccc} E^\times \mathbf{A}_F^\times \backslash \mathbf{A}_E^\times / \mathcal{O}_{\mathbf{A}_E}^\times & \xrightarrow{1:1} & \operatorname{Pic} X' / p^* \operatorname{Pic} X \\ \downarrow \Theta_E & & \downarrow p_* \\ G_F Z_{\mathbf{A}_F} \backslash G_{\mathbf{A}_F} / K & \xrightarrow{1:1} & \mathbf{P}\operatorname{Bun}_2 X\ . \end{array}$$

Since f is right K-invariant, we may take the quotient of the domain of integration by $T_{\mathbf{A}_F} \cap K = \Theta_E(\mathcal{O}_{\mathbf{A}_E}^\times)$ from the right and we obtain the assertion of the theorem for some still undetermined value of c. The value of c is computed by plugging in a constant function for f. □

6.1.3 Remark. We are fortunate to find a torus that has such a particularly simple description. If the basis elements of E over F have nontrivial valuation at some place—which necessarily happens if E is a quadratic extension different from the constant field extension—, then the inverse image of $K = \prod_{x \in |X|} G(\mathcal{O}_x)$ under $\mathbf{A}_E^\times \hookrightarrow G_{\mathbf{A}_F}$ does not equal $\mathcal{O}_{\mathbf{A}_E}^\times$.

It seems very unlikely to me that in the general case, $T_{\mathbf{A}}$ corresponds to the image of $p_* : \operatorname{Pic} X' / p^* \operatorname{Pic} X \to \mathbf{P}\operatorname{Bun}_2 X$, but it is rather the image of a certain extension of the Picard group that captures information about the valuation of the basis elements.

6.1.4 Write $\operatorname{Cl}^{\mathrm{pr}} X$ for the set of divisor classes that are represented by prime divisors and $\operatorname{Cl}^{\mathrm{eff}} X$ for the semigroup they generate, viz. for all classes that are represented by effective divisors. In particular, $\operatorname{Cl}^{\mathrm{eff}} X$ contains 0, the class of the zero divisor, and for all other $[D] \in \operatorname{Cl}^{\mathrm{eff}}$, $\deg D > 0$. Denote by $\operatorname{Cl}^d X$ the set of divisor classes of degree d and by $\operatorname{Cl}^{\geq d} X$ the set of divisor classes of degree at least d. Let g_X be the genus of X.

6.1.5 Lemma.

$$\operatorname{Cl}^{\geq g_X} X \quad \subset \quad \operatorname{Cl}^{\mathrm{eff}} X\ .$$

Proof. Let C be a canonical divisor on X, which is of degree $2g_X - 2$. For a divisor D, define $l(D) = \dim_{\mathbf{F}_q} H^0(X, \mathcal{L}_D)$. We have $[D] \in \mathrm{Cl}^{\mathrm{eff}} X$ if and only if $l(D) > 0$, cf. [28, Section IV.1]. The Riemann-Roch theorem is

$$l(D) - l(D - C) \;=\; \deg D + 1 - g_X \;,$$

cf. [28, Thm. IV.1.3].

If now $[D] \in \mathrm{Cl}^{\geq g_X} X$, then $\deg D \geq g_X$ and the Riemann-Roch theorem implies that $l(D) \geq \deg D + 1 - g_X > 0$. □

6.1.6 Let D be an effective divisor. Then it can be written in a unique way up to permutation of terms as a sum of prime divisors $D = x_1 + \ldots + x_n$. We set $\Phi_D = \Phi_{x_1} \cdots \Phi_{x_n}$. Recall from Lemma 1.4.15 that $\mathcal{H}_K$ is commutative, so Φ_D is well-defined. Further we briefly write $\mathcal{G}_D$ for the graph $\mathcal{G}_{\Phi_D, K}$ of Φ_D, and $\mathcal{U}_D(v)$ for $\mathcal{U}_{\Phi_D, K}(v)$.

Let $[D] \in \mathrm{Cl}\, X$. Recall from paragraph 5.1.3 that $\mathcal{L}_D$ denotes the associated line bundle and from paragraph 5.2.2 that c_D denotes the vertex that is represented by $\mathcal{L}_D \oplus \mathcal{O}_X$. Recall from Proposition 5.3.7 (iv) that $\delta(c_D) = \deg D$, where δ is as defined in paragraph 5.3.1.

6.1.7 Lemma. *Let D be an effective divisor of positive degree.*

(i) *Let $v, v' \in \mathrm{Vert}\, \mathcal{G}_D$. If v' is a Φ_D-neighbour of v, then $|\delta(v') - \delta(v)| \leq \deg D$.*

(ii) *Moreover,*

$$(c_0, c_D, q+1) \;\in\; \mathcal{U}_D(c_0)\;,$$

and for all other edges $(c_0, v, \lambda) \in \mathcal{U}_D(c_0)$, the inequality $\delta(v) < \deg D$ holds.

Proof. We do induction on the number of factors in $\Phi_D = \Phi_{x_1} \cdots \Phi_{x_n}$ with $x_1, \ldots, x_n$ being prime divisors. Put $x = x_n$.

If $n = 1$, then $\Phi_D = \Phi_x$. Assertion (i) follows from Proposition 5.4.2 and assertion (ii) follows from Theorem 5.4.6.

If $n > 1$, we can write $\Phi_D = \Phi_{D'} \Phi_x$ for the effective divisor $D' = x_1 + \cdots + x_{n-1}$, which is of positive degree $\deg D' = \deg D - \deg x$. Assume that (i) and (ii) hold for D'.

We prove (i). Let v' be a Φ_D-neighbour of v. By Proposition 4.1.7, there is a v'' that is a $\Phi_{D'}$-neighbour of v and a Φ_x-neighbour of v', thus the inductive hypothesis and Proposition 5.4.2 imply

$$\big|\delta(v') - \delta(v)\big| \;\leq\; \big|\delta(v') - \delta(v'')\big| + \big|\delta(v'') - \delta(v)\big| \;\leq\; \deg D' + \deg x \;=\; \deg D\;.$$

We prove (ii). By the inductive hypothesis, there is precisely one edge (c_0, v', m) in $\mathcal{U}_{D'}(c_0)$ with $\delta(v') = \deg D'$, namely, $(c_0, c_{D'}, q+1)$. By Theorem 5.4.6, there is precisely one edge $(c_{D'}, v, m')$ in $\mathcal{U}_x(c_{D'})$ with $\delta(v) - \delta(c_{D'}) = \deg x$, namely, $(c_{D'}, c_D, 1)$. Proposition 4.1.7 together with (i) implies (ii). □

6.1.8 Theorem. *The dimension of the space of unramified toroidal automorphic forms is finite, bounded by*

$$\dim \mathcal{A}_{\mathrm{tor}}(E)^K \;\leq\; \#\big(\mathbf{P}\mathrm{Bun}_2\, X - \{c_D\}_{[D] \in \mathrm{Cl}^{\mathrm{eff}} X}\big)\;,$$

where E/F is the constant field extension.

Proof. First remark that given the inequality in the theorem, finite-dimensionality follows since the right hand set is finite. Indeed, by Lemma 6.1.5,

$$\mathbf{P}\mathrm{Bun}_2\,X - \{c_D\}_{[D]\in\mathrm{Cl}^{\mathrm{eff}}X} \;\subset\; \left\{v\in\mathbf{P}\mathrm{Bun}_2\,X \mid \delta(v)\le m_X\right\}$$

since $m_x = \max\{0, 2g_X - 2\} \ge g_X - 1$, and the latter set is finite.

We now proceed to the proof of the inequality. Let $f\in\mathcal{A}_{\mathrm{tor}}(E)^K$. We will show by induction on $d=\deg D$ that the value of f at a vertex c_D with $[D]\in\mathrm{Cl}^{\mathrm{eff}}X$ is uniquely determined by the values of f at the elements of $\mathbf{P}\mathrm{Bun}_2\,X - \{c_D\}_{[D]\in\mathrm{Cl}^{\mathrm{eff}}X}$. This will prove the theorem.

By Proposition 1.5.15 and Theorem 6.1.2, the condition for f to lie in $\mathcal{A}_{\mathrm{tor}}(E)^K$ reads

$$\sum_{[\mathcal{L}]\in(\mathrm{Pic}\,X'/\,p^*\,\mathrm{Pic}\,X)} \Phi(f)([p_*\mathcal{L}]) \;=\; 0\,, \qquad \text{for all } \Phi\in\mathcal{H}.$$

If $d=0$, take Φ as the identity element in $\mathcal{H}_K$. We know from Proposition 5.3.8 that $p_*(\mathrm{Pic}\,X'/\,p^*\,\mathrm{Pic}\,X) = \mathbf{P}\mathrm{Bun}_2^{\mathrm{tr}}\,X\cup\{c_0\}$, so $f(c_0)$ equals a linear combination of values of f at vertices v in $\mathbf{P}\mathrm{Bun}_2^{\mathrm{tr}}\,X$, which all satisfy $\delta(v)<0$. Since the zero divisor class is the only class in $\mathrm{Cl}^{\mathrm{eff}}X$ of degree 0, we have proved the case $d=0$.

Next, let D be an effective divisor of degree $d>0$, and put $\Phi=\Phi_D$. If v is a Φ_D-neighbour of w, then $\delta(v)$ and $\delta(w)$ can differ at most by d (Lemma 6.1.7 (i)). Therefore all Φ_D-neighbours v of vertices in $\mathbf{P}\mathrm{Bun}_2^{\mathrm{tr}}\,X$ have $\delta(v)<d$. The vertex c_D is the only Φ_D-neighbour v of c_0 with $\delta(v)=d$ (Lemma 6.1.7 (ii)). Thus

$$0 \;=\; \sum_{\mathcal{L}\in(\mathrm{Pic}\,X'/\,p^*\,\mathrm{Pic}\,X)} \Phi_D(f)([p_*\mathcal{L}]) \;=\; (q+1)\,f(c_D) \;+\; \sum_{\substack{\mathcal{L}\in(\mathrm{Pic}\,X'/\,p^*\,\mathrm{Pic}\,X),\\ ([p_*\mathcal{L}],v,\lambda)\in\mathcal{U}_D([p_*\mathcal{L}])\\ \delta(v)<d}} \lambda\, f(v)$$

determines $f(c_D)$ as the linear combination of values of f at vertices v with $\delta(v)<d$. By the inductive hypothesis, $f(c_D)$ is already determined by the values of f at vertices that are not contained in $\{c_D\}_{[D]\in\mathrm{Cl}^{\mathrm{eff}}X}$. □

6.1.9 Theorem. *The space $\mathcal{A}_{\mathrm{tor}}^{\mathrm{nr}}$ of unramified toroidal automorphic forms is admissible.*

Proof. This follows from Theorem 6.1.8 by Theorem 3.6.3. □

Recall from Example 5.4.11 that for the function field of the projective line over $\mathbf{F}_q$, every projective line bundle is of the form c_{nx} for some non-negative integer n, where x is a place of degree 1. So Theorems 6.1.8 and 3.6.3 immediately imply:

6.1.10 Theorem. *If X is the projective line over $\mathbf{F}_q$, then $\mathcal{A}_{\mathrm{tor}}^{\mathrm{nr}}(E) = 0$.* □

6.1.11 The finite dimensionality of the space of unramified toroidal automorphic forms implies that $\mathcal{A}_{\mathrm{tor}}^K\subset\mathcal{A}_{\mathrm{adm}}^K$. By Theorem 3.6.3, we obtain the following decomposition, where $\mathcal{E}_{\mathrm{tor}}^K=\mathcal{A}_{\mathrm{tor}}^K\cap\mathcal{E}$ is the space of unramified toroidal automorphic forms in the Eisenstein part, $\mathcal{R}_{\mathrm{tor}}^K=\mathcal{A}_{\mathrm{tor}}^K\cap\mathcal{R}$ is the space of unramified toroidal automorphic forms in the residual part and $\mathcal{A}_{0,\mathrm{tor}}^K=\mathcal{A}_{\mathrm{tor}}^K\cap\mathcal{A}_0$ is the space of unramified toroidal cusp forms.

6.1.12 Corollary. $\mathcal{A}_{\mathrm{tor}}^K \;=\; \mathcal{E}_{\mathrm{tor}}^K\,\oplus\,\mathcal{R}_{\mathrm{tor}}^K\,\oplus\,\mathcal{A}_{0,\mathrm{tor}}^K\,.$

This allows us to investigate the Eisenstein part, the residual part and the cuspidal separately in the follwing three sections.

6.1.13 Example (Finite dimensionality in a ramified case). Let X be the projective line over $\mathbf{F}_q$ and x a place of degree 1. Consider $K' = \{k \in K \mid k_x \equiv \begin{pmatrix} 1 & \\ & 1 \end{pmatrix} \pmod{\pi_x}\}$, which is the same subgroup as in Example 4.3.9. Recall this example, in particular the definitions of c'_0 and $c'_{nx,w}$ for $n \geq 0$ and $w \in \mathbf{P}^1(\kappa_x)$, the definitions of the Hecke operators Φ'_x and $\Phi'_{y,\gamma}$ for a place $y \neq x$ of degree 1 and $\gamma \in G_{\mathbf{F}_q}$ as well as the illustrations of their graphs in Figures 4.9 and 4.8, respectively.

Taking these results on trust, we can prove that $\mathcal{A}^{K'}_{\mathrm{tor}}$ is finite dimensional by the same strategy that we used in the proof of Theorem 6.1.8, namely, we do an induction on d to show that for $d \geq 2$, the value of an $f \in \mathcal{A}^{K'}_{\mathrm{tor}}$ at a vertex v with $\delta(v) = d$ is uniquely determined by the values of f at the vertices v' with $\delta(v') \leq d$.

Let $d = 2$. In the present case, Theorem 6.1.2 yields $f(c'_0) = 0$. If we apply $\Phi'_{y,e}$ to this equation, where e denotes the identity matrix, then we get

$$\sum_{w \in \mathbf{P}^1(\kappa_x)} f(c'_{x,w}) \;=\; 0\,,$$

if we apply Φ'_x to it, we get

$$\sum_{\substack{w \in \mathbf{P}^1(\kappa_x) \\ w \neq [0:1]}} f(c'_{x,w}) \;=\; 0\,.$$

Substracting the latter from the former equation yields $f(c'_{x,[0:1]}) = 0$. If now v is a vertex with $\delta(v) = 2$, i.e. $v = c'_{2x,w}$ for some $w \in \mathbf{P}^1(\kappa_x)$, we choose a $\gamma \in G_{\mathbf{F}_q}$ such that $w = [0:1]\gamma$ and obtain by applying $\Phi'_{y,\gamma}$ to $f(c'_{x,[0:1]}) = 0$ that

$$q\, f(c'_0) + f(c'_{2x,w}) \;=\; 0$$

and thus $f(c'_{2x,w}) = 0$.

If $d > 2$, then $v = c'_{dx,w}$ for some $w \in \mathbf{P}^1(\kappa_x)$. Applying $(\Phi'_{y,e})^{d-2}$ to the equation $f(c'_{2x,w}) = 0$ yields

$$f(c'_{dx,w}) + \sum_{\delta(v)<d} m(v)\, f(v) \;=\; 0$$

for certain numbers $m(v)$. This completes the induction and we have thus shown that $\mathcal{A}^{K'}_{\mathrm{tor}}$ is finite-dimensional.

6.1.14 Remark. It is not difficult to generalise the inductive step to the case of an arbitrary curve X and arbitrary $K' < K$ since the graphs of Hecke operators for unramified places, i.e. for places y such that $K'_y = K_y$, have 'cusps' that behave as in the unramified case. This can be seen, for example, in the illustration of $\Phi'_{y,\gamma}$ in Figure 4.8.

But the initial step causes problems: For Hecke operators relative to proper subgroups K' of K, typically, the class of the identity matrix e is connected to several vertices v with the same value for $\delta(v)$. It is not possible to show finite dimensionality by considering only Hecke operators for unramified places; one also has to consider the more involved Hecke operators for ramified places.

Nevertheless, we state:

6.1.15 Conjecture. *The space of toroidal automorphic forms is admissible.*

6.2 Toroidal Eisenstein series

In this section, we investigate the intersection of the space of toroidal automorphic forms with the Eisenstein part.

6.2.1 Since $\mathcal{A}_{\rm tor}^{\rm nr}$ is admissible, Theorem 3.1.9 implies that $\mathcal{E}_{\rm tor}^{\rm nr} = \mathcal{E} \cap \mathcal{A}_{\rm tor}^{\rm nr}$ is characterised by its unramified elements $\mathcal{E}_{\rm tor}^{K}$. Theorem 3.6.3 implies that $\mathcal{E}_{\rm tor}^{K}$ has a decomposition

$$\mathcal{E}_{\rm tor}^{K} \;=\; \bigoplus_{\substack{\{\chi,\chi^{-1}\}\subset\Xi_0 \\ \chi^2\neq|\;|^{\pm 1}}} \left(\mathcal{E}_{\rm tor}^{K} \cap \widetilde{\mathcal{E}}(\chi)^K\right),$$

where only finitely many terms $\mathcal{E}_{\rm tor}^{K} \cap \widetilde{\mathcal{E}}(\chi)^K$ are nontrivial as $\mathcal{E}_{\rm adm}^{\rm nr}$ is admissible. Each of these terms has a basis of the form

$$\left\{\tilde{E}(\,\cdot\,,\chi), \tilde{E}^{(1)}(\,\cdot\,,\chi), \ldots, \tilde{E}^{(n-1)}(\,\cdot\,,\chi)\right\},$$

where n is its complex dimension.

Thus it suffices to investigate Eisenstein series of the form $E(\,\cdot\,,\chi)$ and their derivatives $E^{(i)}(\,\cdot\,,\chi)$ for unramified quasi-characters χ in order to determine $\mathcal{E}_{\rm tor}^{\rm nr}$. We will, however, state and prove theorems for general quasi-characters $\chi \in \Xi$ where no additional effort is required.

6.2.2 Let E be a separable quadratic field extension of F. Consider an anisotropic torus $T \subset G$, whose F-rational points are the image of $E^\times$ under $\Theta_E : E \to \mathrm{Mat}_2(F)$. Recall from paragraph 1.5.2 that this injective F-linear homomorphism is given by the choice of a basis of E over F and that it extends to $\Theta_E : \mathbf{A}_E^\times \to G_{\mathbf{A}_F}$. Let $\mathrm{N}_{E/F} : \mathbf{A}_E^\times \to \mathbf{A}_F^\times$ be the norm of E over F extended to the ideles. We have that $\det(\Theta_E(t)) = \mathrm{N}_{E/F}(t)$ ([38, Prop. VI.5.6]).

Let h_E denote the class number of E and let q_E be the cardinality of the constant field of E. Consider the $\mathbf{A}_F$-linear projection

$$\begin{array}{rccc} \mathrm{pr}: & \mathrm{Mat}_2\,\mathbf{A}_F & \longrightarrow & \mathbf{A}_F^2\,. \\ & g & \longmapsto & (0,1)g \end{array}$$

The kernel of pr is contained in the upper triangular matrices and does not contain any nontrivial central matrix. Since the only maximal torus of G that is contained in the standard Borel subgroup B is the diagonal torus, the intersection of the upper triangular matrices with $T_{\mathbf{A}}$ is $Z_{\mathbf{A}}$. Thus $\Theta_E(\mathbf{A}_E) \cap \ker \mathrm{pr} = \{0\}$ and the $\mathbf{A}_F$-linear map $\widetilde{\Theta}_E = \mathrm{pr}\circ\Theta_E : \mathbf{A}_E \to \mathbf{A}_F^2$ is injective. We can consider $\widetilde{\Theta}_E$ as a collection of maps $\widetilde{\Theta}_{E,x}$ for $x \in |X|$. For each $x \in |X|$, this map $\widetilde{\Theta}_{E,x}$ is an injective homomorphism of 2-dimensional F_x-vectorspaces and necessarily surjective. This shows that $\widetilde{\Theta}_E$ is an isomorphism of $\mathbf{A}_F$-modules.

In the natural topology as free $\mathbf{A}_F$-modules, $\widetilde{\Theta}_E$ is thus a isomorphism of locally compact groups. Define $\varphi_T : \mathbf{A}_F^2 \to \mathbf{C}$ as $h_E(q_E-1)^{-1}(\mathrm{vol}\,\mathcal{O}_{\mathbf{A}_E})^{-1}$ times the characteristic function of $\mathrm{pr}\big(\Theta_E(\mathcal{O}_{\mathbf{A}_E})\big)$. Since $\widetilde{\Theta}_E$ is a homeomorphism, φ_T and also $\varphi_{T,g} = \varphi_T(\,\cdot\,g)$ are Schwartz-Bruhat functions for all $g \in G_{\mathbf{A}}$.

The following observation of Hecke ([30, p. 201]) was translated by Zagier ([83, pp. 298-299]) into adelic language.

6.2.3 Theorem. *Let T be an anisotropic torus corresponding to a separable field extension E/F. For every $\varphi : \mathbf{A}^2 \to \mathbf{C}$ that is a Schwartz-Bruhat function, $g \in G_{\mathbf{A}}$ and $\chi \in \Xi$, there exists a holomorphic function $e_T(g,\varphi,\chi,s)$ of $s \in \mathbf{C}$ with the following properties:*

(i) *For all $s \in \mathbf{C}$ such that $\chi^2|\,|^{2s} \neq |\,|^{\pm 1}$,*

$$E_T(g,\varphi,\chi,s) \;=\; e_T(g,\varphi,\chi,s)\; L_E(\chi \circ \mathrm{N}_{E/F}, s+1/2)\,.$$

(ii) *For every $g \in G_{\mathbf{A}}$ and $\chi \in \Xi$, there is a Schwartz-Bruhat function $\varphi : \mathbf{A}^2 \to \mathbf{C}$ such that*

$$e_T(g,\varphi,\chi,s) \;=\; \chi(\det g)\,|\det g|^{s+1/2}$$

for all $s \in \mathbf{C}$. If $\chi \in \Xi_0$, then $\varphi = \varphi_{T,g^{-1}}$ satisfies the equation.

Proof. Though this result is known and the following computation is done at several places in the literature (in chronological order: [83], [81], [12]), we will show a proof because of the relevance for this thesis.

For every $\varphi : \mathbf{A}^2 \to \mathbf{C}$ that is a Schwartz-Bruhat function, $g \in G_{\mathbf{A}}$ and $\chi \in \Xi$, both $E_T(g,\varphi,\chi,s)$ and $L_E(\chi \circ \mathrm{N}_{E/F}, s+1/2)$ are meromorphic functions of $s \in \mathbf{C}$. Define $e_T(g,\varphi,\chi,s)$ as their quotient. This is a meromorphic function of s that satisfies (i). Before showing that $e_T(g,\varphi,\chi,s)$ is holomorphic in s, we consider part (ii).

Part (ii) needs more care. Recall from paragraph 1.5.12 that we have made choices of Haar measures that match with the following changes of integrals. Let $\mathrm{Re}\, s > 1/2 - \mathrm{Re}\,\chi$, then Lemma 2.5.4 applies, and we obtain

$$E_T(g,\varphi,\chi,s) \;=\; \int\limits_{T_F Z_{\mathbf{A}} \backslash T_{\mathbf{A}}} \;\; \int\limits_{Z_F \backslash Z_{\mathbf{A}}} \sum_{u \in F^2 - \{0\}} \varphi(uztg)\, \chi(\det(ztg))\, |\det(ztg)|^{s+1/2}\, dz\, dt\,.$$

Since $T_F \backslash T_{\mathbf{A}} \simeq (T_F Z_{\mathbf{A}} \backslash T_{\mathbf{A}}) \times \big(Z_F \backslash Z_{\mathbf{A}}\big)$, we can apply Fubini's theorem (cf. paragraph 1.2.4) to derive

$$E_T(g,\varphi,\chi,s) \;=\; \int\limits_{T_F \backslash T_{\mathbf{A}}} \sum_{u \in F^2 - \{0\}} \varphi(utg)\, \chi(\det(tg))\, |\det(tg)|^{s+1/2}\, dt\,.$$

The map Θ_E identifies $\mathbf{A}_E^\times$ with $T_{\mathbf{A}_F}$. The $\mathbf{A}_F$-linear isomorphism $\widetilde{\Theta}_E$ identifies $\mathbf{A}_E$ with $\mathbf{A}_F^2$ and restricts to a bijection between $E^\times$ and $F^2 - \{0\}$. Thus we can rewrite the integral as

$$\chi(\det g)\, |\det g|^{s+1/2} \int\limits_{E^\times \backslash \mathbf{A}_E^\times} \sum_{u \in E^\times} \varphi(\widetilde{\Theta}_E(ut)g)\, \chi(\mathrm{N}_{E/F}(t))\, \big|\mathrm{N}_{E/F}(t)\big|^{s+1/2}\, dt\,.$$

If we define $\tilde{\varphi}_g = \varphi\big(\widetilde{\Theta}_E(\,\cdot\,)g\big) : \mathbf{A}_E \to \mathbf{C}$ and apply Fubini's theorem again, we get

$$E_T(g,\varphi,\chi,s) \;=\; \chi(\det g)\, |\det g|^{s+1/2} \int\limits_{\mathbf{A}_E^\times} \tilde{\varphi}_g(t)\, \chi \circ \mathrm{N}_{E/F}(t)\, |t|_{\mathbf{A}_E}^{s+1/2}\, dt\,.$$

Note that $\tilde{\varphi}_g : \mathbf{A}_E \to \mathbf{C}$ is a Bruhat-Schwartz function as φ is one. Thus the integral is the Tate integral $L_E(\tilde{\varphi}_g, \chi\circ \mathrm{N}_{E/F}, s+1/2)$.

Theorem 2.2.7 implies that there is a Schwartz-Bruhat function $\psi : \mathbf{A}_E \to \mathbf{C}$ such that

$$L_E(\psi, \chi\circ\mathrm{N}_{E/F}, s+1/2) \;=\; L_E(\chi\circ\mathrm{N}_{E/F}, s+1/2)\,.$$

If we define $\varphi : \mathbf{A}_F^2 \to \mathbf{C}$ to be the Schwartz-Bruhat function such that $\psi = \tilde{\varphi}_g$, then $e_T(g,\varphi,\chi,s) = \chi(\det g)\,|\det g|^{s+1/2}$. If $\chi \in \Xi_0$, then $\chi\circ\mathrm{N}_{E/F}$ is an unramified character of $\mathbf{A}_E$ and

$$\psi = \varphi_{T,g^{-1}}\big(\widetilde{\Theta}_E(\,\cdot\,)g\big) = \varphi_T \circ \widetilde{\Theta}_E$$

yields the desired ψ as it adopts the role of ψ_0 in Theorem 2.2.7. Hence (ii) holds by meromorphic continuation.

Let φ be arbitrary. By Theorem 2.2.7, $L_E(\psi,\chi\circ\mathrm{N}_{E/F}, s+1/2)$ equals a holomorphic multiple of $L_E(\chi\circ\mathrm{N}_{E/F}, s+1/2)$ in $s\in\mathbf{C}$ for any Schwartz-Bruhat function $\psi = \tilde{\varphi}_g$, thus $e_T(g,\varphi,\chi,s)$ is holomorphic in $s\in\mathbf{C}$. □

For any Schwartz-Bruhat function $\varphi : \mathbf{A}^2 \to \mathbf{C}$ and any $g \in G_{\mathbf{A}}$ we have that the automorphic form $g.E(\,\cdot\,,\varphi,\chi)$ is an element of $\mathcal{P}(\chi)$ (paragraph 2.3.22). By the definition of E-toroidality, we obtain as an immediate consequence:

6.2.4 Corollary. *Let $\chi\in\Xi$ such that $\chi^2 \neq |\;|^{\pm 1}$ and let $\varphi : \mathbf{A}^2 \to \mathbf{C}$ be a Schwartz-Bruhat function. Let E/F be a quadratic separable field extension. Then $E(\,\cdot\,,\varphi,\chi)$ is E-toroidal if and only if $L_E(\chi, 1/2) = 0$.*

6.2.5 We establish the analogue of Hecke's theorem for a split torus. Let $T = \{\begin{pmatrix} * & \\ & * \end{pmatrix}\} \subset G$ be the diagonal torus. We again write $\mathbf{A}$ for the adeles of F. Define the Schwartz-Bruhat function $\varphi_T : \mathbf{A}^2 \to \mathbf{C}$ as $h_F(q-1)^{-1}(\mathrm{vol}\,\mathcal{O}_{\mathbf{A}})^{-1}$ times the characteristic function of $\mathcal{O}_{\mathbf{A}}^2$, which is the same as φ_0 as defined in paragraph 2.3.20. Put $\varphi_{T,g} = \varphi_T(\,\cdot\,g)$, which is again a Schwartz-Bruhat function since multiplying with g from the right is an automorphism of the locally compact group $\mathbf{A}_F^2$.

Recall from paragraph 2.3.20 that we defined $f_{\varphi,\chi}(s) \in \mathcal{P}(\chi|\;|^s)$ as

$$f_{\varphi,\chi}(s)(g) = \int_{Z_{\mathbf{A}}} \varphi((0,1)zg)\,\chi(\det(zg))\,|\det(zg)|^{s+1/2}\;dz$$

for $\mathrm{Re}\,s > 1/2 - \mathrm{Re}\,\chi$. Put $e = \begin{pmatrix} 1 & \\ & 1 \end{pmatrix}$ and $w_0 = \begin{pmatrix} & 1 \\ 1 & \end{pmatrix}$.

6.2.6 Lemma. *Let T be the diagonal torus. For every $\varphi : \mathbf{A}^2 \to \mathbf{C}$ that is a Schwartz-Bruhat function, $g\in G_{\mathbf{A}}$ and $\chi\in\Xi$, there exists a holomorphic function $\tilde{e}_T(g,\varphi,\chi,s)$ of $s\in\mathbf{C}$ with the following properties.*

(i) *For all $s\in\mathbf{C}$ such that $\chi^2|\;|^{2s} \neq |\;|^{\pm 1}$,*

$$\int_{T_F Z_{\mathbf{A}}\backslash T_{\mathbf{A}}} \big(E(tg,\varphi,\chi,s) - f_{\varphi,\chi}(s)(tg) - f_{\varphi,\chi}(s)(w_0 tg)\big)\,dt \\ = \tilde{e}_T(g,\varphi,\chi,s)\,\big(L(\chi,s+1/2)\big)^2\,.$$

In particular, the left hand side is well-defined and converges.

(ii) *For every $g \in G_{\mathbf{A}}$ and $\chi \in \Xi$, there is a Schwartz-Bruhat function $\varphi : \mathbf{A}^2 \to \mathbf{C}$ such that*

$$e_T(g,\varphi,\chi,s) \;=\; \chi(\det g)\,|\det g|^{s+1/2}$$

for all $s \in \mathbf{C}$. If $\chi \in \Xi_0$, then $\varphi = \varphi_{T,g^{-1}}$ satisfies the equation.

Proof. Let $\operatorname{Re} s > 1/2 - \operatorname{Re}\chi$, and denote the left hand side of the equation in (i) by I. Recall from paragraph 1.5.12 that we have made choices of Haar measures that match with the following changes of integrals. We choose $\{e, w_0, \left(\begin{smallmatrix}1 & \\ c & 1\end{smallmatrix}\right)\}_{c\in F^\times}$ as a system of representatives of $B_F\backslash G_F$. By definition of $E(tg,\varphi,\chi,s)$,

$$E(tg,\varphi,\chi,s) - f_{\varphi,\chi}(s)(tg) - f_{\varphi,\chi}(s)(w_0 tg) \;=\; \sum_{c\in F^\times} f_{\varphi,\chi}(s)\Big(\left(\begin{smallmatrix}1 & \\ c & 1\end{smallmatrix}\right)tg\Big)\,.$$

Hence

$$I \;=\; \int\limits_{T_F Z_{\mathbf{A}}\backslash T_{\mathbf{A}}} \sum_{c\in F^\times} f_{\varphi,\chi}(s)\Big(\left(\begin{smallmatrix}1 & \\ c & 1\end{smallmatrix}\right)tg\Big)\,dt\,.$$

Note that this is a well-defined expression since

$$f_{\varphi,\chi}(s)\Big(\left(\begin{smallmatrix}1 & \\ c & 1\end{smallmatrix}\right)\left(\begin{smallmatrix}zt_1 & \\ & zt_2\end{smallmatrix}\right)\Big) \;=\; f_{\varphi,\chi}(s)\Big(\left(\begin{smallmatrix}zt_1 & \\ & zt_2\end{smallmatrix}\right)\left(\begin{smallmatrix}1 & \\ ct_1t_2^{-1} & 1\end{smallmatrix}\right)\Big) \;=\; f_{\varphi,\chi}(s)\Big(\left(\begin{smallmatrix}1 & \\ ct_1t_2^{-1} & 1\end{smallmatrix}\right)\Big)$$

for $\left(\begin{smallmatrix}zt_1 & \\ & zt_2\end{smallmatrix}\right) \in T_F Z_{\mathbf{A}}$, so changing the representative of $t \in T_F Z_{\mathbf{A}}\backslash T_{\mathbf{A}}$ only permutes $\{\left(\begin{smallmatrix}1 & \\ c & 1\end{smallmatrix}\right)\}_{c\in F^\times}$. Substituting the definition of $f_{\varphi,\chi}(s)$, we find

$$I \;=\; \int\limits_{T_F Z_{\mathbf{A}}\backslash T_{\mathbf{A}}} \sum_{c\in F^\times} \int\limits_{Z_{\mathbf{A}}} \varphi\big((c,1)ztg\big)\;\chi(\det(ztg))\;|\det(ztg)|^{s+1/2}\,dz\,dt\,.$$

By writing φ_g for the Schwartz-Bruhat function $\varphi(\,\cdot\,g)$, applying Fubini's theorem to

$$\big(T_F Z_{\mathbf{A}}\backslash T_{\mathbf{A}}\big)\times Z_{\mathbf{A}} \;\simeq\; \big(T_F\backslash T_{\mathbf{A}}\big)\times Z_F$$

(cf. paragraph 1.2.4) and observing that we have $\det z \in F^\times \subset \ker(\chi|\;|^{s+1/2})$ for a matrix $z \in Z_F$, we find

$$I \;=\; \int\limits_{T_F\backslash T_{\mathbf{A}}} \sum_{c\in F^\times} \int\limits_{F^\times} \varphi_g((zc,z)t)\;\chi(\det(tg))\;|\det(tg)|^{s+1/2}\,dz\,dt\,.$$

We now replace c by cz^{-1}, replace the sum by the integral over the discrete space $F^\times$ and use

$$\begin{array}{ccc} T_F\backslash T_{\mathbf{A}} & \simeq & (F^\times\backslash\mathbf{A}^\times)\times(F^\times\backslash\mathbf{A}^\times)\,. \\ t & \mapsto & (t_1,t_2) \end{array}$$

Then I equals

$$\chi(\det g)\;|\det g|^{s+1/2}\int\limits_{F^\times\backslash\mathbf{A}^\times}\int\limits_{F^\times\backslash\mathbf{A}^\times}\int\limits_{F^\times}\int\limits_{F^\times} \varphi_g(ct_1,at_2)\;\chi(t_1t_2)\;|t_1t_2|^{s+1/2}\,da\,dc\,dt_1\,dt_2$$

$$=\;\chi(\det g)\;|\det g|^{s+1/2}\int\limits_{\mathbf{A}^\times}\Bigg(\int\limits_{\mathbf{A}^\times} \varphi_g(t_1,t_2)\;\chi(t_1)\;|t_1|^{s+1/2}\,dt_1\Bigg)\,\chi(t_2)\;|t_2|^{s+1/2}\,dt_2\,.$$

Let $U \subset \mathbf{A}^2$ be the compact domain of φ_g. Then $\big\{t_1 \in \mathbf{A}\big|\big(\{t_1\}\times\mathbf{A}\big)\cap U \neq \emptyset\big\}$ is compact. For every t_2, the function $t_1 \to \varphi_g(t_1,t_2)$ is locally constant on $\mathbf{A}\times\{t_2\} \subset \mathbf{A}\times\mathbf{A}$ endowed with the subspace topology. Consequently, $\varphi_g(\,\cdot\,,t_2)$ is a Schwartz-Bruhat function for every t_2 and the expression in brackets that we see in the last equation is a Tate integral, which equals a multiple of $L(\chi,s+1/2)$ (Theorem 2.2.7). Denote the factor by $\tilde{\varphi}_g(t_2)$. For the same reasons as before, but with the roles of t_1 and t_2 reversed, we see that $\varphi_g(t_1,\,\cdot\,)$ is a Schwartz-Bruhat function for every t_1. Hence the value of the Tate integral is locally constant in t_2 and vanishes at all t_2 outside a compact set. Since $L(\chi,s+1/2)$ does not depend on t_2, the factor $\tilde{\varphi}_g$ is locally constant and compact support. Hence $\tilde{\varphi}_g : \mathbf{A}\to\mathbf{C}$ is a Schwartz-Bruhat function. Substituting the Tate integral in the last equation by $\tilde{\varphi}_g(t_2)L(\chi,s+1/2)$ yields

$$I \;=\; \chi(\det g)\;|\det g|^{s+1/2}\,L(\chi,s+1/2)\int\limits_{\mathbf{A}^\times}\tilde{\varphi}_g(t_2)\chi(\det g)\,|t_2|^{s+1/2}\,dt_2\;,$$

where we see again a Tate integral, which equals a multiple of $L(\chi,s+1/2)$.

We end up with the right hand side of the equation in (i) if $\tilde{e}_T(g,\varphi,\chi,s)$ is suitably defined. In particular, the left hand side is a well-defined and converging expression, which is meromorphic in $s\in\mathbf{C}$, and $\tilde{e}_T(g,\varphi,\chi,s)$ is meromorphic as the quotient of meromorphic functions. Hence (i) holds.

By Theorem 2.2.7, there is a Schwartz-Bruhat function $\psi:\mathbf{A}\to\mathbf{C}$ such that we have $L(\psi,\chi,s+1/2) = L(\chi,s+1/2)$. If we define $\varphi:\mathbf{A}^2\to\mathbf{C}$ to be the Schwartz-Bruhat function such that $\varphi_g(t_1,t_2)=\psi(t_1)\cdot\psi(t_2)$. Then $\tilde{e}_T(g,\varphi,\chi,s)=\chi(\det g)\,|\det g|^{s+1/2}$. If $\chi\in\Xi_0$, then $\varphi_{T,g^{-1}}$ satisfies the equality by Theorem 2.2.7. Hence (ii) holds by meromorphic continuation.

By Theorem 2.2.7, $L(\psi,\chi,s+1/2)$ equals a holomorphic multiple of $L(\chi,s+1/2)$ in $s\in\mathbf{C}$ for any Schwartz-Bruhat function ψ, thus $e_T(g,\varphi,\chi,s)$ is holomorphic in $s\in\mathbf{C}$ for an arbitrary Schwartz-Bruhat function φ. □

6.2.7 Recall from paragraph 2.3.9 that

$$E_N(g,\varphi,\chi,s) \;=\; f_{\varphi,\chi}(s)(g) \;+\; M_\chi(s)\,f_{\varphi,\chi}(s)(g)$$

and from Theorems 2.3.13 and 2.3.14 that there is a flat section $\hat{f}_{\varphi,\chi}(s)$ and a function $c(\chi,s)$ such that

$$M_\chi(s)\,f_{\varphi,\chi}(s) \;=\; c(\chi,s)\,\hat{f}_{\varphi,\chi^{-1}}(-s) \quad\text{and}\quad E\big(\,\cdot\,,f_{\varphi,\chi}(s)\big) = c(\chi,s)\,E\big(\,\cdot\,,\hat{f}_{\varphi,\chi^{-1}}(-s)\big)$$

By paragraph 2.3.22, there is a Schwartz-Bruhat function $\hat{\varphi}$ such that

$$\hat{f}_{\varphi,\chi^{-1}}(s) \;=\; f_{\hat{\varphi},\chi^{-1}}(s) \qquad\text{and}\qquad E(\,\cdot\,,\hat{\varphi},\chi^{-1},s) \;=\; E(\,\cdot\,,\hat{f}_{\varphi,\chi^{-1}}(s))\,.$$

Recall from Theorem 2.2.2 that for every $\chi\in\Xi$ there is a holomorphic function $\epsilon(\chi,s)$ of $s\in\mathbf{C}$ such that

$$L(\chi,1/2+s) = \epsilon(\chi,s)L(\chi^{-1},1/2-s)\,.$$

Let $T \subset G$ be a split torus. Then T_F is given as the image of $\Theta_E : E^\times \to G_F$, where $E = F \oplus F$. Recall from paragraph 1.5.12 that in this case, we defined

$$f_T(g) = \int\limits_{T_F Z_{\mathbf{A}} \backslash T_{\mathbf{A}}} \Big(f - \frac{1}{2}(f_N + f_{N^T})\Big)(tg)\, dt$$

for $f \in \mathcal{A}$, where

$$f_{N^T}(g) = \int\limits_{N_F^T \backslash N_{\mathbf{A}}^T} f(ng)\, dn = \int\limits_{N_F \backslash N_{\mathbf{A}}} f(w_0 n w_0 g)\, dn = f_N(w_0 g)\,.$$

Proposition 1.5.3 implies that there is a $\gamma \in G_F$ such that $T = \gamma^{-1} T_0 \gamma$, where T_0 is the diagonal torus. Recall the definition of φ_{T_0} for the diagonal torus T_0 from paragraph 6.2.5. Define $\varphi_T = \varphi_{T_0,\gamma}$. Note that this definition does not depend on γ because the only matrices that leave T_0 invariant by conjugation are e and w_0. But $\varphi_{T_0}(\,\cdot\, w_0\gamma) = \varphi_{T_0}(\,\cdot\,\gamma)$ by the definition of φ_{T_0}.

We now state the analogue of Theorem 6.2.3 for split tori, which is also the adelic translation of a long-known formula ([83, eq. (30)]).

6.2.8 Theorem. *Let T be a split torus. For every Schwartz-Bruhat function $\varphi : \mathbf{A}^2 \to \mathbf{C}$, $g \in G_{\mathbf{A}}$ and $\chi \in \Xi$, there exists a holomorphic function $e_T(g,\varphi,\chi,s)$ of $s \in \mathbf{C}$ with the following properties.*

(i) *For all $s \in \mathbf{C}$ such that $\chi^2|\ |^{2s} \neq |\ |^{\pm 1}$,*

$$E_T(g,\varphi,\chi,s) = e_T(g,\varphi,\chi,s)\,\big(L(\chi,s+1/2)\big)^2\,.$$

(ii) *If $\chi \in \Xi_0$, then $e_T(e,\varphi_T,\chi,s) = 1$ for all $s \in \mathbf{C}$.*

Proof. First, let T be the diagonal torus. Let $\chi \in \Xi_0$ and $s \in \mathbf{C}$ such that $\chi^2|\ |^{2s} \neq |\ |^{\pm 1}$. We calculate:

$$\begin{aligned}
&2\,E_T(g,\varphi,\chi,s)\\
&= \int\limits_{T_F Z_{\mathbf{A}} \backslash T_{\mathbf{A}}} \big(2\,E(tg,\varphi,\chi,s) - E_N(tg,\varphi,\chi,s) - E_{N^T}(tg,\varphi,\chi,s)\big)\, dt\\
&= \int\limits_{T_F Z_{\mathbf{A}} \backslash T_{\mathbf{A}}} \Big(2\,E(tg,\varphi,\chi,s) - f_{\varphi,\chi}(s)(tg) - M_\chi(s)\, f_{\varphi,\chi}(s)(tg)\\
&\qquad\qquad - f_{\varphi,\chi}(s)(w_0 tg) - M_\chi(s)\, f_{\varphi,\chi}(s)(w_0 tg)\Big)\, dt\\
&= \int\limits_{T_F Z_{\mathbf{A}} \backslash T_{\mathbf{A}}} \Big(\big(E(tg,\varphi,\chi,s) - f_{\varphi,\chi}(s)(tg) - f_{\varphi,\chi}(s)(w_0 tg)\big)\\
&\qquad + c(\chi,s)\,\big(E(tg,\hat\varphi,\chi^{-1},-s) - f_{\hat\varphi,\chi^{-1}}(-s)(tg) - f_{\hat\varphi,\chi^{-1}}(-s)(w_0 tg)\big)\Big)\, dt\,,
\end{aligned}$$

where we applied the formulas of the previous paragraph and the functional equation, cf. Theorem 2.3.14. By Lemma 6.2.6, we can split the last integral into two and obtain:

$$\tilde{e}_T(g,\varphi,\chi,s)\,\big(L(\chi,s+1/2)\big)^2 \;+\; c(\chi,s)\,\tilde{e}_T(g,\hat{\varphi},\chi^{-1},-s)\,\big(L(\chi^{-1},-s+1/2)\big)^2 \,.$$

We apply the functional equation to $L(\chi^{-1},-s+1/2)$, cf. Theorem 2.2.2, and obtain (i) for the diagonal torus if we put

$$e_T(g,\varphi,\chi,s) \;=\; \frac{1}{2}\,\tilde{e}_T(g,\varphi,\chi,s) \;+\; \frac{1}{2}\,\epsilon(\chi,s)^{-2}\,c(\chi,s)\,\tilde{e}_T(g,\hat{\varphi},\chi^{-1},-s)\,.$$

This defines $e_T(g,\varphi,\chi,s)$ as holomorphic function of $s\in\mathbf{C}$ since $\epsilon(\chi,s)$ is non-vanishing as a function at s (Theorem 2.2.2).

If T is any split torus, define $e_T(g,\varphi,\chi,s)=e_{T_0}(\gamma g,\varphi,\chi,s)$. By Proposition 1.5.3, there is a $\gamma\in G_F$ such that $T=\gamma T_0\gamma^{-1}$, where T_0 is the diagonal torus. Recall from Remark 1.5.14 that $f_T(g)=f_{T_0}(\gamma g)$. This reduces the case of the general split torus to the case of the diagonal torus. Thus (i) holds.

Regarding (ii), let $\chi\in\Xi_0$ and $s\in\mathbf{C}$ be such that $\chi^2|\;|^{2s}\neq|\;|^{\pm1}$. Since we may replace χ by $\chi|\;|^s$, we assume that $s=0$ without loss of generality. Recall from paragraph 2.3.22 that $E(\,\cdot\,,\varphi_0,\chi)=E(\,\cdot\,,\chi)$. Put $f_\chi=f_{\varphi_0,\chi}(0)\in\mathcal{P}(\chi)$ and $f_{\chi^{-1}}=f_{\varphi_0,\chi^{-1}}(0)\in\mathcal{P}(\chi^{-1})$. By paragraph 2.3.9, we have

$$E_N(g,\chi) \;=\; f_\chi(g)+M_\chi(0)f_\chi(g) \quad\text{and}\quad E_N(g,\chi^{-1}) \;=\; f_{\chi^{-1}}(g)+M_{\chi^{-1}}(0)f_{\chi^{-1}}(g)\,,$$

where N is the unipotent radical of the standard Borel subgroup.

Observe that for $T=\gamma^{-1}T_0\gamma$, we have $e_T(e,\varphi_T,\chi,s)=e_{T_0}(\gamma,\varphi_{T_0,\gamma^{-1}},\chi,s)$. As in the proof of (i), we may restrict to the diagonal torus $T=T_0$ without loss of generality. We follow the lines of the calculation in the proof of (i), where we make use of the functional equation for $E(\,\cdot\,,\chi)$ (Theorem 2.3.14), the functional equation for $L(\chi,1/2)$ (Theorem 2.2.2) and Lemma 6.2.6 (ii):

$$\begin{aligned}
2\,E_T(e,\chi) \;&=\; \int_{T_F Z_{\mathbf{A}}\backslash T_{\mathbf{A}}} \big(2\,E(t,\chi)\;-\;E_N(t,\chi)\;-\;E_{N^T}(t,\chi)\big)\,dt\\
&=\; \int_{T_F Z_{\mathbf{A}}\backslash T_{\mathbf{A}}} \Big(\big(E(t,\chi)\;-\;f_\chi(t)\;-\;f_\chi(w_0 t)\big)\\
&\qquad\qquad +\;\chi^2(\mathrm{c})\,\big(E(t,\chi^{-1})\;-\;f_{\chi^{-1}}(t)\;-\;f_{\chi^{-1}}(w_0 t)\big)\Big)\,dt\,,\\
&=\; \tilde{e}_T(e,\varphi_T,\chi,0)\,\big(L(\chi,1/2)\big)^2\;+\;\tilde{e}_T(e,\varphi_T,\chi^{-1},0)\,\chi^2(\mathrm{c})\,\big(L(\chi^{-1},1/2)\big)^2\\
&=\; \big(L(\chi,1/2)\big)^2\;+\;\chi^2(\mathrm{c})\,\chi^{-2}(\mathrm{c})\,\big(L(\chi,1/2)\big)^2\\
&=\; 2\big(L(\chi,1/2)\big)^2\,.
\end{aligned}$$

By holomorphic continuation, we find $e_T(e,\varphi_0,\chi,s)=1$ for all $s\in\mathbf{C}$. □

For any Schwartz-Bruhat function $\varphi:\mathbf{A}^2\to\mathbf{C}$ and any $g\in G_{\mathbf{A}}$ we have that the automorphic form $g.E(\,\cdot\,,\varphi,\chi)$ is an element of $\mathcal{P}(\chi)$ (paragraph 2.3.22). By the definition of $F\oplus F$-toroidality, we obtain as an immediate consequence:

6.2.9 Corollary. *Let $\chi \in \Xi_0$ such that $\chi^2 \neq |\,|^{\pm 1}$ and let $\varphi : \mathbf{A}^2 \to \mathbf{C}$ be a Schwartz-Bruhat function. Then $E(\,\cdot\,, \varphi, \chi)$ is $F \oplus F$-toroidal if and only if $L(\chi, 1/2) = 0$.*

6.2.10 Let $T \subset G$ be a maximal torus defined by $\Theta_E : E^\times \to G_F$. If E is a field, then the reciprocity map (cf. paragraph 2.2.9) assigns to the nontrivial character of $\mathrm{Gal}(E/F)$ a character of $\mathbf{A}_F^\times$, which we denote by $\chi_T = \chi_E$. This character is of order two and its kernel is precisely $\mathrm{N}_{E/F}(\mathbf{A}_E^\times)$. By Lemma 2.2.10,

$$L_E(\chi \circ \mathrm{N}_{E/F}, s) \;=\; L_F(\chi, s)\; L_F(\chi\chi_T, s)\,.$$

If $E = F \oplus F$, then define $\chi_T = \chi_E$ as the trivial character. Furthermore, for every maximal torus T of G, set

$$e_T^{(i)}(g, \varphi, \chi) \;:=\; \frac{d^i}{ds^i}\, e_T(g, \varphi, \chi, s)\Big|_{s=0}\,.$$

6.2.11 Theorem. *Let T be a maximal torus in G and n a positive integer. For all $g \in G_{\mathbf{A}}$ and $\chi \in \Xi_0$ such that $\chi^2 \neq |\,|^{\pm 1}$,*

$$E_T^{(n)}(g, \varphi, \chi) \;=\; \sum_{\substack{i+j+k=n \\ i,j,k \geq 0}} \frac{n!}{i!\,j!\,k!}\, e_T^{(i)}(g, \varphi, \chi)\; L^{(j)}(\chi, 1/2)\; L^{(k)}(\chi\chi_T, 1/2)\,.$$

Proof. Observe that in both the case of an anisotropic torus and the case of a split torus, we are taking integrals over functions with compact support, so the derivatives with respect to s commute with the integrals. Everything follows at once from applying the Leibniz rule to the formulas in Theorems 6.2.3 and 6.2.8. □

6.2.12 Let $\chi \in \Xi_0$ such that $\chi^2 \neq |\,|^{\pm 1}$. We say that χ is a zero of $L(\,\cdot\,, 1/2)$ of order n if $L(\chi, s + 1/2)$ vanishes of order n at $s = 0$. By the previous theorem, we see that if χ is a zero of $L(\,\cdot\,, 1/2)$ of order n, then all the functions $E(\,\cdot\,, \chi), \ldots, E^{(n-1)}(\,\cdot\,, \chi)$ are toroidal.

The functional equation for L-series (Theorem 2.2.2) implies that zeros come in pairs: χ is a zero of order n if and only if χ^{-1} is a zero of order n, and if $\chi = \chi^{-1}$, then χ is a zero of even order (Lemma 2.5.6). We call $\{\chi, \chi^{-1}\}$ a *pair of zeros of order n* if χ is a zero of order n in case $\chi \neq \chi^{-1}$, or if χ is a zero of order $2n$ in case $\chi = \chi^{-1}$.

Recall from paragraph 3.6.1 the definition of the space $\widetilde{\mathcal{E}}(\chi)^K$ and its basis elements $\tilde{E}^{(i)}(\,\cdot\,, \chi)$, where $\chi \in \Xi_0$ and $i \geq 0$ is an integer. Because $\widetilde{\mathcal{E}}(\chi)^K = \widetilde{\mathcal{E}}(\chi^{-1})^K$ and because in case $\chi = \chi^{-1}$, we defined $\tilde{E}^{(i)}(\,\cdot\,, \chi)$ as $E^{(2i)}(\,\cdot\,, \chi)$, we obtain that if $\{\chi, \chi^{-1}\}$ is a pair of zeros of order n, then $\tilde{E}(\,\cdot\,, \chi), \ldots, \tilde{E}^{(n-1)}(\,\cdot\,, \chi)$ are toroidal and span an n-dimensional $\mathcal{H}_K$-module provided that $\chi^2 \neq |\,|^{\pm 1}$.

We summarise this.

6.2.13 Corollary. *Let $\chi \in \Xi_0$ such that $\chi^2 \neq |\,|^{\pm 1}$ and $i \geq 0$.*

(i) *Let E/F be a separable quadratic algebra extension. Then $\tilde{E}^{(i)}(\,\cdot\,, \chi)$ is E-toroidal if and only if $\{\chi, \chi^{-1}\}$ is a zero of $L(\,\cdot\,, 1/2)L(\,\cdot\,\chi_E, 1/2)$ that is at least of order i.*

(ii) *If $\{\chi, \chi^{-1}\}$ is a pair of zeros of $L(\,\cdot\,, 1/2)$ of order n, then $\tilde{E}(\,\cdot\,, \chi), \ldots, \tilde{E}^{(n-1)}(\,\cdot\,, \chi)$ are toroidal.* □

6.2.14 Theorem. *The dimension of $\mathcal{E}_{\mathrm{tor}}^K$ is at least $(g_F-1)h_F+1$, where g_F is the genus and h_F the class number of F.*

Proof. Fix an idele $a_1 \in \mathbf{A}^\times$ of degree 1 and let $\omega_1,\dots,\omega_{h_F} \in \Xi_0$ be the characters that are trivial on $\langle a_1 \rangle$. Assume that ω_1 is the trivial character. Then for every $\chi \in \Xi_0$, there is a unique $j \in \{1,\dots,h_F\}$ and $s \in \mathbf{C}/(2\pi\mathrm{i}/\ln q)\mathbf{Z}$ such that $\chi = \omega_j \,|\;|^s$, cf. Corollary 2.1.4. Proposition 2.2.11 proves the existence of a finite abelian unramified extension F'/F of order h_F such that

$$\prod_{i=1}^{h_F} L_F(\omega_i, s+1/2) \;=\; \zeta_{F'}(s+1/2)\,.$$

In particular the zeros of both hand sides as functions of s are in one-to-one correspondence.

From Theorem 2.2.8, we know that this zeta function is of the form

$$\zeta_{F'}(s) = \frac{\mathfrak{L}_{F'}(q^{-s})}{(1-q^{-s})(1-q^{1-s})}$$

for some polynomial $\mathfrak{L}_{F'}(T) \in \mathbf{Z}[T]$ of degree $2g_{F'}$ that has no zero at $T=1$ or $T=q^{-1}$. This means that the sum over the orders of all pairs of zeros of $L(\,\cdot\,,1/2)$ sums up to $g_{F'}$, and that we find $g_{F'}$ linearly independent toroidal automorphic forms in $\mathcal{E}^K$. Note that for a quasi-character $\chi = \omega_i \,|\;|^s$ with $\chi^2 = |\;|^{\pm 1}$, we have that $\zeta_{F'}(s+1/2) \neq 0$ because $\mathfrak{L}(T)$ has no zero at $T=q^0$ or $T=q^{-1}$. Hence if χ is a zero of $L(\,\cdot\,,1/2)$, then $\tilde{E}(\,\cdot\,,\chi)$ is not a residuum.

Finally, we apply Hurwitz' theorem ([28, Cor. 2.4]) to the unramified extension F'/F and obtain:

$$2g_{F'}-2 \;=\; h_F\,(2g_F-2) \qquad \text{and thus} \qquad g_{F'} \;=\; (g_F-1)\,h_F+1\,. \qquad \square$$

6.2.15 Conjecture. *The dimension of $\mathcal{E}_{\mathrm{tor}}^K$ equals $(g_F-1)h_F+1$.*

6.2.16 Remark. The question whether the dimension of $\mathcal{E}_{\mathrm{tor}}^K$ can be larger than the lower bound $(g_F-1)h_F+1$ depends on the question whether there is a character χ_0 on the divisor class group $F^\times \backslash \mathbf{A}^\times / \mathcal{O}_{\mathbf{A}}^\times$ such that for all maximal tori T the L-functions $L(\chi_0\chi_T, s)$ have a common zero s.

Theorems 1.1 and 5.2 in [67] state that for every global function field F of characteristic different from 2, there is integer n_0 such that for every $n \geq n_0$ the occurrence of such a common zero is excluded for the constant field extension $F_n = \mathbf{F}_{q^n} F$. This means that for every quasicharacter $\chi_0 : \mathbf{A}_{F^n}^\times \to \mathbf{C}^\times$ that is trivial on F_n and for every $s \in \mathbf{C}$, there is a separable quadratic field extension E_n/F_n such that $L(\chi_0\chi_T, s) \neq 0$. (Note that in our particular case all exceptional situations of [67, para. 5.1] can be easily excluded since χ_0 is not symplectically self-dual and $\#o = 1$.)

If the genus of F is 1 and either its characteristic is different from 2 or h_F is different from $q+1$, then in Theorem 8.3.11 we will show by a different method that such a common zero cannot occur.

One way to prove this result for all global function fields of arbitrary characteristic, genus and class number is by proving a non-vanishing result for double Dirichlet series as considered in [18] and [19].

6.3 Toroidal residues of Eisenstein series

In this section, we will prove that residues of Eisenstein series are not toroidal (Theorem 6.3.8).

6.3.1 Let $T \subset G$ be a maximal torus corresponding to the quadratic algebra extension E/F. Let χ_T be the character on $\mathbf{A}^\times$ that we defined in paragraph 6.2.10. Theorems 6.2.3 and 6.2.8 defined for every Schwartz-Bruhat function $\varphi : \mathbf{A}^2 \to \mathbf{C}$, $g \in G_\mathbf{A}$ and $\chi \in \Xi$ a holomorphic function $e_T(g,\varphi,\chi,s)$ of $s \in \mathbf{C}$ such that $E_T(g,\varphi,\chi,s)$ equals $e_T(g,\varphi,\chi,s)$ times a certain L-function provided the Eisenstein series has no pole at s.

Our aim is to the investigate toroidal integrals of the residues $R(g,\varphi,\chi)$ of $E(g,\varphi,\chi,s)$ at $s=0$, where $\chi^2 = |\ |^{\pm 1}$. Note that

$$R_T(g,\varphi,\chi) \;=\; \big(\mathrm{Res}_{s=0}\, E(g,\varphi,\chi,s)\big)_T \;=\; \mathrm{Res}_{s=0}\, E_T(g,\varphi,\chi,s)$$

since we integrate functions with compact support in $T_F Z_\mathbf{A} \backslash T_\mathbf{A}$.

6.3.2 Lemma. *Let T be an anisotropic torus and $\chi = \omega |\ |^{\pm 1/2} \in \Xi$ with $\omega^2 = 1$. For every Schwartz-Bruhat function $\varphi : \mathbf{A}^2 \to \mathbf{C}$ and $g \in G_\mathbf{A}$,*

$$R_T(g,\varphi,\chi) \;=\; e_T(g,\varphi,\chi,0)\ \mathrm{Res}_{s=0}\ L_E(\chi \circ \mathrm{N}_{E/F}, s+1/2)\,.$$

Proof. With help of Theorem 6.2.3, we calculate

$$\begin{aligned} R_T(g,\varphi,\chi) &= \lim_{s\to 0}\ s\ E_T(g,\varphi,\chi,s) \\ &= \lim_{s\to 0}\ s\ e_T(g,\varphi,\chi,s)\ L_E(\chi\circ \mathrm{N}_{E/F}, s+1/2) \\ &= e_T(g,\varphi,\chi,0)\ \mathrm{Res}_{s=0}\ L_E(\chi\circ \mathrm{N}_{E/F}, s+1/2)\,. \quad \square \end{aligned}$$

6.3.3 Let $T \subset G$ be a maximal torus and $\chi = \omega |\ |^{\pm 1/2} \in \Xi$ with $\omega^2 = 1$. Let $\varphi : \mathbf{A}^2 \to \mathbf{C}$ a Schwartz-Bruhat function. By Theorem 2.4.2, $R(\,\cdot\,,\varphi,\chi)$ is a multiple of $\omega \circ \det$. In particular, $R(g,\varphi,\chi) = 0$ for any $g \in G_\mathbf{A}$ if and only if $R(e,\varphi,\chi) = 0$.

6.3.4 Lemma. *Let T be an anisotropic torus and $\chi = \omega |\ |^{\pm 1/2} \in \Xi$ with $\omega^2 = 1$. There is a Schwartz-Bruhat function φ such that $R_T(e,\varphi,\chi) \neq 0$ if and only if $\omega = 1$ or $\omega = \chi_T$.*

Proof. Observe that the residuum of

$$L_E(\chi\circ \mathrm{N}_{E/F}, s+1/2) \;=\; L_F(\omega, s+1/2 \pm 1/2)\ L_F(\omega\chi_T, s+1/2\pm 1/2)$$

at $s=0$ is nontrivial if and only if one of the two factors is the zeta function of F, and this happens if $\omega = 1$ or $\omega = \chi_T^{-1} = \chi_T$.

If $\mathrm{Res}_{s=0} L_E(\chi\circ \mathrm{N}_{E/F}, s+1/2) = 0$, then $R_T(e,\varphi,\chi) = 0$ for all Schwartz-Bruhat functions φ by Lemma 6.3.2.

If not, then $R(e,\varphi_T,\chi) = 1 \cdot \mathrm{Res}_{s=0} L_E(\chi\circ \mathrm{N}_{E/F}, s+1/2)$ (Theorem 6.2.3 (ii)) does not vanish. $\square$

6.3.5 Lemma. *Let T be an anisotropic torus and $\chi = \omega |\ |^{\pm 1/2} \in \Xi$ with $\omega^2 = 1$. If $R_T(e,\varphi,\chi) = 0$ for all Schwartz-Bruhat functions φ, then there exists a Schwartz-Bruhat function φ such that $R_T^{(1)}(e,\varphi,\chi) \;\neq\; 0$.*

Proof. By Lemma 6.3.4, we have that $R_T(g,\varphi,\chi)=0$ for all φ and $g\in G_{\mathbf{A}}$ if and only if $L_E(\omega\circ \mathrm{N}_{E/F},\,\cdot\,)$ has no pole at 0 or 1. With the help of Theorem 6.2.3, we calculate

$$\begin{aligned} R_T^{(1)}(e,\varphi_T,\chi) &= \lim_{s\to 0}\ \frac{d}{ds}\ s\ E_T(e,\varphi_T,\chi,s) \\ &= \lim_{s\to 0}\ \frac{d}{ds}\ s\ e_T(e,\varphi_T,\chi,s)\,L_E(\chi\circ \mathrm{N}_{E/F},s+1/2) \\ &= \lim_{s\to 0}\Big(e_T(e,\varphi_T,\chi,s)\,L_E(\chi\circ \mathrm{N}_{E/F},s+1/2) \\ &\qquad\qquad + s\ \frac{d}{ds}\ e_T(e,\varphi_T,\chi,s)\,L_E(\chi\circ \mathrm{N}_{E/F},s+1/2)\Big) \\ &= e_T(e,\varphi_T,\chi,0)\,L_E(\omega\circ \mathrm{N}_{E/F},1/2\pm 1/2)\ , \end{aligned}$$

which does not vanish by Theorem 6.2.3 (ii) and Corollary 2.2.12. □

6.3.6 Let T be a split torus and $\chi=\omega|\ |^{\pm 1/2}\in\Xi$ with $\omega^2=1$. Let N be the unipotent radical of a Borel subgroup $B\subset G$. Then

$$(\omega\circ\det)_N(g)\ =\ \int\limits_{N_F\backslash N_{\mathbf{A}}}\omega\circ\det(ng)\,dn\ =\ \omega\circ\det(g)\ .$$

Consequently

$$R_T(e,\varphi,\chi)\ =\ 0$$

for every Schwartz-Bruhat function φ by the definition of the toroidal integral for a split torus.

We summarise:

6.3.7 Theorem. *Let E be a quadratic separable algebra extension of F, χ_E the character from paragraph 6.2.10 and $\chi=\omega|\ |^{\pm 1/2}\in\Xi$ with $\omega^2=1$.*

(i) *If ω is trivial, then $R(\,\cdot\,,\chi)\in\mathcal{A}_{\mathrm{tor}}(E)$ if and only if $E\simeq F\oplus F$.*

(ii) *If ω is nontrivial, then $R(\,\cdot\,,\chi)\in\mathcal{A}_{\mathrm{tor}}(E)$ if and only if $\omega\neq\chi_E$.*

(iii) *If E is a field and $n\geq 1$, then $R^{(n)}(\,\cdot\,,\chi)\notin\mathcal{A}_{\mathrm{tor}}(E)$.* □

6.3.8 Theorem. $\mathcal{R}_{\mathrm{tor}}=\{0\}$. □

6.4 Remarks on toroidal cusp forms

6.4.1 Remark. Waldspurger calculated toroidal integrals of cusp forms over number fields. So assume for a moment that F is a number field, π an irreducible unramified cuspidal representation and $f\in\pi$ an unramified cusp form. Let $L(\pi,s)$ be the L-function of π. Let $T\subset G$ be an torus corresponding to a quadratic field extension E of F and χ_T the character corresponding to T by class field theory. Then the square of the absolute value of

$$\int_{T_F Z_{\mathbb{A}} \backslash T_{\mathbb{A}}} f(t)\,dt$$

equals a harmless factor times $L(\pi,1/2)L(\pi\chi_T,1/2)$, cf. [71, Prop. 7].

These integrals are nowadays called Waldspurger periods of f, and it is translated in some cases to global function fields, cf. [44]. This leads to the conjecture:

6.4.2 Conjecture. *A cusp form f of an irreducible unramified cuspidal subrepresentation π of the space of automorphic forms is toroidal if and only if $L(\pi,1/2)=0$.*

By the multiplicity one theorem (3.5.3), this conjecture implies

6.4.3 Conjecture. *The dimension of $\mathcal{A}^K_{0,\mathrm{tor}}$ equals the number of isomorphism classes of irreducible unramified cuspidal representations π with $L(\pi,1/2)=0$.*

6.4.4 Remark. In Theorem 8.3.1, we will prove by a different method that $\mathcal{A}^K_{0,\mathrm{tor}}=\{0\}$ if $g_F=1$.

6.5 Some history around the Riemann hypothesis

Ich setzte nun $s=\frac{1}{2}+ti$ und

$$\Pi\left(\frac{s}{2}\right)(s-1)\pi^{-\frac{s}{2}}\zeta(s) \;=\; \xi(t)\,, \quad [\ldots]$$

Die Anzahl der Wurzeln von $\xi(t)=0$, deren reeller Theil zwischen 0 und T liegt, ist etwa

$$=\frac{T}{2\pi}\log\frac{T}{2\pi}-\frac{T}{2\pi}\,;$$

denn das Integral $\int d\log\xi(t)$ positiv um den Inbegriff der Werthe von t erstreckt, deren imaginärer Theil zwischen $\frac{1}{2}\mathrm{i}$ und $-\frac{1}{2}\mathrm{i}$ und deren reeller Theil zwischen 0 und T liegt, ist (bis auf einen Bruchtheil von der Ordnung der Grösse $\frac{1}{T}$) gleich $\left(T\log\frac{T}{2\pi}-T\right)\mathrm{i}$; dieses Integral aber ist gleich der Anzahl der in diesem Gebiet liegenden Wurzeln von $\xi(t)=0$, multiplicirt mit $2\pi\mathrm{i}$. Man findet nun in der That etwa so viel reelle Wurzeln innerhalb dieser Grenzen, und es ist sehr wahrscheinlich, dass alle Wurzeln reell sind. Hiervon wäre allerdings ein strenger Beweis zu wünschen; [...]

Bernard Riemann, [54]

Though the function

$$\zeta(s) \;=\; \sum_{n=1}^{\infty}\frac{1}{n^s} \qquad \text{for } \mathrm{Re}\, s>1$$

was studied by Leonard Euler for real values of s long before the cited article by Bernard Riemann from 1859 ([54]) was written, it was this article that gave $\zeta(s)$ the name ‘Riemann zeta function’ and that gave the conjecture that the zeros of $\xi(s)$ are real the name

'Riemann hypothesis'. This hypothesis is equivalent with the more common formulation that the nontrivial zeros of $\zeta(s)$, i.e. the zeros of $\zeta(s)$ that are not negative even integers, have real part $1/2$. The proof that Riemann asks for is still an open problem today.

In his article, Riemann was concerned with the approximation of the number of primes $\pi(x)$ up to a given $x \in \mathbf{R}$ by the logarithmic integral function

$$\mathrm{Li}(x) = \int_0^\infty \frac{1}{\ln t}\, dt \ ,$$

a connection that was first observed by Gauß. It states that—though the occurrence of a prime number, which corresponds to a jump in $\pi(x)$, seems unpredictable—the value of $\pi(x)$ is of a comparable size to $\mathrm{Li}(x)$. Riemann writes in the same article [54]:

> Die bekannte Näherungsformel $F(x) = \mathrm{Li}(x)$ ist also nur bis auf Grössen von der Ordnung $x^{\frac{1}{2}}$ richtig [...].

The precise relation between this approximation and the Riemann hypothesis was given by Helge von Koch in 1901 ([68]), namely, that the Riemann hypothesis is equivalent to the statement that

$$\lim_{x\to\infty} \frac{\pi(x) - \mathrm{Li}(x)}{x^{1/2+\epsilon}}$$

is bounded for every $\epsilon > 0$. (See Don Zagier's inaugural lecture in Bonn [82] for a more comprehensive overview over these connections).

In his doctoral thesis from 1924 ([2], [3]), Emil Artin defined a zeta function for the function field of an elliptic or a hyperelliptic curve over a finite field. He calculated the zeta functions for about 40 function fields and found out that in these cases the analogue of the Riemann hypothesis holds true, i.e. that all zeros have real part $1/2$.

Whereas Artin's zeta function was the strict analogue of the Riemann zeta function, Friedrich Karl Schmidt introduced in 1929 ([57]) a more intrinsic definition of a zeta function that included an additional factor for the place at infinity, which was singled out in Artin's definition. Since this factor is invertible, it does not change the validity of the Riemann hypothesis. Schmidt further extended the definition to all global function fields F, and it is this kind of zeta function that we denote by ζ_F in the present thesis.

Helmut Hasse proved in 1933 ([29]) that for the function field of an elliptic curve over a finite field, the Riemann hypothesis holds true, and André Weil proved in 1948 ([76]) the Riemann hypothesis for the function fields of an arbitrary curve over a finite field by methods from algebraic geometry, which he developed for this purpose over a period of several years. We refer to this result as the Hasse-Weil theorem, cf. Theorem 6.7.1.

Weil defined in 1949 ([77]) a zeta function for more general varieties over finite fields and stated his famous conjectures, which included the Riemann hypothesis for this class of zeta functions. Many mathematicians like Dwork and Grothendieck worked on these conjectures and could solve parts of them, but it was not before 1973 that Pierre Deligne succeeded in proving the Riemann hypothesis for projective nonsingular varieties over finite fields ([16]). However, it is still unclear if the known methods generalise from global function fields to number fields.

Meanwhile, there are alternative proofs of the Hasse-Weil theorem that use less algebraic geometry, e.g. the proof by Yuri Manin in special cases in 1956 ([45]), by Serguei Stepanov in 1969 ([63]) and by Wolfgang Schmidt in 1973 ([58]). Finally, Enrico Bombieri simplified this proof in 1974 ([7]). But it also failed to be translated to $\mathbf{Q}$ yet.

There are further approaches by formulating conditions that imply the Riemann hypothesis for $\mathbf{Q}$. To name a few, there is Weil's criterion from 1952 ([78]) or Li's criterion from 1997 ([42]). In 1999, Alain Connes ([13]) showed that a certain trace formula is equivalent to the Riemann hypothesis.

At the Bombay Colloquium in January 1979, Don Zagier ([83]) observed that if the kernel of certain operators on automorphic forms turns out to give a unitarizable representation, formulas of Hecke imply the Riemann hypothesis. Zagier called elements of this kernel toroidal automorphic forms.

In the following section, we elaborate the analogue of Zagier's idea for global function fields, namely, the implications of unitarizability of the space of toroidal automorphic forms for the Hasse-Weil theorem. In the last section, we show the impact of the Hasse-Weil theorem on the unitarizability of the space of toroidal automorphic forms.

6.6 From unitarizability to the Riemann hypothesis

This section translates the observation of Don Zagier ([83, pp. 295–296]) that unitarizability of the space of toroidal automorphic forms implies the Riemann hypothesis to the setting of global function fields. Recall the definition of the restricted tensor product of representations and of the principal series representations $\mathcal{P}_x(\chi_x)$ from paragraph 3.1.5.

6.6.1 Definition. An irreducible subrepresentation $V \subset \mathcal{A}^{\mathrm{nr}}_{\mathrm{adm}}$ is called *unitarizable* if for all $x \in |X|$, there is an unramified quasi-character $\chi_x : F_x \to \mathbf{C}^\times$ such that

$$V \;\simeq\; \bigotimes_{x\in|X|}{}' \mathcal{P}_x(\chi_x)$$

as $G_{\mathbf{A}}$-representation and for all $x \in |X|$, the quasi-character χ_x either is a character or equals $|\;|_x^{s_x}$ for $s_x \in (-1/2, 1/2)$ or $(s_x - \pi \mathrm{i}/\ln q) \in (-1/2, 1/2)$.

A unitarizable representation is called a *tempered representation* if for all $x \in |X|$, the quasi-character χ_x is a character. Otherwise it is called a *complementary series representation*.

An unramified representation is called a unitarizable/tempered/complementary series representation if it decomposes into a direct sum of irreducible unitarizable / tempered / complementary series representations.

6.6.2 Remark. Let $V \subset \mathcal{A}^{\mathrm{nr}}_{\mathrm{adm}}$ be an irreducible subspace. Then V is unitarizable if and only if there is a $G_{\mathbf{A}}$-invariant Hermitian product on V. Bearing in mind that the definition of unitarizability is of local nature, i.e. it refers to properties of certain representations $\mathcal{P}_x(\chi_x)$ of G_x for every $x \in |X|$, and that every irreducible representations of G_x is isomorphic to some principal series if not 1-dimensional (Theorem 3.1.8), the assertion follows from [11, Thm. 4.6.7]. The consequence that the Hilbert space completion of a unitarizable invariant $V \subset \mathcal{A}^{\mathrm{nr}}_{\mathrm{adm}}$ is a unitary representation of $G_{\mathbf{A}}$ explains the naming.

6.6.3 Lemma. *Let $\chi \in \Xi_0$, then the following are equivalent.*

(i) *$\mathcal{P}(\chi)$ is a tempered representation.*

(ii) $\operatorname{Re}\chi = 0$.

(iii) *$\lambda_x(\chi) \in [-2q_x^{1/2}, 2q_x^{1/2}]$ for all $x \in |X|$.*

(iv) *$\lambda_x(\chi) \in [-2q_x^{1/2}, 2q_x^{1/2}]$ for some $x \in |X|$.*

Proof. Recall from paragraph 3.1.5 that

$$\mathcal{P}(\chi) \;\simeq\; {\bigotimes_{x\in|X|}}' \mathcal{P}(\chi_x)$$

with $\chi_x = \chi|_{F_x}$, which all are characters if and only if χ is a character. This is the case if $\operatorname{im}\chi \subset \mathbf{S}^1$, or equivalently if $\operatorname{Re}\chi = 0$. Thus the equivalence of (i) and (ii).

Assume (ii). Then $\operatorname{im}\chi \subset \mathbf{S}^1$, and $\lambda_x(\chi) = q_x^{1/2}\big(\chi^{-1}(\pi_x) + \chi(\pi_x)\big)$ for every $x \in |X|$. But $\chi^{-1}(\pi_x)$ is the complex conjugate of $\chi(\pi_x)$, therefore $\chi^{-1}(\pi_x) + \chi(\pi_x) \in [-2,2]$. Thus (iii). The implication from (iii) to (iv) is trivial.

Conversely, $\chi^{-1}(\pi_x) + \chi(\pi_x) \in [-2,2]$ only if $\chi^{-1}(\pi_x)$ is the complex conjugate of $\chi(\pi_x)$, thus $\chi(\pi_x) \in \mathbf{S}^1$. But by Lemma 3.7.2, $\operatorname{ev}_x^{-1}(\chi(\pi_x))$ contains only multiples of χ by characters, and since this fibre contains a character of the form $|\ |^s$ for some purely imaginary s, χ itself must be a character. Thus (iv) implies (ii). □

6.6.4 Lemma. *Let $\chi \in \Xi_0$, then the following are equivalent.*

(i) *$\mathcal{P}(\chi)$ is unitarizable.*

(ii) $\operatorname{Re}\chi = 0$ *or* $\chi = \omega|\ |^s$ *for some* $\omega \in \Xi_0$ *with* $\omega^2 = 1$ *and some* $s \in \mathbf{C}$ *such that* $s \in (-1/2,\ 1/2)$ *or* $(s - \pi\mathrm{i}/\ln q) \in (-1/2,\ 1/2)$.

(iii) *$\lambda_x(\chi) \in (-(q_x+1),\ q_x+1)$ for all $x \in |X|$.*

(iv) *There exists a subset $S \subset |X|$ that generates $\operatorname{Cl} F$ such that for all $x \in S$, $\lambda_x(\chi) \in (-(q_x+1),\ q_x+1)$.*

Proof. Assume (i) and choose a place x. Note that as $F_x^\times / \mathcal{O}_x^\times \simeq \mathbf{Z}$, χ_x is of the form $|\ |^{s_x}$ for some $s_x \in \mathbf{C}$, and thus $\lambda_x(\chi) = q_x^{1/2}(q_x^{s_x} + q_x^{-s_x})$. If the restriction χ_x of χ to $F_x^\times$ is a character, then $\lambda_x(\chi) = q_x^{1/2}\big(\chi_x^{-1}(\pi_x) + \chi_x(\pi_x)\big) \in [-2q_x^{1/2},\ 2q_x^{1/2}]$. If not, observe that

$$\begin{aligned} s_x \in (-1/2,\ 1/2) \quad &\text{if and only if} \quad (q_x^{s_x} + q_x^{-s_x}) \in [2,\ q_x^{1/2} + q_x^{-1/2}) \text{ and} \\ (s_x - \pi\mathrm{i}/\ln q) \in (-1/2,\ 1/2) \quad &\text{if and only if} \quad (q_x^{s_x} + q_x^{-s_x}) \in (-q_x^{1/2} - q_x^{-1/2},\ -2]\,. \end{aligned}$$

This proves (iii) from (i). The implication (iii) to (iv) is clear.

Assume (iv). If for one $x \in |X|$, we have $\lambda_x(\chi) \in [-2q_x^{1/2},\ 2q_x^{1/2}]$, then Lemma 6.6.3 implies that $\operatorname{Re}\chi = 0$. If not, then $s_x \in (-1/2,\ 1/2)$ or $(s_x - \pi\mathrm{i}/\ln q) \in (-1/2,\ 1/2)$. By Lemma 3.7.2, all quasi-characters χ' with $\chi'(\pi_x) = \chi(\pi_x)$ are of the form $\chi' = \omega\chi$ with a character ω that satisfies $\omega(\pi_x) = 1$. In particular, $|\ |^{s_x}$ is of this form, and to have for all $x \in |X|$ that $\omega(\pi_x)\,|\pi_x| \in (-(q_x+1),\ -2q_x^{1/2}) \cup (2q_x^{1/2},\ q_x+1)$, it must hold true that $\omega(\pi_x) = \pm 1$, and thus $\omega^2 = 1$. Hence (ii).

Statement (i) follows from (ii) by the definition of a unitarizable representation. □

6.6.5 Theorem (Zagier). *If every irreducible subrepresentation of $\mathcal{A}_{\text{tor}}^{\text{nr}}$ is a tempered representation, then all zeros of ζ_F have real part $1/2$. If furthermore, $\mathcal{A}_{\text{tor}}^{\text{nr}}$ is itself a tempered representation, then ζ_F has only simple zeros.*

Proof. By Theorem 6.2.11, we know that every zero $1/2+s$ of order n of ζ_F yields that $\tilde{E}(\cdot,|\,|^s),\ldots,\tilde{E}^{(n-1)}(\cdot,|\,|^s)$ are toroidal. Only $\tilde{E}(\cdot,|\,|^s)$ generates an irreducible representation. If this representation is tempered, then the real part of s is 0 by Lemma 6.6.3.

If furthermore $\mathcal{A}_{\text{tor}}^{\text{nr}}$ is the direct sum of irreducible tempered subrepresentations, then no derivative of an Eisenstein series can occur and the zeros of ζ_F must be of order 1. □

By Lemma 6.6.3, we obtain:

6.6.6 Corollary. *If there is a place x such that the eigenvalue of every Φ_x-eigenfunction in $\mathcal{A}_{\text{tor}}^K$ lies in the interval $[-2q_x^{1/2}, 2q_x^{1/2}]$, then all zeros of ζ_F have real part $1/2$.* □

6.6.7 Remark. We will see in Chapter 8 that the developed methods are strong enough to prove that the space of unramified toroidal automorphic forms for global function fields of genus 1 contains only unitarizable subquotients, without using the Hasse-Weil theorem or the Ramanujan-Petersson conjecture.

A proof of unitarizability for the unramified toroidal automorphic forms over **Q** would imply that the zeros of the Riemann zeta function ζ either lie in the interval $(0,1)$ or have real part $1/2$. Since we know that ζ has no zero in $(0,1)$, cf. [66, Formula (2.12.4)], the Riemann hypothesis for **Q** indeed follows from unitarizability ([83, pp. 295–296]).

This result, however, is peculiar to **Q**: The zeta function ζ_F of the function field F of the elliptic curve over $\mathbf{F}_4$ that is defined by the Weierstrass equation $\underline{Y}^2+\underline{Y}=\underline{X}^3+\alpha$, where α is an element in $\mathbf{F}_4-\mathbf{F}_2$, has a zero of order 2 at $1/2$.

Note that unitarizability implies in particular that no derivatives of Eisenstein series occur, so this further implies the simplicity of the zeros of ζ_F. It seems indeed likely that the Riemann zeta function ζ has simple zeros. For an overview over the research related to this question, see [50, §2]. The above example shows that simplicity of the zeros is also not a general phenomenon.

The milder assumption of that every irreducible subquotient of the space of unramified toroidal automorphic forms is unitarizable still implies the Riemann hypothesis for **Q**, but it allows multiple zeros of ζ.

6.7 From the Riemann hypothesis to unitarizability

The implication of Theorem 6.6.5 is of hypothetical nature as the Riemann hypothesis is proven for global function fields (Theorem 6.7.1). We can, however, make use of the Riemann hypothesis to prove the hypothesis of Theorem 6.6.5. This proof uses admissibility (Theorem 6.1.9) and the Ramanujan-Petersson conjecture (Theorem 6.7.3).

6.7.1 Theorem (Hasse-Weil, [76]). *If $\zeta_F(s)=0$, then* $\operatorname{Re} s = 1/2$.

6.7.2 Corollary. *Let* $\chi \in \Xi$. *If* $L_F(\chi, s) = 0$, *then* $\operatorname{Re} s = 1/2 - \operatorname{Re}\chi$.

Proof. This follows immediately from the theorem and Proposition 2.2.11. □

The Ramanujan-Petersson conjecture holds true for GL_2 over global function fields.

6.7.3 Theorem (Drinfeld, [17]). *Every irreducible subrepresentation* V *of* $\mathcal{A}_0$ *is a tempered representation.*

6.7.4 Recall from paragraph 2.3.16 that $E(\,\cdot\,,\chi)$ generates a subrepresentation of $\mathcal{A}$ that is isomorphic to $\mathcal{P}(\chi)$. Furthermore, if $V \subset \mathcal{A}$ is generated by $E(\,\cdot\,,\chi), \dots, E^{(n)}(\,\cdot\,,\chi)$ as $G_{\mathbf{A}}$-module, and $V' \subset \mathcal{A}$ by $E(\,\cdot\,,\chi), \dots, E^{(n-1)}(\,\cdot\,,\chi)$, then by Proposition 3.3.3 also the quotient representation $V\,/\,V'$ is isomorphic to $\mathcal{P}(\chi)$. Thus the isomorphism types of all irreducible subquotients of $\mathcal{E}^{\mathrm{nr}}_{\mathrm{tor}}$ are determined by the irreducible subrepresentations of $\mathcal{E}^{\mathrm{nr}}_{\mathrm{tor}}$.

This has the following implication. Let $V \subset \widetilde{\mathcal{E}}^{\mathrm{nr}}$ be an invariant subspace and let $S \subset |X|$ be a subset that generates $\mathrm{Cl}\,F$. If for every $\mathcal{H}_K$-eigenfunction $f \in V$ with eigencharacter λ_f and $x \in S$, the eigenvalue $\lambda_f(\Phi_x) \in (-(q_x+1),\ q_x+1)$, then every irreducible subquotient of V is unitarizable. If for every $\mathcal{H}_K$-eigenfunction $f \in V$ with eigencharacter λ_f, there is a place $x \in |X|$ such that $\lambda_f(\Phi_x) \in [-2q_x^{1/2}, 2q_x^{1/2}]$, then every irreducible subquotient of V is a tempered representation.

6.7.5 Theorem. *The irreducible unramified subquotients of the representation space of toroidal automorphic forms are tempered representations.*

Proof. Every subrepresentation of $\mathcal{A}^{\mathrm{nr}}_{\mathrm{tor}}$ is generated by elements in $\mathcal{A}^{K}_{\mathrm{tor}}$, and by Theorem 6.1.8, we know that $\mathcal{A}^{K}_{\mathrm{tor}}$ is finite-dimensional. It is thus contained in $\mathcal{A}^{K}_{\mathrm{adm}}$, which decomposes by Theorem 3.6.3 into the three parts $\mathcal{E}^{K}$, $\mathcal{R}^{K}$ and $\mathcal{A}^{K}_{0}$.

Concerning $\mathcal{E}^{K}$, we know from Corollary 6.2.4 that only for the $\chi \in \mathbf{C}$ such that $\zeta_E(\chi \circ N_{E/F}, 1/2) = 0$ for any quadratic field extension E of F, the Eisenstein series $E(\,\cdot\,,\chi)$ can be toroidal, which generates a subrepresentation of $\mathcal{A}^{\mathrm{nr}}_{\mathrm{tor}}$ that is isomorphic to $\mathcal{P}(\chi)$. By Corollary 6.7.2, $\operatorname{Re}\chi = 0$, and by Lemma 6.6.3, $\mathcal{P}(\chi)$ is thus a tempered representation. By paragraph 6.7.4, there will not occur any other isomorphism types for further irreducible subquotients of $\mathcal{E}^{\mathrm{nr}}$ than those generated by Eisenstein series, thus we showed that all irreducible subquotients of $\mathcal{E}^{\mathrm{nr}}_{\mathrm{tor}}$ are tempered representations.

By Theorem 6.3.8, $\mathcal{R}^{\mathrm{nr}}_{\mathrm{tor}} = 0$, and Theorems 3.5.2 and 6.7.3 yield that $\mathcal{A}^{\mathrm{nr}}_{0,\mathrm{tor}}$ decomposes into a direct sum of tempered representations. □

CHAPTER 7

Graphs for genus 1

This chapter determines the graphs of Hecke operators of degree 1 if the curve is of genus 1. Atiyah's classification of vector bundles over an elliptic curve over an algebraic closed field ([5]) can be used to investigate the vertices. The calculation of the edges makes use of both Atiyah's work and methods from Chapter 5.

7.1 Vertices

In this section, we determine all isomorphism classes of projective line bundles for a curve X over $\mathbf{F}_q$ of genus 1. Propositions 5.2.3 and 5.2.4 already give a characterisation of $\mathbf{P}\mathrm{Bun}_2^{\mathrm{dec}} X$ and $\mathbf{P}\mathrm{Bun}_2^{\mathrm{tr}} X$, respectively, in terms of the class groups of X, and of its quadratic extension $X' = X \otimes \mathbf{F}_{q^2}$. We are left with understanding the structure of $\mathbf{P}\mathrm{Bun}_2^{\mathrm{gi}} X$.

7.1.1 Let X be a curve of genus 1 over $\mathbf{F}_q$ with function field F, $\mathrm{Cl}\, X$ the divisor class group and h_X the class number. The canonical sheaf ω_X is isomorphic to the structure sheaf $\mathcal{O}_X$ ([28, Ch. IV, Ex. 1.3.6]). The map $X(\mathbf{F}_q) \to \mathrm{Cl}^1 X$ obtained by considering an $\mathbf{F}_q$-rational point as a prime divisor, is a bijection ([28, Ch. IV, Ex. 1.3.7]). We identify these sets. The choice of an $x_0 \in X(\mathbf{F}_q)$ defines the bijection

$$\begin{array}{ccc} X(\mathbf{F}_q) & \longrightarrow & \mathrm{Cl}^0 X \; . \\ x & \longmapsto & [x]-[x_0] \end{array}$$

So $X(\mathbf{F}_q)$ inherits a group structure and X becomes an elliptic curve. For this reason, a global function field of genus 1 is also called an elliptic function field.

7.1.2 The Riemann-Roch theorem reduces to $\dim_{\mathbf{F}_q} \Gamma(\mathcal{L}) - \dim_{\mathbf{F}_q} \Gamma(\mathcal{L}^{-1}) = \deg \mathcal{L}$. Since $\Gamma(\mathcal{L})$ is non-zero if and only if $\mathcal{L}$ is associated to an effective divisor ([28, Prop. II.7.7(a)]), we obtain:

$$\dim_{\mathbf{F}_q} \Gamma(\mathcal{L}) \;=\; \begin{cases} 0 & \text{if } \deg \mathcal{L} \leq 0 \text{ and } \mathcal{L} \not\simeq \mathcal{O}_X, \\ 1 & \text{if } \mathcal{L} \simeq \mathcal{O}_X, \text{ and} \\ \deg \mathcal{L} & \text{if } \deg \mathcal{L} > 0. \end{cases}$$

By Serre duality, $\operatorname{Ext}^1(\mathcal{O}_X,\mathcal{O}_X) \simeq \operatorname{Hom}(\mathcal{O}_X,\mathcal{O}_X) \simeq \Gamma(\mathcal{O}_X)$ is one-dimensional. Thus $\mathbf{P}\operatorname{Ext}^1(\mathcal{O}_X,\mathcal{O}_X)$ contains only one element. This determines a rank 2 bundle $\mathcal{M}_0$ by paragraph 5.3.3. Since $\delta(\mathcal{O}_X,\mathcal{M}_0) = 0$, Lemma 5.3.6 implies that $\delta(\mathcal{M}_0) = 0$, and since $\mathcal{M}_0 \not\simeq \mathcal{O}_X \oplus \mathcal{O}_X$, the vector bundle $\mathcal{M}_0$ is indecomposable. Proposition 5.3.8, in turn, implies that $[\mathcal{M}_0] \notin \mathbf{P}\mathrm{Bun}_2^{\mathrm{tr}} X$, hence $[\mathcal{M}_0] \in \mathbf{P}\mathrm{Bun}_2^{\mathrm{gi}} X$. We call this class s_0.

Recall that $\mathcal{L}_x$ denotes the line bundle associated to the divisor class $[x] \in \operatorname{Cl} X$. For a place x of degree 1, the $\mathbf{F}_q$-vectorspace $\operatorname{Ext}^1(\mathcal{O}_X,\mathcal{L}_x) \simeq \operatorname{Hom}(\mathcal{O}_X,\mathcal{L}_x) \simeq \Gamma(\mathcal{L}_x)$ is also one-dimensional, and defines a rank 2 bundle $\mathcal{M}_x$ (cf. paragraph 5.3.3). In this case, $\delta(\mathcal{O}_X,\mathcal{M}_x) = \deg \mathcal{O}_X - \deg \mathcal{L}_x = -1$, because if $\mathcal{M}_x$ would have a subbundle $\mathcal{L} \to \mathcal{M}_x$ of degree 1, Lemma 5.3.6 would imply that $\mathcal{M}_x$ decomposes into $\mathcal{O}_X \oplus \mathcal{L}$. Such a decomposition cannot exist, since $\mathcal{M}_x$ was chosen to be a nontrivial extension of $\mathcal{L}_x$ by $\mathcal{O}_X$. Because $\delta(\mathcal{M}_x) \equiv \delta(\mathcal{O}_X,\mathcal{M}_x) \pmod 2$, we obtain that $\delta(\mathcal{M}_x) = -1$, and by Proposition 5.3.8, that $[\mathcal{M}_x] \in \mathbf{P}\mathrm{Bun}_2^{\mathrm{gi}} X$. We denote this class by s_x.

7.1.3 Remark. Note that the notation for the vector bundle $\mathcal{M}_x$ of the previous paragraph is the same as the notation for the stalk of some vector bundle $\mathcal{M}$ at x. To avoid confusion, we will reserve the notation $\mathcal{M}_x$ strictly for the vector bundle defined in the last paragraph throughout the whole chapter.

Since $X(\mathbf{F}_q) = \operatorname{Cl}^1 X$, the graph of Φ_x depends only on the divisor class of x. The vertices c_D and $t_{D'}$, where $[D] \in \operatorname{Cl} X$ and $[D'] \in \operatorname{Cl} X'$ depend also only on the divisor classes of D and D' (Propositions 5.2.3 and 5.2.4), respectively. This justifies that there will arise no ambiguity if we allow ourselves to substitute $[D] \in \operatorname{Cl} X$ by $D \in \operatorname{Cl} X$ and $[D'] \in \operatorname{Cl} X'$ by $D' \in \operatorname{Cl} X'$ for better readability.

7.1.4 Proposition.

$$\mathbf{P}\mathrm{Bun}_2^{\mathrm{gi}} X \;=\; \left\{\, s_x \;\middle|\; x \in \operatorname{Cl}^1 X \,\right\} \;\amalg\; \left\{\, s_0 \,\right\},$$

and $s_x = s_y$ *if and only if* $(x-y) \in 2\operatorname{Cl}^0 X$.

Proof. Let $\mathcal{B}_n^d(Y)$ be the set of isomorphism classes of geometrically indecomposable rank n bundles over Y that have degree d. The symbol Y denotes one of X, X', or $\overline{X} = X \otimes \overline{\mathbf{F}}_q$ with $\overline{\mathbf{F}}_q$ being the algebraic closure of $\mathbf{F}_q$. Observe that we have inclusions $\mathcal{B}_n^d(X) \subset \mathcal{B}_n^d(X') \subset \mathcal{B}_n^d(\overline{X})$, cf. 5.2.1. For a rank 1 bundle $\mathcal{L}$ over Y, the map

$$\begin{array}{ccc} \mathcal{B}_n^d(Y) & \longrightarrow & \mathcal{B}_n^{d+rn}(Y) \\ \mathcal{M} & \longmapsto & \mathcal{M} \otimes \mathcal{L}^r \end{array}$$

defines a bijection of sets for every $d,r \in \mathbf{Z}$ and $n \geq 1$. We have to determine the orbits under $\operatorname{Pic}^0 X$ of $\mathcal{B}_2^0(X)$ and $\mathcal{B}_2^1(X)$ to verify the proposition. We already know that $\mathcal{M}_0 \in \mathcal{B}_2^0(X)$ and $\mathcal{M}_x \in \mathcal{B}_2^1(X)$ for all $x \in \operatorname{Cl}^1 X$.

For the case $d = 0$, we use the following result of Atiyah.

7.1.5 Theorem ([5, Thm. 5 (ii)]). *For all* $\mathcal{M},\mathcal{M}' \in \mathcal{B}_n^0(\overline{X})$*, there exists a unique* $\mathcal{L} \in \operatorname{Pic}^0 \overline{X}$ *such that* $\mathcal{M} \simeq \mathcal{M}' \otimes \mathcal{L}$.

This implies that for every $\mathcal{M} \in \mathcal{B}_2^0(X)$, there exists a unique $\mathcal{L} \in \operatorname{Pic}^0 \overline{X}$ such that $\mathcal{M} \simeq \mathcal{M}_0 \otimes \mathcal{L}$. But the action of $\operatorname{Pic}^0 \overline{X}$ and $\operatorname{Gal}(\overline{\mathbf{F}}_q/\mathbf{F}_q)$ on vector bundles over $\overline{X}$ commute, and thus for every $\sigma \in \mathrm{Gal}(\overline{\mathbf{F}}_q / \mathbf{F}_q)$,

$$\mathcal{M}_0 \otimes \mathcal{L}^\sigma \simeq (\mathcal{M}_0 \otimes \mathcal{L})^\sigma \simeq \mathcal{M}^\sigma \simeq \mathcal{M} \simeq \mathcal{M}_0 \otimes \mathcal{L} .$$

By uniqueness, $\mathcal{L}^\sigma \simeq \mathcal{L}$, and thus $\mathcal{L} \in \mathrm{Pic}^0 X$. Hence $[\mathcal{M}] = s_0 \in \mathbf{P}\mathrm{Bun}_2^{\mathrm{gi}} X$.

For $d = 1$, we restate Atiyah's classification of indecomposable vector bundles over $\overline{X}$.

7.1.6 Theorem ([5, Thm. 7]). *There are bijections* $\varphi_n^d : \mathcal{B}_n^d(\overline{X}) \to \mathrm{Pic}^0(\overline{X})$ *such that the diagrams*

$$\begin{array}{ccc} \mathcal{B}_n^d(\overline{X}) & \xrightarrow{\varphi_n^d} & \mathrm{Pic}^0(\overline{X}) \\ \downarrow{\scriptstyle \det} & & \downarrow{\scriptstyle (n,d)} \\ \mathcal{B}_1^d(\overline{X}) & \xrightarrow{\varphi_1^d} & \mathrm{Pic}^0(\overline{X}) \end{array}$$

commute for all $d \in \mathbf{Z}$ *and* $n \geq 1$. *Here,* (n,d) *denotes multiplication with the greatest common divisor of* n *and* d.

This means that $\det : \mathcal{B}_2^1(\overline{X}) \to \mathcal{B}_1^1(\overline{X})$ is a bijection, and consequently the restriction $\det : \mathcal{B}_2^1(X) \to \mathcal{B}_1^1(X)$ is still injective. Because every element of $\mathcal{B}_1^1(X)$ is of the form $\mathcal{L}_x$ for some place x of degree 1 and because $\det(\mathcal{M}_x) \simeq \mathcal{L}_x \in \mathcal{B}_1^1(X)$, we obtain that $\mathcal{B}_2^1(X) = \{\mathcal{M}_x | x \in \mathrm{Cl}^1 X\}$.

By the injectivity of the determinant map, $\mathcal{M}_x \simeq \mathcal{M}_y \otimes \mathcal{L}$ for some $\mathcal{L} \in \mathrm{Pic}^0 X$ if and only if $\det \mathcal{M}_x \simeq \det(\mathcal{M}_y \otimes \mathcal{L}) \simeq (\det \mathcal{M}_y) \otimes \mathcal{L}^2$, or, equivalently, $(x - y) \in 2\,\mathrm{Cl}^0 X$. This proves Proposition 7.1.4. □

7.1.7 Remark. This proposition shows that projective line bundles that are geometrically indecomposable behave differently from those that decompose after extension of constants, cf. Lemma 5.2.5. If $x - y \notin 2\,\mathrm{Cl}^0 X$, then s_x and s_y are not isomorphic. However there is a finite constant extension of $Y \to X$ such that $x - y \in 2\,\mathrm{Cl}^0 Y$, since geometrically the class group of an elliptic curve is divisible. Thus s_x and s_y become isomorphic over Y. For a concrete example, consider $X = X_6$, and $Y = X_6'$ as in paragraph 7.3.3.

7.1.8 Corollary. *If a rank* 2 *bundle* $\mathcal{M}$ *has* $\delta(\mathcal{M}) = -1$ *and* $\det \mathcal{M} \simeq \mathcal{L}_x$, *then* $\mathcal{M}$ *represents* s_x.

Proof. A rank 2 bundle $\mathcal{M}$ with $\delta(\mathcal{M}) = -1$ must be geometrically indecomposable. The corollary follows from the fact that every element of $\mathcal{B}_2^1(X)$ is characterised by its determinant. □

7.1.9 Corollary. *Let* $x \in X(\mathbf{F}_q)$. *Then the nucleus* $\mathcal{N}_x$ *of the graph* $\mathcal{G}_x$ *consists of the vertices*

$$\mathrm{Vert}\,\mathcal{N}_x = \{t_D\}_{D \in \mathrm{Cl}\,X'} \amalg \{s_x\}_{x \in \mathrm{Cl}^1 X} \amalg \{s_0\} \amalg \{c_D\}_{D \in \mathrm{Cl}^0 X \cup \mathrm{Cl}^1 X} .$$

Proof. By definition, the nucleus contains all vertices $v \in \mathrm{Vert}\,\mathcal{G}_x$ with $\delta(v) \leq 1$. In particular, $\mathcal{N}_x$ contains $\mathbf{P}\mathrm{Bun}_2^{\mathrm{tr}} X$, which is described in Proposition 5.2.4, $\mathbf{P}\mathrm{Bun}_2^{\mathrm{gi}} X$, which is described in Proposition 7.1.4, and the vertices $v \in \mathbf{P}\mathrm{Bun}_2^{\mathrm{dec}} X$ with $\delta(v) \leq 1$, which are as in the corollary by Proposition 5.2.3. □

7.2 Edges

Let x be a place of degree 1 and let Φ_x be defined as in 1.4.2. Theorem 5.4.9 and Proposition 7.1.4 determine the graph $\mathcal{G}_x$ of Φ_x up to the edges of the nucleus $\mathcal{N}_x$ as illustrated in Figure 7.1.

For an elliptic curve, the divisor classes of degree 0 can be represented by the difference of two divisors of degree 1, or more precisely $z-x$ runs through $\mathrm{Cl}^0 X$ as z varies through all degree 1 places while x is fixed. We characterise all the missing edges of $\mathcal{G}_x$.

7.2.1 Theorem. *Let x be a prime divisor of degree 1 and $h_2 = \#\mathrm{Cl}^0 X[2]$ the cardinality of the 2-torsion of the class group. Then the edges with origin in $\mathcal{N}_x$ are given by the following list.*

$$
\begin{aligned}
\mathcal{U}_x(c_0) &= \{(c_0, c_x, q+1)\}\,,\\
\mathcal{U}_x(c_x) &= \{(c_x, c_{2x}, 1), (c_x, c_0, 1), (c_x, s_0, q-1)\}\,,\\
\mathcal{U}_x(c_y) &= \{(c_y, c_{y+x}, 1), (c_y, c_{y-x}, q)\} \qquad \textit{if } y \neq x,\\
\mathcal{U}_x(c_{y-x}) &= \{(c_{y-x}, c_y, 2), (c_{y-x}, s_y, q-1)\} \qquad \textit{if } y \neq x, \textit{ but } y-x \in (\mathrm{Cl}\, X)[2],\\
\mathcal{U}_x(c_{y-x}) &= \{(c_{y-x}, c_y, 1), (c_{y-x}, c_{2x-y}, 1), (c_{y-x}, s_y, q-1)\} \qquad \textit{if } y-x \notin (\mathrm{Cl}\, X)[2],\\
\mathcal{U}_x(s_0) &= \{(s_0, c_x, 1), (s_0, s_x, q)\}\,,\\
\mathcal{U}_x(t_D) &= \{(t_D, s_{x+D+\sigma D}, q+1)\} \qquad \textit{for } D \in \mathrm{Cl}^0 X' - \mathrm{Cl}^0 X, \textit{ and}\\
\mathcal{U}_x(s_y) &= \{(s_y, s_0, h_2) \mid \textit{if } y \equiv x \pmod{2\,\mathrm{Cl}^0 X}\}\\
&\quad \cup \left\{ (s_y, c_{z-x}, \tfrac{1}{2}h_2) \;\middle|\; \substack{\textit{if } (z-x)\in(\mathrm{Cl}^0 X)[2],\\ z\neq x, \textit{ and } (z-y)\in 2\,\mathrm{Cl}^0 X} \right\}\\
&\quad \cup \left\{ (s_y, c_{z-x}, h_2) \;\middle|\; \substack{\textit{if } (z-x)\notin(\mathrm{Cl}^0 X)[2],\\ \textit{and } (z-y)\in 2\,\mathrm{Cl}^0 X} \right\}\\
&\quad \cup \left\{ (s_y, t_D, \tfrac{1}{2}h_2) \;\middle|\; \substack{\textit{if } D\in(\mathrm{Cl}^0 X'-\mathrm{Cl}^0 X), 2D\in\mathrm{Cl}^0 X,\\ \textit{and } y\equiv D+\sigma D+x \pmod{2\,\mathrm{Cl}^0 X}} \right\}\\
&\quad \cup \left\{ (s_y, t_D, h_2) \;\middle|\; \substack{\textit{if } D\in(\mathrm{Cl}^0 X'-\mathrm{Cl}^0 X), 2D\notin\mathrm{Cl}^0 X,\\ \textit{and } y\equiv D+\sigma D+x \pmod{2\,\mathrm{Cl}^0 X}} \right\} \qquad \textit{for } y \in \mathrm{Cl}^1 X.
\end{aligned}
$$

Remark on illustrations: There are illustrations of these sets at the appropriate places in the proof. We draw vertices v from left to right in order of increasing value of $\delta(v)$. At the end of this section and in section 7.3 one finds illustrations of entire graphs.

Proof. We recall some results that we will use in the proof without further reference. If v and w are Φ_x-neighbours, then $\delta(w) = \delta(v) \pm 1$ (Lemma 5.4.2). The weights of all Φ_x-neighbours of each vertex sum up to $q+1$ (Proposition 4.2.4). The Φ_x-neighbours v' of a vertex $v = [\mathcal{M}]$ with $\delta(v') = \delta(v) + 1$ counted with multiplicity are in bijection with the maximal subbundles of $\mathcal{M}$ (Lemma 5.4.4). This bijection is given by taking a maximal subbundle $\mathcal{L} \to \mathcal{M}$ (paragraph 5.4.3) to its associated sequence. Recall that $\mathcal{J}_x$ is the kernel of $\mathcal{O}_X \to \mathcal{K}_x$ (paragraph 5.4.1) and that $\mathcal{L}_x$ denotes the line bundle associated to the divisor x. We prove the theorem case by case.

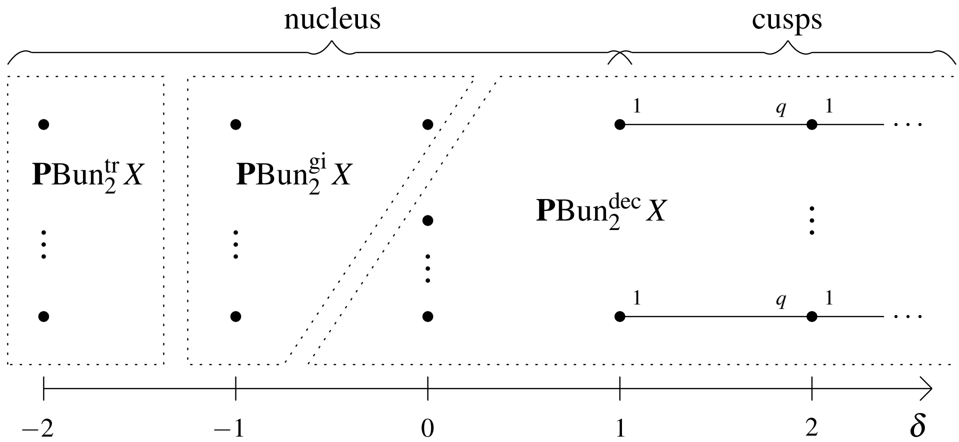
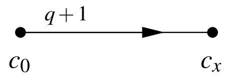
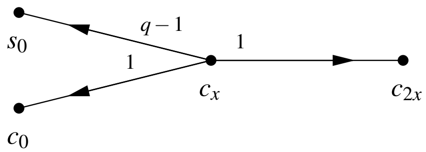


Figure 7.1: $\mathcal{G}_x$ up to a finite number of edges

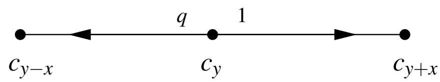
- Theorem 5.4.6 describes $\mathcal{U}_x(c_0)$ completely:



- Let $\mathcal{M} = \mathcal{L}_x \oplus \mathcal{O}_X$ represent c_x. We know from Theorem 5.4.6 that c_{2x} is the only neighbour $\mathcal{M}'$ with $\delta(\mathcal{M}') = 2$. It has multiplicity 1 and is given by the sequence associated to $\mathcal{L}_x \to \mathcal{M}$. By Lemma 5.4.13, the sequence associated to $\mathcal{O}_X \to \mathcal{M}$ gives $\mathcal{O}_X \oplus \mathcal{O}_X$ as neighbour. For all other $q-1$ neighbours $\mathcal{M}'$, neither $\mathcal{L}_x \to \mathcal{M}$ nor $\mathcal{O}_X \to \mathcal{M}$ lifts to $\mathcal{M}'$, but then $\mathcal{L}_x \mathcal{J}_x \subset \mathcal{L}_x \to \mathcal{M}$ lifts to a subbundle $\mathcal{O}_X \simeq \mathcal{L}_x \mathcal{J}_x \to \mathcal{M}'$. We have that $\det \mathcal{M}' \simeq (\det \mathcal{M})\mathcal{J}_x \simeq \mathcal{O}_X$, but $\mathcal{O}_X \to \mathcal{M}'$ cannot have a complement, since otherwise $\mathcal{O}_X \to \mathcal{M}$ would lift. Thus $\mathcal{M}'$ must represent s_0. This describes $\mathcal{U}_x(c_x)$:

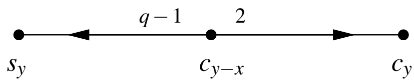


- Let $\mathcal{M} = \mathcal{L}_y \oplus \mathcal{O}_X$ represent c_y with $y \neq x$. Again, we know that c_{y+x} is the only neighbour $\mathcal{M}'$ with $\delta(\mathcal{M}') = 2$, and it has multiplicity 1. For all other q neighbours, $\mathcal{L}_y \mathcal{J}_x \to \mathcal{M}'$ is a subbundle, and $\mathcal{M}' / \mathcal{L}_y \mathcal{J}_x \simeq \mathcal{O}_X$. But since $\mathcal{L}_y \mathcal{J}_x \not\simeq \mathcal{O}_X$, we have that $\mathrm{Ext}^1(\mathcal{L}_y \mathcal{J}_x, \mathcal{O}_X) = 0$ (paragraph 7.1.2), and thus $\mathcal{M}'$ decomposes. We obtain for $\mathcal{U}_x(c_y)$:

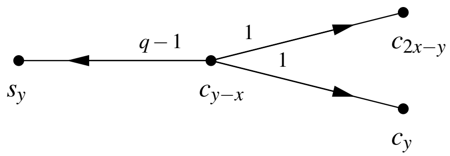


- Let $\mathcal{M} = \mathcal{L}_y \oplus \mathcal{L}_x$ represent c_{y-x} with $y \neq x$. Then the sequences associated to the two maximal subbundles $\mathcal{L}_y \to \mathcal{M}$ and $\mathcal{L}_x \to \mathcal{M}$ determine two neighbours $\mathcal{L}_y \oplus \mathcal{O}_X$

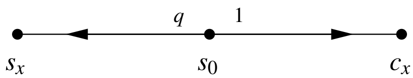
and $\mathcal{L}_y \mathcal{J}_x \oplus \mathcal{L}_X$. They both decompose by Lemma 5.4.13 and represent c_y and c_{2x-y}, respectively. For all other $q-1$ neighbours $\mathcal{M}'$, no maximal line bundle lifts, and thus $\delta(\mathcal{M}') = -1$. Since $\det \mathcal{M}' \simeq \mathcal{L}_y \mathcal{L}_x \mathcal{J}_x \simeq \mathcal{L}_y$, by Corollary 7.1.8, $\mathcal{M}'$ represents s_y. We have $c_{2x-y} = c_y$ if and only if $\mathcal{L}_x^2 \mathcal{L}_y^{-1} \simeq \mathcal{L}_y$, or equivalently, $\left(\mathcal{L}_x \mathcal{L}_y^{-1}\right)^2 \simeq \mathcal{O}_X$. This means that these two neighbours are the same if and only if $x - y \in (\mathrm{Cl}\, X)[2]$. If this is the case, we get for $\mathcal{U}_x(c_{y-x})$:



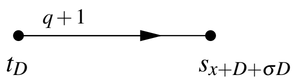
- If $x - y \notin (\mathrm{Cl}\, X)[2]$, $\mathcal{U}_x(c_{y-x})$ looks like:



- Let $\mathcal{M}$ be the bundle $\mathcal{M}_0$ of paragraph 7.1.2, which represents s_0. Then it has a unique maximal subbundle $\mathcal{O}_X \to \mathcal{M}$ and an associated neighbour $\mathcal{M}'$ with $\delta(\mathcal{M}') = 1$, which decomposes. Because its maximal subbundle is $\mathcal{O}_X \to \mathcal{M}'$, $\det \mathcal{M}' \simeq \mathcal{J}_x$, and we recognise it as $\mathcal{O}_X \oplus \mathcal{J}_x$. Thus $\mathcal{M}'$ represents c_x. All q other neighbours $\mathcal{M}'$ have $\delta(\mathcal{M}') = -1 = \delta(\mathcal{M}' \otimes \mathcal{L}_x)$, and $\det(\mathcal{M}' \otimes \mathcal{L}_x) \simeq \mathcal{J}_x \mathcal{L}_x^2 \simeq \mathcal{L}_x$. By Corollary 7.1.8, $\mathcal{M}' \otimes \mathcal{L}_x$ and thus also $\mathcal{M}'$ represent s_x, and $\mathcal{U}_x(s_0)$ is as follows:



- Let $\mathcal{M}$ represent t_D for a $D \in \mathrm{Cl}^0 X' - \mathrm{Cl}^0 X$. Since $\delta(t_D) = -2$, every neighbour $\mathcal{M}'$ of $\mathcal{M}$ must have $\delta(\mathcal{M}') = -1$. It is determined by its determinant, which we can calculate by extending constants to X'. We have $\det \mathcal{M}' \simeq \mathcal{J}_x \det(\mathcal{L}_D \oplus \mathcal{L}_{\sigma D}) \simeq \mathcal{J}_x \mathcal{L}_D \mathcal{L}_{\sigma D}$. Because $-x + D + \sigma D \equiv x + D + \sigma D \pmod{2\,\mathrm{Cl}\, X}$, Corollary 7.1.8 implies that $\mathcal{M}'$ represents $s_{x+D+\sigma D}$. We obtain for $\mathcal{U}_x(t_D)$:



- The most subtle part is to determine the neighbours of s_y for $y \in \mathrm{Cl}^1 X$. We choose $\mathcal{M}_y$ as representative for s_y, see paragraph 7.1.2, and recall that it was defined by a nontrivial element in $\mathrm{Ext}^1(\mathcal{O}_X, \mathcal{L}_y)$. Thus $\det(\mathcal{M}_y) = \mathcal{L}_y$, and $\delta(\mathcal{M}_y) = -1$. Look at an exact sequence

$$0 \longrightarrow \mathcal{M}' \longrightarrow \mathcal{M}_y \longrightarrow \mathcal{K}_x \longrightarrow 0 \ .$$

Then $\det(\mathcal{M}') \simeq (\det \mathcal{M}_y)\mathcal{J}_x \simeq \mathcal{L}_{y-x} \in \mathrm{Pic}^0 X$, and $\delta(\mathcal{M}') \in \{-2, 0\}$. By Proposition 4.4.3, s_y must also be a neighbour of $[\mathcal{M}']$. But we have already determined the neighbours of vertices v with these properties. We find that for $(z - x) \in \mathrm{Cl}^0 X - \{0\}$, c_{z-x} is a neighbour of s_y if and only if $y \equiv z \pmod{2\,\mathrm{Cl}^0 X}$, t_D with $D \in \mathrm{Cl}^0 X' - \mathrm{Cl}^0 X$ is a neighbour of s_y if and only if $y \equiv x + D + \sigma D \pmod{2\,\mathrm{Cl}^0 X}$, and s_0 is a neighbour of s_y if and only if $y \equiv x \pmod{2\,\mathrm{Cl}^0 X}$, but c_0 is never a neighbour of s_y. This shows that the theorem lists precisely the neighbours of s_y. There is still some work to be done to determine the weights. We begin with an observation.

7.2.2 Lemma. *Up to isomorphism with fixed $\mathcal{M}_y$, there is at most one exact sequence $0 \to \mathcal{M}' \to \mathcal{M}_y \to \mathcal{K}_x \to 0$ for a fixed $\mathcal{M}'$.*

Proof. Suppose there are two. We derive a contradiction as follows. If $\delta(\mathcal{M}') \neq 0$, then $\mathcal{M}'$ must be a trace of a line bundle $\mathcal{L}$ defined over X'. By extending constants to $\mathbf{F}_{q^2}$, we may thus assume that $\delta(\mathcal{M}') = 0$ and that there are $\mathcal{L}, \mathcal{L}' \in \mathrm{Pic}^0 X$ such that $\mathcal{M}'$ is an extension of $\mathcal{L}'$ by $\mathcal{L}$. The composition $\mathcal{L} \to \mathcal{M}' \to \mathcal{M}$ defines a maximal subbundle of $\mathcal{M}$ because $\delta(\mathcal{L}, \mathcal{M}) = -1$. We get back the inclusion $\mathcal{M}' \to \mathcal{M}$ by taking the associated sequence. Since we assume we have two different inclusions of $\mathcal{M}'$ into $\mathcal{M}$, we get two different subbundles of the form $\mathcal{L} \to \mathcal{M}$, thus an inclusion $\mathcal{L} \oplus \mathcal{L} \to \mathcal{M}$. The cokernel is a torsion sheaf of degree 1 defined over $\mathbf{F}_{q^2}$, say $\mathcal{K}_{x'}$ for a place x' of $\mathbf{F}_{q^2} F$, and we obtain an exact sequence

$$0 \longrightarrow \mathcal{L} \oplus \mathcal{L} \longrightarrow \mathcal{M}_y \longrightarrow \mathcal{K}_{x'} \longrightarrow 0 \;;$$

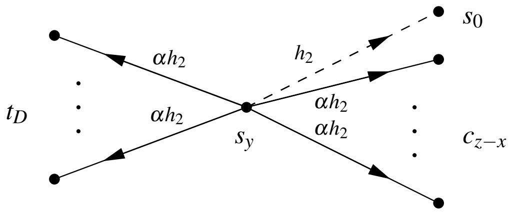
$c_0 = [\mathcal{L} \oplus \mathcal{L}]$ is thus an $\Phi_{x'}$-neighbour of $\mathcal{M}_y$. This is a contradiction as s_y is not a neighbour of c_0. □

We consider a second neighbour $\mathcal{M}''$ of $\mathcal{M}_y$ that represents the same element as $\mathcal{M}'$ in $\mathbf{P}\mathrm{Bun}\,X$, i.e. $\mathcal{M}'' \simeq \mathcal{M}' \otimes \mathcal{L}_0$ for some $\mathcal{L}_0 \in \mathrm{Pic}\,X$. Since they have the same determinant, $\mathcal{L}_0^2 \simeq \det(\mathcal{M}' \otimes \mathcal{L}_0)(\det \mathcal{M}')^{-1} \simeq (\det \mathcal{M}'')(\det \mathcal{M}')^{-1} \simeq \mathcal{O}_x$, meaning $\mathcal{L}_0 \in (\mathrm{Pic}\,X)[2]$. On the other hand, Theorem 7.1.6 tells us that for $\mathcal{M}_y \in \mathcal{B}_2^1(X)$, $\mathcal{M}_y \otimes \mathcal{L}_0 \simeq \mathcal{M}_y$ if and only if $\mathcal{L}_0 \in (\mathrm{Pic}\,X)[2]$. Thus $(\mathrm{Pic}\,X)[2]$ acts on the sequences that we investigate. By Lemma 7.2.2, we find that the multiplicity of a neighbour $\mathcal{M}'$ of $\mathcal{M}_y$ equals the number of isomorphism classes that $\mathcal{M}' \otimes \mathcal{L}_0$ meets as $\mathcal{L}_0$ varies through $(\mathrm{Pic}\,X)[2] = (\mathrm{Pic}^0 X)[2]$.

We begin with the case of a neighbour $\mathcal{M}'$ that is associated to a maximal subbundle $\mathcal{L} \to \mathcal{M}_y$. Then $\delta(\mathcal{L}, \mathcal{M}') = 0$. If $\mathcal{M}'/\mathcal{L} \simeq \mathcal{L}$, the only possibility with these properties is s_0. But then $\mathcal{L} \to \mathcal{M}'$ is the only maximal subbundle, so all $\mathcal{L} \otimes \mathcal{L}_0$ with $\mathcal{L}_0 \in (\mathrm{Pic}^0 X)[2]$ have different associated sequences, and the multiplicity of s_0 is therefore $h_2 = \#(\mathrm{Pic}^0 X)[2]$.

If $\mathcal{L}' := \mathcal{M}'/\mathcal{L} \not\simeq \mathcal{L}$, then $\mathcal{M}'$ represents c_{z-x} for the divisor $(z-x) \in \mathrm{Cl}^0 X$ that satisfies $\mathcal{L}_{z-x} \simeq \mathcal{L}'\mathcal{L}^{-1}$. Since $\mathcal{L}_{y-x} \simeq \det \mathcal{M}' \simeq \mathcal{L}\mathcal{L}'$, we have $z \equiv y \pmod{2\,\mathrm{Cl}^0 X}$. The rank 2 bundle $\mathcal{M}'$ has two different maximal subbundles, and it could happen that $\mathcal{M}' \simeq \mathcal{M}' \otimes \mathcal{L}_0$ for some $\mathcal{L}_0 \in (\mathrm{Pic}^0 X)[2] - \{\mathcal{O}_X\}$. This only happens if $\mathcal{L}' \simeq \mathcal{L}\mathcal{L}_0$, so $\mathcal{L}'\mathcal{L}^{-1} \in (\mathrm{Pic}^0 X)[2]$, or equivalently, $(z-x) \in (\mathrm{Cl}^0 X)[2]$. Thus the multiplicity of c_{z-x} as a neighbour of s_y is $h_2/2$ if $(z-x) \in (\mathrm{Cl}^0 X)[2] - \{0\}$ and h_2 if $(z-x) \notin (\mathrm{Cl}^0 X)[2]$.

The last case is that of $\delta(\mathcal{M}') = -2$, where $\mathcal{M}'$ is the trace of a line bundle $\mathcal{L}_D$, where $D \in \mathrm{Cl}^0 X' - \mathrm{Cl}^0 X$. If we lift the situation to X', then $\mathcal{M}' \simeq \mathcal{L}_D \oplus \mathcal{L}_{\sigma D}$, and we see as in the preceding case that $\mathcal{M}' \simeq \mathcal{M}' \otimes \mathcal{L}_0$ for some $\mathcal{L}_0 \in (\mathrm{Pic}^0 X)[2] - \{\mathcal{O}_X\}$ if and only if $D - \sigma D \in (\mathrm{Cl}^0 X)[2]$. This is equivalent to the two conditions $D - \sigma D \in \mathrm{Cl}^0 X$ and $2D - 2\sigma D = 0$, or $2D = (D - \sigma D) + (D + \sigma D) \in \mathrm{Cl}^0 X$ and $2D = \sigma(2D)$, respectively, both saying that $2D \in \mathrm{Cl}^0 X$. This finally gives for $D \in \mathrm{Cl}^0 X' - \mathrm{Cl}^0 X$ that t_D has multiplicity $h_2/2$ as neighbour of s_y if $2D \in \mathrm{Cl}^0 X$ and h_2 if $2D \notin \mathrm{Cl}^0 X$. We illustrate this below. The dashed arrow only occurs if $y-x \in 2\,\mathrm{Cl}^0 X$. The indices z and D take all possible values as in the theorem, and $\alpha \in \{1/2, 1\}$ depends on the particular edge.



This completes the proof of the theorem. □

7.2.3 Remark. In Remark 5.1.13, we explained the connection between the graphs that Serre considers in [60] and graphs of Hecke operators. In [64], Takahashi classified Serre's graphs for places x of degree 1 and genus 1 by elementary matrix calculations. If the class number is odd, then Serre's graph coincides with $\mathcal{G}_x$ without weights. If, however, the class number is even, then these two notions of graphs produce different objects.

When we calculate the space of unramified toroidal automorphic forms, we also need to evaluate $\mathcal{G}_x$ for different places x. Namely, we will use of the graphs $\mathcal{G}_x$ for all h_X places x of degree 1. It is not visible from [64] how the vertices of the graphs for various places of degree 1 relate to each other, but Theorem 7.2.1 makes this clear.

7.2.4 (Odd class number) Let the class number $h = h_X$ be odd and x a place of degree 1. Then $\mathcal{G}_x$ has only one component. We write $\{x, z_2, \dots, z_h\} = \mathrm{Cl}^1 X$ where the z_i's are ordered such that $z_{2i} - x = x - z_{2i+1}$ for $i = 1, \dots, (h-1)/2$ and $\{t_1, \dots, t_{r'}\} = \mathbf{P}\mathrm{Bun}_2^{\mathrm{tr}} X$. Then we can illustrate the graph of Φ_x as in Figure 7.2.

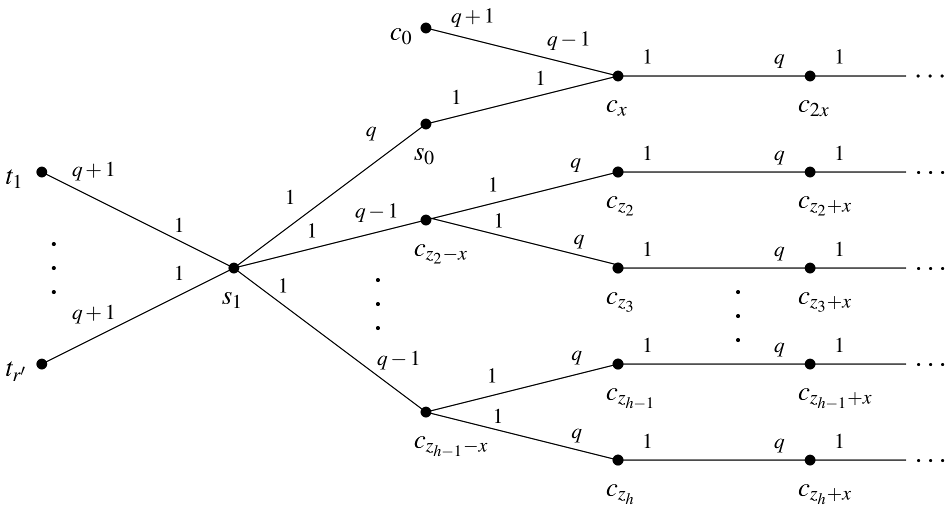


Figure 7.2: $\mathcal{G}_x$ for a degree one place x of an elliptic curve with odd class number

7.3 Examples

This section provides some examples of illustrations of graphs of Hecke operators.

7.3.1 Example. The easiest examples are given by elliptic curves with only one rational point x and can be found in the literature, cf. [15], [60, 2.4.4 and Ex. 3 of 2.4] or [64]. There are up to isomorphism three such elliptic curves: X_2 over $\mathbf{F}_2$ defined by the Weierstrass equation $\underline{Y}^2 + \underline{Y} = \underline{X}^3 + \underline{X} + 1$, X_3 over $\mathbf{F}_3$ defined by the Weierstrass equation $\underline{Y}^2 = \underline{X}^3 + \underline{2}X + 2$ and X_4 over $\mathbf{F}_4$ defined by the Weierstrass equation $\underline{Y}^2 + \underline{Y} = \underline{X}^3 + \alpha$ with $\mathbf{F}_4 = \mathbf{F}_2(\alpha)$. Since the class number is 1, $\mathbf{P}\mathrm{Bun}_2^{\mathrm{dec}} X_q = \{c_{nx}\}_{n\geq 0}$ and $\mathbf{P}\mathrm{Bun}_2^{\mathrm{gi}} X_q = \{s_0, s_x\}$ for $q \in \{2,3,4\}$. One calculates that $\mathrm{Cl}^0(X_2 \otimes \mathbf{F}_4) \simeq \mathbf{Z}/5\mathbf{Z}$, $\mathrm{Cl}^0(X_3 \otimes \mathbf{F}_9) \simeq \mathbf{Z}/7\mathbf{Z}$ and $\mathrm{Cl}^0(X_4 \otimes \mathbf{F}_{16}) \simeq \mathbf{Z}/9\mathbf{Z}$, thus $\mathbf{P}\mathrm{Bun}_2^{\mathrm{tr}} X_q$ has q different elements $t_1, \dots, t_q$ for $q \in \{2,3,4\}$. We obtain Figure 7.3.

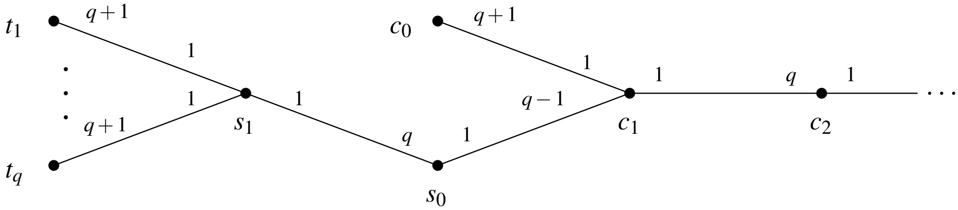


Figure 7.3: $\mathscr{G}_x$ for the unique degree one place x of the elliptic curves X_q for $q = 2,3,4$

We give two examples for elliptic curves with even class number. Both examples are elliptic curves over $\mathbf{F}_3$ with class number 4, but with respective 2-torsion $\mathbf{Z}/4\mathbf{Z}$ and $(\mathbf{Z}/2\mathbf{Z})^2$.

7.3.2 Example. The first example is the elliptic curve X_5 over $\mathbf{F}_3$ defined by the Weierstrass equation $\underline{Y}^2 = \underline{X}^3 + \underline{X} + 2$, which has class group $\mathrm{Cl}^0 X_5 \simeq \mathbf{Z}/4\mathbf{Z} = \{x, y, z, z'\}$, where $x - y$ is the element of order 2. The number of components is $h_2 = 2$, and $\mathbf{P}\mathrm{Bun}_2^{\mathrm{gi}} X$ is given by s_0, $s_x = s_y$ and $s_z = s_{z'}$. The class group of $X_5' = X_5 \otimes \mathbf{F}_9$ is $\mathrm{Cl}^0 X_5' \simeq (\mathbf{Z}/4\mathbf{Z})^2$, thus $\mathrm{Cl}^0 X_5' / \mathrm{Cl}^0 X_5 \simeq \mathbf{Z}/4\mathbf{Z}$. Let $\{0, D, D', D''\}$ be representatives such that D is the divisor with $2D \in \mathrm{Cl}^0 X_5$. Then $\mathbf{P}\mathrm{Bun}_2^{\mathrm{tr}} X_5$ contains the two elements t_D and $t_{D'} = t_{D''}$. We do not need to calculate the norm map $\mathrm{Cl}^0 X_5' \to \mathrm{Cl}^0 X_5$ as we can find out to which of t_D and $t_{D'}$ the vertices s_x and s_z are connected by the constraint that the weights around s_x and s_z, respectively, sum up to 4. The graph is illustrated in Figure 7.4.

7.3.3 Example. The second example X_6 over $\mathbf{F}_3$ is defined by the Weierstrass equation $\underline{Y}^2 = \underline{X}^3 + 2\underline{X}$, and has class group $\mathrm{Cl}^0 X_6 \simeq (\mathbf{Z}/2\mathbf{Z})^2 = \{x, y, z, w\}$. Here $h_2 = 4$, and s_x, s_y, s_z and s_w are pairwise distinct vertices. For $X_6' = X_6 \otimes \mathbf{F}_9$, $\mathrm{Cl}^0 X_6' \simeq (\mathbf{Z}/4\mathbf{Z})^2$, thus $\mathrm{Cl}^0 X_6' / \mathrm{Cl}^0 X_6 \simeq (\mathbf{Z}/2\mathbf{Z})^2$, which we represent by $\{0, D_1, D_2, D_3\}$, each of the D_i being of order 2. Again, by the constraint that weights around each vertex sum up to 4, we find that $\mathbf{P}\mathrm{Bun}_2^{\mathrm{tr}} X_6$ contains three different traces of the line bundles corresponding to D_1, D_2 and D_3, which we denote by t_y, t_z and t_w, and which are connected to s_y, s_z and s_w, respectively. The graph is illustrated in Figure 7.5.

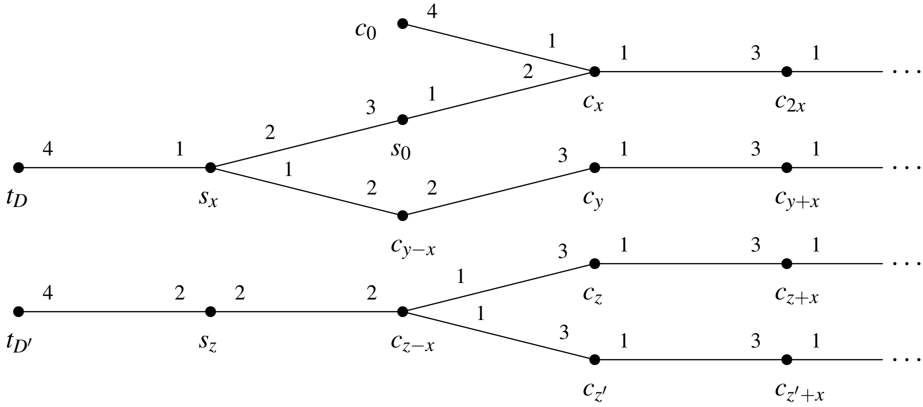


Figure 7.4: $\mathcal{G}_x$ for a degree one place x of the elliptic curves X_5

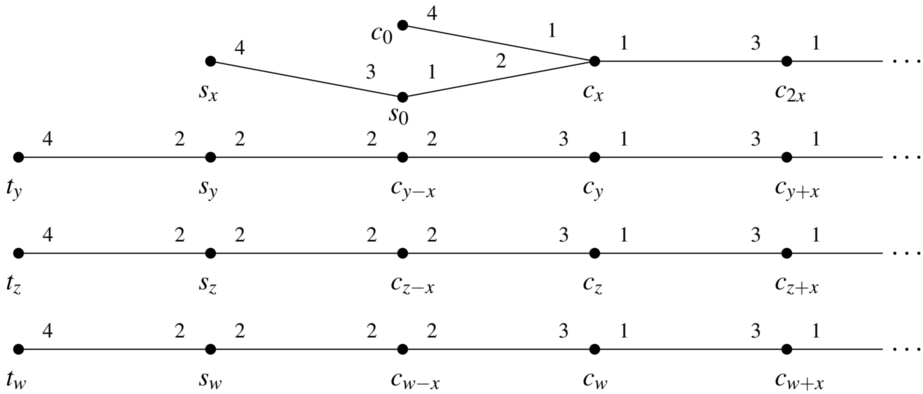


Figure 7.5: $\mathcal{G}_x$ for a degree one place x of the elliptic curves X_6

CHAPTER 8

Toroidal automorphic forms for genus 1

The aim of this chapter is to investigate the space of unramified toroidal automorphic forms for a global function field F of genus 1. The strategy is as follows: If F' denotes the quadratic constant field extension of F, then the results from Chapter 6 provide precise conditions on the functions in the completed Eisenstein part to be F'-toroidal. Theorem 6.1.2 yields a translation of the toroidal condition for F' as a linear equation in the values of an unramified F'-toroidal automorphic form, which can be interpreted in terms of values at vertices of the graph of an unramified Hecke operator. Together with eigenvalue equations for various Hecke operators, which are calculated using the results from Chapter 7, this excludes the existence of nontrivial toroidal cusp forms. This finally leads to the conclusion that at least one and at most two of the unramified F'-toroidal automorphic forms are toroidal. In the last section, we discuss how close the developed methods get to a proof of the Riemann hypothesis for function fields of genus 1, i.e. the theorem of Hasse ([29, §4]).

8.1 Eigenvalue equations

In this section, we formulate eigenvalue equations for $\mathcal{H}_K$-eigenfunctions, which can be extracted from the graphs that we determined in the previous chapter. First, we fix some notation that will be used throughout this chapter.

8.1.1 Let F be an elliptic function field, i.e. the function field of a curve X of genus 1. Let $\mathbf{F}_q$ be the the field of constants, $\operatorname{Cl} X$ the divisor class group and h_X the class number. Recall from paragraph 7.1.1 that the set of $\mathbf{F}_q$-rational points $X(\mathbf{F}_q) = \{x_1, \dots, x_{h_X}\}$ of X, considered as prime divisors, is in one-to-one correspondence with the set of divisor classes $\operatorname{Cl}^1 X$ of degree 1 on X. We identify these sets. Let $p : X' = X \otimes \mathbf{F}_{q^2} \to X$ be the covering by the constant extension of degree 2 and let $F' = \mathbf{F}_{q^2} F$ be the function field of X'. The map $p^* : \operatorname{Cl} X \to \operatorname{Cl} X'$ is injective, so we may and will consider $\operatorname{Cl} X$ as a subgroup of $\operatorname{Cl} X'$. Let σ denote the nontrivial element of the Galois group of F'/F.

As in Chapter 7, we write $D \in \operatorname{Cl} X$, where we, strictly speaking, want D to denote a divisor and not a divisor class. But since no ambiguity arises as explained in Remark 7.1.3, we allow ourselves this misuse of notation in favour of better readability. If D is a divisor of degree 0, then there is for any chosen $x \in \operatorname{Cl}^1 X$ a unique $z \in \operatorname{Cl}^1 X$ such that $z - x$ represents the same divisor class as D. If we fix x and write $z - x \in \operatorname{Cl}^0 X$, we make implicit use of this fact.

Let x be a place of degree 1 and define the following numbers:

$$\begin{aligned} h &= h_X = \#\mathrm{Cl}^0 X = \#\{c_D\}_{D\in\mathrm{Cl}^1 X}, & h' &= \#(\mathrm{Cl}^0 X'/\mathrm{Cl}^0 X),\\ h_2 &= \#\mathrm{Cl}^0 X[2] = \#\{s_y\}_{y\in X(\mathbf{F}_q)}, & h'_2 &= \#(\mathrm{Cl}^0 X'/\mathrm{Cl}^0 X)[2],\\ r &= (h+h_2)/2-1 = \#\{c_D\}_{D\in\mathrm{Cl}^0 X-\{0\}}, & r' &= (h'+h'_2)/2-1 = \#\{t_D\}_{D\in\mathrm{Cl}\, X'-\mathrm{Cl}\, X}. \end{aligned}$$

The equality in the definition of h_2 follows from Proposition 7.1.4, the equality in definition of h and r from Proposition 5.2.3 and the equality in definition of r' from Proposition 5.2.4. Figure 8.1 shows certain subsets of Vert $\mathcal{G}_x$. Each dashed subset of Vert $\mathcal{G}_x$ is defined by the set written underneath. The integer written to the right is its cardinality. A line between two dashed areas indicate that there is at least one edge in $\mathcal{G}_x$ between two vertices in the corresponding subsets.

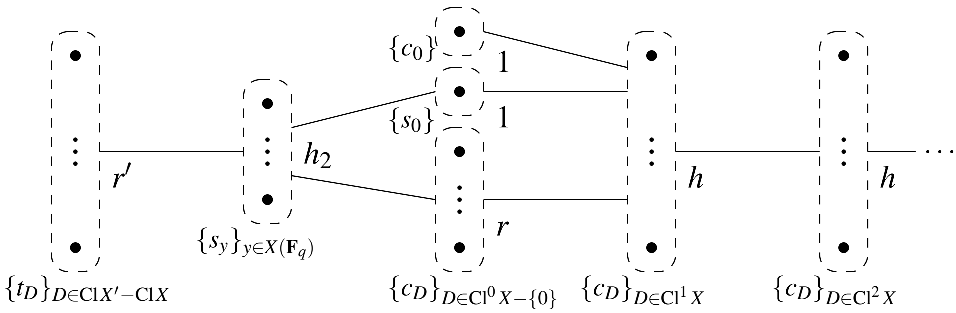


Figure 8.1: Certain subsets of Vert $\mathcal{G}_x$ and their cardinalities

8.1.2 Lemma. $h' = 2(q+1)-h$.

Proof. Fix a place x of degree 1 and consider $\mathcal{G}_x$. We count the weights around the h_2 vertices s_y, where y varies through $\mathrm{Cl}^1 X$ modulo adding a class in $2\,\mathrm{Cl}^0 X$. By Proposition 4.2.4, the weights around each of the h_2 vertices add up to $q+1$. On the other hand, Theorem 7.2.1 tells us precisely which vertices occur as Φ_x-neighbours of the s_y's and with which weight. We count all weights around the s_y's:

- The vertex s_0 occurs with weight h_2.
- The vertex c_{z-x} occurs with weight h_2 if $z-x\in\mathrm{Cl}^0 X-\{0\}$ and $z-x\neq x-z$.
- The vertex c_{z-x} occurs with weight $h_2/2$ if $z-x\in\mathrm{Cl}^0 X-\{0\}$ and $z-x=x-z$.
- The vertex t_D occurs with weight h_2 if $D\in\mathrm{Cl}\,X'-\mathrm{Cl}\,X$ and $2D\notin\mathrm{Cl}\,X$.
- The vertex t_D occurs with weight $h_2/2$ if $D\in\mathrm{Cl}\,X'-\mathrm{Cl}\,X$ and $2D\in\mathrm{Cl}\,X$.

Since $c_{z-x}=c_{x-z}$, the sum of the weights of the c_{z-x}'s is $(h_2/2)(h-1)$. Since $t_D=t_{-D}$ and t_D depends only on the class of D modulo $\mathrm{Cl}\,X$, the sum of the weights of the t_D's is $(h_2/2)(h'-1)$. Adding up all these contributions gives

$$h_2(q+1) \;=\; h_2 + (h_2/2)(h-1) + (h_2/2)(h'-1)\,,$$

which implies the relation of the lemma. □

8.1.3 Remark. This result can also be obtained from the equality $\zeta_{F'}(s) = \zeta_F(s) L_F(\chi, s)$, where $\chi = |\ |^{\pi i/\ln q}$ is the quasi-character corresponding to F' by class field theory. For curves of genus 1, these function can be written out explicitly as

$$\frac{q^2T^4 + (hh' - q^2 - 1)T^2 + 1}{(1-T^2)(1-q^2T^2)} \quad = \quad \frac{qT^2 + (h-q-1)T + 1}{(1-T)(1-qT)} \cdot \frac{qT^2 - (h-q-1)T + 1}{(1+T)(1+qT)},$$

where $T = q^{-s}$. Comparing the coefficients of the numerators of these rational functions in T yields an alternative proof of the lemma.

8.1.4 Remark. Over an algebraically closed field, the 2-torsion of an elliptic curve is isomorphic to $\mathbf{Z}/2\mathbf{Z} \times \mathbf{Z}/2\mathbf{Z}$ if the characteristic is not 2 and it is either trivial or isomorphic to $\mathbf{Z}/2\mathbf{Z}$ if the characteristic is 2, depending on whether the curve is supersingular or not ([62, Cor. 6.4,Thm. 3.1]). Hence h_2 and h'_2 can be 1, 2 or 4, where the last case only occurs if q is odd.

The previous lemma implies that h is odd if and only if h' is odd. Since an abelian group has trivial 2-torsion precisely when its order is odd, this implies that $h_2 = 1$ if and only if $h'_2 = 1$.

In characteristic 2, we thus always have that $h'_2 = h_2$. It is not clear to me whether it can happen in odd characteristic that one of both h_2 and h'_2 is 2, while the other one is 4.

8.1.5 Let $f \in \mathcal{A}^K$ be an $\mathcal{H}_K$-eigenfunction with eigencharacter λ_f. For all $x \in |X|$, put $\lambda_x = \lambda_f(\Phi_x)$. The unramified automorphic form f can be seen as a function on the vertices of the graph of an unramified Hecke operator, so we can evaluate the eigenvalue equations

$$\Phi_x\, f \quad = \quad \lambda_x\, f$$

with help of Theorem 7.2.1 at each vertex for each place x of degree 1.

For every place x of degree 1 we obtain the following equations for the vertices in the nucleus of Φ_x. Note that the expressions in the right-most column are labels, which will be used for the purpose of reference.

$$\begin{aligned}
\lambda_x f(t_D) &= (q+1) f(s_{D+\sigma D+x}) && \text{for } D \in \mathrm{Cl}\, X' - \mathrm{Cl}\, X, && (x, t_D)\\
\lambda_x f(s_0) &= q f(s_x) + f(c_x)\,, && && (x, s_0)\\
\lambda_x f(c_0) &= (q+1) f(c_x)\,, && && (x, c_0)\\
\lambda_x f(c_{z-x}) &= (q-1) f(s_z) + f(c_z) + f(c_{2x-z}) && \text{for } z \in X(\mathbf{F}_q) - \{x\}, && (x, c_{z-x})\\
\lambda_x f(c_x) &= (q-1) f(s_0) + f(c_0) + f(c_{2x})\,, && && (x, c_x)\\
\lambda_x f(c_z) &= q f(c_{z-x}) + f(c_{z+x}) && \text{for } z \in X(\mathbf{F}_q) - \{x\}, && (x, c_z)\\
\lambda_x f(s_y) &= \alpha f(s_0) + (h_2/2) \sum_{\substack{(z-x)\in \mathrm{Cl}^0 X\\ (z-x)\neq 0\\ (z-y)\in 2\,\mathrm{Cl}^0 X}} f(c_{z-x}) + (h_2/2) \sum_{\substack{[D]\in \mathrm{Cl}\, X'/\mathrm{Cl}\, X\\ [D]\neq \mathrm{Cl}\, X\\ D-\sigma D+x-y\in 2\,\mathrm{Cl}^0 X}} f(t_D) && && (x, s_y)
\end{aligned}$$

for $y \in X(\mathbf{F}_q)$, where $\alpha = \begin{cases} h_2 & \text{if } (y-x) \in 2\,\mathrm{Cl}^0 X, \\ 0 & \text{if } (y-x) \notin 2\,\mathrm{Cl}^0 X. \end{cases}$

If we add up all the eigenvalue equations evaluated in the vertices s_y, where we let y range over all of $X(\mathbf{F}_q) = \mathrm{Cl}^1 X$, then we obtain that

$$\sum_{y \in X(\mathbf{F}_q)} \lambda_y f(s_y) \;=\; hf(s_0) + (h/2) \sum_{\substack{(z-x)\in \mathrm{Cl}^0 X \\ (z-x)\neq 0}} f(c_{z-x}) + (h/2) \sum_{\substack{[D]\in \mathrm{Cl}\, X'/\,\mathrm{Cl}\, X \\ [D]\neq \mathrm{Cl}\, X}} f(t_D)\,. \qquad (x, \textstyle\sum s_y)$$

8.2 The space of cusp forms

Let f be a $\mathcal{H}_K$-eigenfunction that is contained in $\mathcal{A}_0^K$. In particular, f is not trivial. We make no assumption of toroidality on f in this section. The cusp form f satisfies the eigenvalue equations of the previous paragraph and additionally $f(v) = 0$ if $\delta(v) \geq 1$. These equations make it possible to explicitly calculate the space of unramified cusp forms as functions on $\mathbf{P}\mathrm{Bun}_2 X$. We get the following result.

8.2.1 Theorem.

(i) *The dimension of* $\mathcal{A}_0^K$ *is* $r' + 1 - h_2$.

(ii) *The support of* $f \in \mathcal{A}_0^K$ *is contained in* $\{t_D, s_0, c_0\}_{D\in \mathrm{Cl}\, X' - \mathrm{Cl}\, X}$.

(iii) *If* x *is a place of odd degree, then* $\Phi_x(f) = 0$.

Proof. Observe that from Theorems 3.2.2 and 3.5.1 and Corollary 3.5.4, it follows that

$$\mathcal{A}_0^K \;=\; \bigoplus_{\lambda\in\mathbf{C}} \mathcal{A}_0(\Phi_x, \lambda)^K$$

for every place x, where both sides are finite dimensional complex vector spaces, thus in particular $\mathcal{A}_0(\Phi_x, \lambda)^K = 0$ for all but finitely many $\lambda \in \mathbf{C}$. Let $f \in \mathcal{A}_0^K$ be a $\mathcal{H}_K$-eigenfunction and x a place of degree 1. We first show that the eigenvalue λ_x of f under Φ_x equals 0.

Assume that $\lambda_x \neq 0$, then we conclude successively:

- $f(c_0) = 0$ by equation (x, c_0).
- $f(s_0) = 0$ by equation (x, c_x).
- $f(c_{z-x}) = 0$ for all places $z \neq x$ of degree 1 by equation (x, c_z).
- $f(s_y) = 0$ for all places y of degree 1 by equations (x, s_0) and (x, c_{z-x}).
- $f(t_D) = 0$ for all $D \in \mathrm{Cl}\, X' - \mathrm{Cl}\, X$ by equation (x, t_D).

Thus f must be trivial, which contradicts the fact that it is an $\mathcal{H}_K$-eigenfunction. This means that $\mathcal{A}_0(\Phi_x, \lambda)^K = 0$ if $\lambda \neq 0$. Therefore $\mathcal{A}_0^K = \mathcal{A}_0(\Phi_x, 0)^K$.

So we know that $\lambda_x = 0$ for all places x of degree 1. We make the following successive conclusions, where we always an appropriate place x of degree 1 in every step:

- $f(s_y) = 0$ for all places y of degree 1 by equation (x, t_D).
- $f(c_{z-x}) = 0$ for all places $z \neq x$ of degree 1 by equation (x, c_z).

- $f(c_0)+(q-1)f(s_0)=0$ by equation (x,c_x).
- $\alpha f(s_0)+(h_2/2)\sum_{\substack{[D]\in\operatorname{Cl}X'/\operatorname{Cl}X\\ [D]\neq\operatorname{Cl}X\\ D-\sigma D+x-y\in 2\operatorname{Cl}^0 X}} f(t_D)=0$ for all places y of degree 1 by equation (x,s_y), where $\alpha=h_2$ if $(y-x)\in 2\operatorname{Cl}^0 X$ and $\alpha=0$ otherwise.

This means that the support of f is contained in $\{t_D,s_0,c_0\}_{D\in\operatorname{Cl}X'-\operatorname{Cl}X}$, which proves (ii).

We have h_2+1 linearly independent equations for f. There are no further restrictions on the values of f given by the eigenvalue equations since equation (x,c_0) becomes trivial. Hence the dimension of $\mathcal{A}_0^K=\mathcal{A}_0(\Phi_x,0)^K$ equals

$$\#\{t_D,s_0,c_0\}_{D\in\operatorname{Cl}X'-\operatorname{Cl}X}-(h_2+1) \;=\; (r'+2)-(h_2+1) \;=\; r'+1-h_2\,,$$

which proves (i).

Assertion (iii) follows since the support of f contains only vertices v with $\delta(v)$ even and thus Lemma 5.4.2 implies that $\Phi_x(f)=0$ for every place x of odd degree. □

8.2.2 Remark. The dimension formula also follows from calculations with theta series, cf. Schleich [56, Satz 3.3.2] and Harder, Li and Weisinger [27, Thm. 5.1].

8.2.3 Proposition. *If $f\in\mathcal{A}_0^K$ is an $\mathcal{H}_K$-eigenfunction, then $f(c_0)\neq 0$.*

Proof. Let $f\in\mathcal{A}_0^K$ is an $\mathcal{H}_K$-eigenfunction with eigencharacter λ_f such that $f(c_0)=0$. We will deduce that f must be the zero function, which is not an $\mathcal{H}_K$-eigenfunction by definition. This will prove the lemma.

First we conclude from Theorem 8.2.1 and equation (x,c_x) that $f(s_0)=0$. The only other vertices that are possibly contained in the support of f are of the form t_D for a $D\in\operatorname{Cl}X'-\operatorname{Cl}X$. We fix an arbitrary $D\in\operatorname{Cl}X'-\operatorname{Cl}X$ for the rest of the proof.

Since $X'(\mathbf{F}_{q^2})=\operatorname{Cl}^1 X'$ maps surjectively to $\operatorname{Cl}X'/\operatorname{Cl}X$, and t_D only depends on the class $[D]\in\operatorname{Cl}X'/\operatorname{Cl}X$, there is a $z\in X'(\mathbf{F}_{q^2})$ such that $t_D=t_z$. The covering $p:X'\to X$ maps z as well as its conjugate σz to a place $y\in|X|$ of degree 2. As classes in $\operatorname{Cl}X'$, we have $y=z+\sigma z$.

In the following, we will investigate the graph of the Hecke operator Φ_y with the help of the graphs of the Hecke operators Φ_z and $\Phi_{\sigma z}$, which are defined over F'. Recall from Lemma 5.2.5 that the map $p^*:\mathbf{P}\mathrm{Bun}_2 X\to\mathbf{P}\mathrm{Bun}_2 X'$ restricts to an injective map

$$p^*:\;\mathbf{P}\mathrm{Bun}_2^{\mathrm{dec}} X\amalg\mathbf{P}\mathrm{Bun}_2^{\mathrm{tr}} X\;\hookrightarrow\;\mathbf{P}\mathrm{Bun}_2^{\mathrm{dec}} X'\,,$$

and p^* maps $\mathbf{P}\mathrm{Bun}_2^{\mathrm{gi}} X$ to $\mathbf{P}\mathrm{Bun}_2^{\mathrm{gi}} X'$. We will denote the elements in $\mathbf{P}\mathrm{Bun}_2^{\mathrm{dec}} X'$ by c'_D with $D\in\operatorname{Cl}X'$. Then we have in particular that $c'_0=p^*(c_0)$, that $c'_{z+\sigma z}=p^*(c_y)$ and that $c'_{z-\sigma z}=p^*(t_z)$, and in each case, there is no other vertex $\mathbf{P}\mathrm{Bun}_2 X$ that is mapped to c'_0, $c'_{z+\sigma z}$, and $c'_{z-\sigma z}$, respectively.

Recall from paragraph 5.1.8 that $\mathcal{K}_y$ denotes the sheaf on X whose stalks are trivial except for the one at y, which equals κ_y. If we denote by $\mathcal{K}_z$ and $\mathcal{K}_{\sigma z}$ the corresponding sheaves on X', we have that $p^*\mathcal{K}_y\simeq\mathcal{K}_z\oplus\mathcal{K}_{\sigma z}$.

Let $\mathcal{M},\mathcal{M}'\in\mathrm{Bun}_2 X$ fit into an exact sequence

$$0\longrightarrow\mathcal{M}'\longrightarrow\mathcal{M}\longrightarrow\mathcal{K}_y\longrightarrow 0\;.$$

Extension of constants is an exact functor, thus we obtain an exact sequence

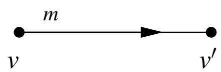
$$0 \longrightarrow p^*\mathcal{M}' \longrightarrow p^*\mathcal{M} \longrightarrow \mathcal{K}_z \oplus \mathcal{K}_{\sigma z} \longrightarrow 0 \ ,$$

which splits into two exact sequences

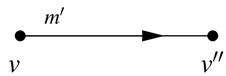
$$0 \to \mathcal{M}'' \to p^*\mathcal{M} \to \mathcal{K}_z \to 0 \qquad \text{and} \qquad 0 \to p^*\mathcal{M}' \to \mathcal{M}'' \to \mathcal{K}_{\sigma z} \to 0\,,$$

where $\mathcal{M}'' \in \operatorname{Bun}_2 X'$ is the kernel of $p^*\mathcal{M} \to \mathcal{K}_z$.

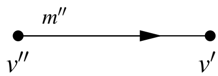
In the language of graphs, this means that for every edge



between vertices $v, v' \in \mathbf{P}\operatorname{Bun}_2 X$ in Edge $\mathcal{G}_y$, there are a vertex $v'' \in \mathbf{P}\operatorname{Bun}_2 X'$, and edges

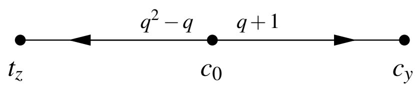


and



in Edge $\mathcal{G}_z$ and Edge $\mathcal{G}_{\sigma z}$, respectively.

We apply this observation to find out all possibilities of Φ_y-neighbours of c_0. The only Φ_z-neighbour of c_0 is c_z, and since $z \neq \sigma z$, the $\Phi_{\sigma z}$-neighbours of c_z are $c_{z-\sigma z} = p^*(t_z)$ and $c_{z+\sigma z} = p^*(c_y)$. This means that the only possible Φ_y-neighbours of c_0 are t_z and c_y. Theorem 5.4.6 says that c_y has multiplicity $q+1$. Thus, by Proposition 4.2.4, the neighbour t_z has multiplicity $(q^2+1)-(q+1) = q^2-q$, hence $\mathcal{U}_y(c_0)$ can be illustrated as



By the assumptions on f, it vanishes both at c_0 and at c_y. Thus the eigenvalue equation

$$\lambda_f(\Phi_y)\, f(c_0) \;=\; (q+1)\, f(c_y) + (q^2-q)\, f(t_z)$$

implies that $f(t_D) = f(t_z) = 0$, which completes the proof. □

8.3 The space of toroidal automorphic forms

Let $F' = \mathbf{F}_{q^2} F$ be the constant field extension of F and let $T' \subset G$ be a torus corresponding to F'. Let $p : X' \to X$ be the map of curves that corresponds to F'/F. Recall from Definition 1.5.13 that we defined an $f \in \mathcal{A}$ to be F'-toroidal if $f_{T'}(g) = 0$ for all $g \in G_{\mathbf{A}}$. Theorem 6.1.2 states that for an automorphic form $f \in \mathcal{A}_{\text{tor}}(F')^K$,

$$f(c_0) \;+ \sum_{\substack{[D]\in \operatorname{Cl} X'/\operatorname{Cl} X \\ [D]\neq \operatorname{Cl} X}} f(t_D) \;=\; 0\,. \qquad (T)$$

We will determine the space of unramified F'-toroidal automorphic forms in this section and draw conclusions about the space of unramified toroidal automorphic forms.

8.3.1 Theorem. *Let $F' = \mathbf{F}_{q^2}F$ be the constant field extension of F. Then the space of unramified F'-toroidal cusp forms is trivial.*

Proof. Since the support of unramified cusp forms is contained in $\mathbf{P}\mathrm{Bun}_2^{\mathrm{tr}} X \cup \{s_0, c_0\}$ (Theorem 8.2.1), after multiplying by $2/h$ equation $(x, \sum s_y)$ simplifies to

$$0 \;=\; 2\,f(s_0) \;+ \sum_{\substack{[D]\in \mathrm{Cl}\,X'/\,\mathrm{Cl}\,X \\ [D]\neq \mathrm{Cl}\,X}} f(t_D)\,.$$

Subtracting equation (T) from it yields

$$0 \;=\; 2\,f(s_0) \;-\; f(c_0)\,.$$

For cusp forms, equation (x, c_x) reads

$$0 \;=\; (q-1)\,f(s_0) + f(c_0)$$

and this implies that $f(c_0) = f(s_0) = 0$, thus by Proposition 8.2.3, there is no $\mathcal{H}_K$-eigenfunction in the $\mathcal{H}_K$-invariant space $(\mathcal{A}_0 \cap \mathcal{A}_{\mathrm{tor}})^K$, and f must be zero. □

8.3.2 Remark. If the analogue of Waldspurger's formula in [71, Prop. 7] for elliptic function fields is true, we deduce the following corollary: for every irreducible unramified cuspidal representation π over an elliptic function field, $L(\pi, 1/2) \neq 0$.

8.3.3 Theorem 6.2.11 puts the unramified F'-toroidal Eisenstein series in connection with the zeros of the zeta-function of F'. Let $\chi_{F'} = |\;|^{\pi\mathrm{i}/\ln q}$, then by class field theory (Lemma 2.2.10),

$$\zeta_{F'}(s) \quad=\quad \zeta_F(s)\cdot L_F(\chi_{F'}, s) \quad=\quad \zeta_F(s)\cdot \zeta_F\Big(s + \frac{\pi\mathrm{i}}{\ln q}\Big)\,,$$

where we regard s as an element in $\mathbf{C}\,/\,\frac{2\pi\mathrm{i}}{\ln q}\mathbf{Z}$.

For a curve of genus 1,

$$\zeta_F(s) \;=\; \frac{qT^2 + \big(h-(q+1)\big)T + 1}{(1-T)(1-qT)}\,,$$

where $T = q^{-s}$ ([55, Thm. 5.9]). This means that

$$\zeta_F(s) = 0 \qquad \text{if and only if} \qquad qT^2 + \big(h-(q+1)\big)T + 1 \;=\; 0\,.$$

8.3.4 Let s be a zero of ζ_F and recall the notion of a pair of zeros from paragraph 6.2.12. Then $\{s, 1-s\}$ is the only pair of zeros of ζ_F, which is of order 1, and $\{s - \frac{\pi\mathrm{i}}{\ln q}, 1-s+\frac{\pi\mathrm{i}}{\ln q}\}$ is the only pair of zeros of $L_F(\chi_{F'}, \cdot\,)$, which is also of order 1.

Note that F' is also of genus 1, but has larger constant field $\mathbf{F}_{q^2}$. This means that $\zeta_{F'}$ has only one pair of zeros modulo $\frac{\pi\mathrm{i}}{\ln q}\mathbf{Z}$, but it vanishes at both pairs of zeros $\{s, 1-s\}$ and $\{s - \frac{\pi\mathrm{i}}{\ln q}, 1-s+\frac{\pi\mathrm{i}}{\ln q}\}$ as function of $s \in \mathbf{C}\,/\,\frac{2\pi\mathrm{i}}{\ln q}\mathbf{Z}$.

8.3.5 Lemma. *The following are equivalent.*

(i) $\zeta_{F'}$ *has a pair of zeros of order* 2.

(ii) $\frac{1}{2}+\frac{\pi\mathrm{i}}{2\ln q}$ *is a zero of* ζ_F.

(iii) $h=q+1$.

Proof. Let s be a zero of ζ_F. Then (i) holds if and only if $1-s\equiv s+\frac{\pi\mathrm{i}}{\ln q} \pmod{\frac{2\pi\mathrm{i}}{\ln q}\mathbf{Z}}$, which is equivalent to (ii).

Put $s=\frac{1}{2}+\frac{\pi\mathrm{i}}{2\ln q}$ and $T=q^{-s}=\mathrm{i}q^{-1/2}$. Then

$$\zeta_F(s) \;=\; \frac{q\mathrm{i}^2q^{-1}+\big(h-(q+1)\big)\,\mathrm{i}q^{-1/2}+1}{(1-\mathrm{i}q^{-1/2})(1-\mathrm{i}q^{1/2})} \;=\; \big(h-(q+1)\big)\underbrace{\frac{\mathrm{i}q^{-1/2}}{(1-\mathrm{i}q^{-1/2})(1-\mathrm{i}q^{1/2})}}_{\neq 0}$$

is zero if and only if $h=q+1$, hence the equivalence of (ii) and (iii). □

8.3.6 Theorem. *Let* $s+1/2$ *be a zero of* ζ_F *and* $W\subset\Xi_0$ *be set of all* $\chi=\omega|\;|^{1/2}$ *such that* $\omega^2=1$, *but* $\omega|_{\mathrm{Cl}^0 X}\neq 1$. *If* $h\neq q+1$, *then* $\mathcal{A}_{\mathrm{tor}}(F')^K$ *is generated by*

$$\Big\{E(\,\cdot\,,|\;|^s),\,E(\,\cdot\,,|\;|^{s+\pi\mathrm{i}/\ln q}),\,R(\,\cdot\,,\chi)\Big\}_{\chi\in W}$$

and if $h=q+1$, *then* $\mathcal{A}_{\mathrm{tor}}(F')^K$ *is generated by*

$$\Big\{E(\,\cdot\,,|\;|^s),\,E^{(1)}(\,\cdot\,,|\;|^s),\,R(\,\cdot\,,\chi)\Big\}_{\chi\in W}.$$

In particular, $\dim\mathcal{A}_{\mathrm{tor}}(F')^K=2h_2$.

Proof. By Theorem 8.3.1, we have $\big(\mathcal{A}_{\mathrm{tor}}(F')\cap\mathcal{A}_0\big)^K=0$ and by Theorem 6.3.7, we have $\big(\mathcal{A}_{\mathrm{tor}}(F')\cap\mathcal{R}\big)^K=\{R(\,\cdot\,,\chi)\}_{\chi\in W}$. By definition, $\chi=\omega|\;|^{1/2}\in W$ if and only if ω is an unramified character that factors through $\mathrm{Cl}\,X\,/\,2\,\mathrm{Cl}\,X$ and that is nontrivial restricted to $\mathrm{Cl}^0 F$. As we have explained in the proof of Proposition 4.4.11 and in the beginning of section 4.5, $\mathrm{Cl}\,X\,/\,2\,\mathrm{Cl}\,X$ is a group of order $2h_2$. The character group of $\mathrm{Cl}\,X\,/\,2\,\mathrm{Cl}\,X$ is of the same order. There are two quadratic characters such that $\omega|_{\mathrm{Cl}^0 X}$ is trivial, namely, the trivial character and $|\;|^{\pi\mathrm{i}/\ln q}$. Consequently, the cardinality of W is $2h_2-2$.

By Theorem 2.2.8, the L-functions $L_{F'}(\chi,\,\cdot\,)$ are constant for quasi-characters χ of $\mathbf{A}_{F'}^\times$ that are not of the form $|\;|^s$. Hence the only toroidal Eisenstein series correspond to the pairs of zeros of $\zeta_{F'}$. These are $\{1/2+s,1/2-s\}$ and $\{1/2+s-\frac{\pi\mathrm{i}}{\ln q},1/2-s+\frac{\pi\mathrm{i}}{\ln q}\}$, and by the previous lemma, they are different and of order 1 if $h\neq q+1$ and they are equal and thus of order 2 if $h=q+1$. This determines $\big(\mathcal{A}_{\mathrm{tor}}(F')\cap\mathcal{E}\big)^K$ as indicated and proves $\dim\mathcal{A}_{\mathrm{tor}}(F')^K=2h_2$. □

8.3.7 Remark. In case $q=p^a$ for a prime $p\neq 2,3$ and an odd integer a, the curve X is isomorphic to a supersingular elliptic curve if and only if $h=q+1$, cf. [73, Thm. 4.1]. For these q, an elliptic curve with function field F is thus not supersingular if and only if the space of $\mathbf{F}_{q^2}F$-toroidal automorphic forms admits a basis of $\mathcal{H}_K$-eigenfunctions. However, for other q, there are supersingular elliptic curves with $h\neq q+1$.

8.3.8 Proposition. *Let $s+1/2$ be a zero of ζ_F. Then $\big(\mathcal{E}\cap\mathcal{A}_{\rm tor}^{\rm nr}(F\oplus F)\big)^K$ is 2-dimensional and generated by*

$$\big\{E(\,\cdot\,,|\,|^s),\, E^{(1)}(\,\cdot\,,|\,|^s)\big\}\,.$$

Proof. Let $T\subset G$ be the diagonal torus and $\chi\in\Xi_0$ such that $\chi^2\neq|\,|^{\pm1}$. By Theorem 6.2.8 (ii),

$$E_T(e,\chi)\;=\;\big(L(\chi,1/2)\big)^2\,.$$

The only pair of zeros of $L(\chi,1/2)$ is $\{|\,|^s,|\,|^{-s}\}$ and it is of order 1. Thus $\{|\,|^s,|\,|^{-s}\}$ is the only pair of zeros of $\big(L(\chi,1/2)\big)^2$ and it is of order 2. □

8.3.9 Let F be an elliptic function field with even class number. Then the class group has a nontrivial character χ_0 of order 2, which can be extended to a character χ of the divisor class group of order 2 by Proposition 2.1.6. Equivalently, χ is an unramified quasi-character of $\mathbf{A}^\times$ of order 2 that is trivial on $F^\times$ and whose kernel does not contain $\mathbf{A}_0^\times$.

By class field theory, there is an unramified quadratic field extension E/F such that $N_{E/F}(\mathbf{A}_E)=\ker\chi$. Let $\chi_E\in\Xi_0$ be the quasi-character that corresponds to E, cf. paragraph 6.2.10. Then $\chi=\chi_E$. Note that χ_E is not equal to $|\,|^s$ for any $s\in\mathbf{C}$, hence E is not the constant field extension, but a separable geometric field extension of F, i.e. the constant field of E equals the constant field of F.

8.3.10 Proposition. *Let F be an elliptic function field with even class number. Then there exist a separable geometric quadratic unramified field extension E/F. Let χ_E be the corresponding quasi-character. Let $s+1/2$ be a zero of ζ_F. Then $\big(\mathcal{E}\cap\mathcal{A}_{\rm tor}^{\rm nr}(E)\big)^K$ is 2-dimensional and generated by*

$$\big\{E(\,\cdot\,,|\,|^s),\, E(\,\cdot\,,\chi_E\,|\,|^s)\big\}\,.$$

Proof. If the class group of F is of even order, then there exists a separable geometric quadratic unramified field extension E/F as explained before.

By Corollary 6.2.4, $E_T(e,\chi)$ is E-toroidal if and only if χ is a zero of

$$L(\chi,1/2)L(\chi\chi_E,1/2)\,.$$

The only pair of zeros of $L(\chi,1/2)$ is $\{|\,|^s,|\,|^{-s}\}$ and it is of order 1. The only pair of zeros of $L(\chi\chi_E,1/2)$ is $\{\chi_E\,|\,|^s,\chi_E^{-1}\,|\,|^{-s}\}$ and it is of order 1. □

8.3.11 Theorem. *Let F be a elliptic function field with class number h and constants $\mathbf{F}_q$. Let $s+1/2$ be a zero of ζ_F.*

(i) *If either the characteristic of F is odd or $h\neq q+1$, then $\mathcal{A}_{\rm tor}^K$ is 1-dimensional and spanned by the Eisenstein series $E(\,\cdot\,,|\,|^s)$.*

(ii) *If the characteristic of F is 2 and $h=q+1$, then $\mathcal{A}_{\rm tor}^K$ is either 1-dimensional and spanned by $E(\,\cdot\,,|\,|^s)$ or 2-dimensional and spanned by $\big\{E(\,\cdot\,,|\,|^s),\,E^{(1)}(\,\cdot\,,|\,|^s)\big\}$.*

Proof. By Theorem 6.2.11, $\mathcal{A}_{\rm tor}^K$ contains $E(\,\cdot\,,|\,|^s)$. The space $\mathcal{A}_{\rm tor}^K$ is the intersection of the spaces $\mathcal{A}_{\rm tor}^K(E)$ for all quadratic separable algebra extensions E/F. From Theorem

6.3.8, we know that $\mathcal{A}_{\rm tor}$ does not contain residues of Eisenstein series. Theorem 8.3.6 describes $\mathcal{A}_{\rm tor}^K(F')$ for the constant field extension F', which narrows down the possibilities to a 1 or 2-dimensional space spanned by certain functions from the Eisenstein part. In particular, the description of these functions in Theorem 8.3.6 implies (ii).

If $h \neq q+1$, then $\zeta_{F'}$ has simple zeros by Lemma 8.3.5. Hence the intersection of $\mathcal{A}_{\rm tor}^K$ with $\big(\mathcal{E} \cap \mathcal{A}_{\rm tor}^{\rm nr}(F \oplus F)\big)^K$ as described in Proposition 8.3.8 is 1-dimensional and spanned by $E(\,\cdot\,, |\,|^s)$.

If, however, $h = q+1$, but the characteristic of F is odd, then $h = q+1$ is even. There is thus a separable geometric quadratic unramified field extension E/F. Since χ_T restricts to a nontrivial character on the class group, the intersection of $\mathcal{A}_{\rm tor}^K$ with $\big(\mathcal{E} \cap \mathcal{A}_{\rm tor}^{\rm nr}(E)\big)^K$ as described in Proposition 8.3.10 is 1-dimensional and spanned by $E(\,\cdot\,, |\,|^s)$. □

8.3.12 Corollary. *Let F be an elliptic function field with constant field $\mathbf{F}_q$ and class number h. If either the characteristic of F is not 2 or $h \neq q+1$, then there is for every $\chi \in \Xi_0$ and for every $s \in \mathbf{C}$ a quadratic character $\omega \in \Xi$ such that $L(\chi\omega, s) \neq 0$.*

Note that a proof of Conjecture 6.2.15 implies:

8.3.13 Conjecture. *Let F be an elliptic function field of genus 1 and $s + 1/2$ a zero of $\zeta_F(s)$. The space $\mathcal{A}_{\rm tor}^K$ is 1-dimensional and spanned by $E(\,\cdot\,, |\,|^s)$.*

8.3.14 Remark. The proof of the last theorem depends on many results from the theory for toroidal automorphic forms as developed in this thesis, including the proof of admissibility of $\mathcal{A}_{\rm tor}^{\rm nr}$. In the particular case that the class number is 1, however, it is possible to deduce the theorem comparatively quickly from results in the literature ([15]).

8.4 Impact on the Riemann hypothesis

As explained in Theorem 6.6.6, the analogue of the Riemann hypothesis for function fields of genus 1 curves over finite fields follows from narrowing down the possibilities for the eigenvalues of unramified toroidal $\mathcal{H}_K$-eigenfunctions under one Hecke operator Φ_x to lie in $[-2q_x^{1/2}, 2q_x^{1/2}]$. We will investigate the eigenvalue equations enriched by the toroidal condition and see how close we can come to this goal. For this, we will neither use the explicit form of the zeta function nor the decomposition theory of the space of automorphic forms. We only apply the theory of graphs of Hecke operators to the connection between zeros of the zeta function, toroidal Eisenstein series and unitarizable or tempered representations as explained in Chapter 6.

8.4.1 If $\mathcal{A}_{\rm tor}^K$ is trivial, then it does not contain any Eisenstein series and therefore the zeta function of F has no zero (Corollary 6.2.13). Then the analogue of the Riemann hypothesis holds for trivial reasons. Hence assume that $\mathcal{A}_{\rm tor}^K$ is not trivial. By finite-dimensionality (Theorem 6.1.8), $\mathcal{A}_{\rm tor}^K$ contains an $\mathcal{H}_K$-eigenfunction, which we denote by f. Let further $x \in |X|$ and $\Phi_x(f) = \lambda_x f$. Since 0 lies in $[-2q_x^{1/2}, 2q_x^{1/2}]$, it is no restriction to assume that $\lambda_x \neq 0$ for some $x \in X(\mathbf{F}_q)$.

In the following lemma we will refer to the eigenvalue equations from paragraph 8.1.5 and the begin of section 8.3. Let F'/F be the constant field extension and $p: X' \to X$ the corresponding map of curves.

8.4.2 Lemma. *If f is a toroidal $\mathcal{H}_K$-eigenfunction such that there exist a place x of degree 1 and a $\lambda \neq 0$ with $\Phi_x(f) = \lambda f$, then there are complex numbers a_0, a_1, b, f_0, f_0', f_1, where $b \neq 0$, and a character $\omega : \mathrm{Cl}\, X \to \{\pm 1\}$ such that for all $y, z \in X(\mathbf{F}_q)$ with $z \neq x$ and all $D \in \mathrm{Cl}\, X' - \mathrm{Cl}\, X$, the equalities*

$$f(t_D) = \omega(D + \sigma D)b \qquad f(s_y) = \omega(y)a_1 \qquad f(s_0) = a_0$$
$$f(c_0) = f_0 \qquad f(c_{z-x}) = \omega(z-x)f_0' \qquad f(c_x) = \omega(x)f_1$$
$$\lambda_z = \omega(z)\lambda$$

hold. The eigenvalue equations and the toroidal condition can be formulated as

$$\begin{aligned} \lambda b &= (q+1)a_1 && \text{by } (x, t_D), \\ \lambda a_0 &= q a_1 + f_1 && \text{by } (x, s_0), \\ \lambda f_0 &= (q+1) f_1 && \text{by } (x, c_0), \\ \lambda f_0' &= (q-1)a_1 + 2 f_1 && \text{by } (x, c_{z-x}), \\ f_0 + (h'-1)b &= 0 && \text{by } (T) \text{ if } \omega|_{\mathrm{Cl}^0 X} \text{ is trivial}, \\ f_0 - b &= 0 && \text{by } (T) \text{ if } \omega|_{\mathrm{Cl}^0 X} \text{ is not trivial}. \end{aligned}$$

Proof. Let f be a toroidal $\mathcal{H}_K$-eigenfunction, x a place of degree 1 and a $\lambda \neq 0$ such that $\Phi_x(f) = \lambda f$. First we reason that f cannot vanish at all vertices of the form t_D.

Assume that $f(t_D) = 0$ for all $D \in \mathrm{Cl}\, X' - \mathrm{Cl}\, X$, then we conclude successively:

- $f(c_0) = 0$ by equation (T).
- $f(s_y) = 0$ for all places y of degree 1 by equation (x, t_D).
- $f(c_x) = 0$ for all places x of degree 1 by equation (x, c_0).
- $f(s_0) = 0$ by equation (x, s_0).
- $f(c_{z-x}) = 0$ for all places $z \neq x$ of degree 1 by equation (x, c_{z-x}).

Hence f also vanishes on the cusps and is trivial. But an $\mathcal{H}_K$-eigenfunction is not trivial by definition. This is a contradiction.

Let $D \in \mathrm{Cl}\, X' - \mathrm{Cl}\, X$ such that $f(t_D) \neq 0$. Then equation (x, t_D) implies that we have $(q+1) f(s_{D-\sigma D+x}) = \lambda_x f(t_D) \neq 0$. For every $z \in X(\mathbf{F}_q)$ and $D' \in \mathrm{Cl}\, X' - \mathrm{Cl}\, X$ such that $D' - \sigma D' + z - D + \sigma D - x \in 2\,\mathrm{Cl}^0 X$,

$$\lambda_z f(t_{D'}) = (q+1) f(s_{D'-\sigma D'+z}) = (q+1) f(s_{D-\sigma D+x}) = \lambda_x f(t_D)\,.$$

In particular if we take $z = x$, then $f(t_{D'}) = f(t_D)$ if $D' - \sigma D' - D + \sigma D \in 2\,\mathrm{Cl}^0 X$ and if we take $D' = D$, then $\lambda_z = \lambda_x$ if $z - x \in 2\,\mathrm{Cl}^0 X$. By exchanging the roles of x and z, we obtain

$$\lambda_x f(t_{D'}) = (q+1) f(s_{D'+\sigma D'+x}) = (q+1) f(s_{D+\sigma D+z}) = \lambda_z f(t_D)$$

and by multiplying both equalities, we get

$$\lambda_x^2 f(t_D) f(t_{D'}) = \lambda_z^2 f(t_D) f(t_{D'}) ,$$

which means that $\lambda_z = \pm\lambda_x$, and thus by the previous equation, $f(t_{D'}) = \pm f(t_D)$. This holds for all $z \in X(\mathbf{F}_q)$ and $D \in \mathrm{Cl}\, X'$, since we can find a D' for every z and a z for every D' such that $D' - \sigma D' + z - D + \sigma D - x \in 2\,\mathrm{Cl}^0\, X$.

Define $\omega(z-x) := \lambda_z/\lambda_x$ to be the sign by which λ_z and λ_x differ. Note that ω is well-defined even if we vary x, since if $z'-x' = z-x$ and $D'-\sigma D'+z-D+\sigma D-x \in 2\,\mathrm{Cl}^0\, X$, then also $D'-\sigma D'+z'-D+\sigma D-x' \in 2\,\mathrm{Cl}^0\, X$ and thus $\lambda_{z'} f(t_{D'}) = \lambda_{x'} f(t_D)$, which implies that $\lambda_{z'}/\lambda_{x'} = \lambda_z/\lambda_x$. Clearly, ω is multiplicative, and if we put $\omega(x) = 1$ for some $x \in X(\mathbf{F}_q)$, we obtain a character $\omega : \mathrm{Cl}\, X \to \{\pm 1\}$.

This means that if we define $\lambda = \omega(x)\lambda_x$ for one $x \in X(\mathbf{F}_q)$, then we have $\lambda_x = \omega(x)\lambda$ for all $x \in X(\mathbf{F}_q)$ and if we define $b = \omega(D-\sigma D) f(t_D)$ for one $D \in \mathrm{Cl}\, X' - \mathrm{Cl}\, X$, then we have $f(t_D) = \omega(D-\sigma D)b$ for all $D \in \mathrm{Cl}\, X' - \mathrm{Cl}\, X$.

From the equations

$$\omega(D-\sigma D+x)\lambda b = \lambda_x f(t_D) = (q+1) f(s_{D-\sigma D+x})$$

it follows that if we define $a_1 = \omega(y) f(s_y)$ for one $y \in X(\mathbf{F}_q)$, then $f(s_y) = \omega(y)a_1$ for all $y \in X(\mathbf{F}_q)$. In this notation, equation (x, t_D) becomes $\lambda b = (q+1)a_1$.

Put $a_0 = f(s_0)$ and $f_0 = f(c_0)$. Equation (x, c_0) implies

$$\omega(x)\lambda f_0 = \lambda_x f(c_0) = (q+1) f(c_x)$$

for every $x \in X(\mathbf{F}_q)$, hence if we define $f_1 = \omega(x) f(c_x)$ for one $x \in X(\mathbf{F}_q)$, then we have $f(c_x) = \omega(x) f_1$ for all $x \in X(\mathbf{F}_q)$, and equation (x, c_0) becomes $\lambda f_0 = (q+1) f_1$.

For all $x, z \in X(\mathbf{F}_q)$ such that $x \neq z$, equation (x, c_{z-x}) implies

$$\begin{aligned} \omega(x)\lambda f(c_{z-x}) &= \lambda_x f(c_{z-x}) \\ &= (q-1) f(s_z) + f(c_z) + f(c_{2x-z}) = (q-1)\omega(z)a_1 + 2\omega(z) f_1 , \end{aligned}$$

hence if we put $f_0' = \omega(z-x) f(c_{z-x})$ for one choice of $z - x \in \mathrm{Cl}^0\, X - \{0\}$, then $f(c_{z-x}) = \omega(z-x) f_0'$ for all choices of $z-x \in \mathrm{Cl}^0\, X - \{0\}$, and equation (x, c_{z-x}) becomes $\lambda f_0' = (q-1)a_1 + 2f_1$.

Observe that if ω is trivial on $\mathrm{Cl}^0\, X$, then equation (T) becomes $f_0 + (h'-1)b = 0$, but if ω is not trivial on $\mathrm{Cl}^0\, X$, it becomes $f_0 - b = f_0 + (h'/2-1)b + (-1)(h'/2)b = 0$.

This proves everything. □

In the following discussion, we keep the notation of the lemma.

8.4.3 In the case that ω is trivial on $\mathrm{Cl}^0\, X$, we conclude successively that

- $f_0 = -(h'-1)b$ by equation (T).
- $a_1 = \frac{\lambda}{q+1} b$ by equation (x, t_D).
- $f_1 = -\frac{(h'-1)\lambda}{q+1} b$ by equation (x, c_0).
- $a_0 = \frac{q}{q+1} b - \frac{h'-1}{q+1} b$ by equation (x, s_0).

- $f'_0 = \frac{q-1}{q+1}b - 2\frac{h'-1}{q+1}b$ by equation (x, c_{z-x}).

So far, we did not touch equation $(x, \sum s_y)$. Since ω is trivial on $\mathrm{Cl}^0 X$, together with the preceding identities, this equation yields that

$$h\lambda \frac{\lambda}{q+1} b \;=\; h\lambda a_1 \underset{(x,\sum s_y)}{=} ha_0 + \frac{h}{2}(h-1) f'_0 + \frac{h}{2}(h'-1)b$$
$$= h\Big(\frac{q}{q+1}b - \frac{h'-1}{q+1}b\Big) + \frac{h}{2}(h-1)\Big(\frac{q-1}{q+1}b - 2\frac{h'-1}{q+1}b\Big) + \frac{h}{2}(h'-1)b$$

If we use that $h' = 2(q+1) - h$ (Lemma 8.1.2), divide through b (which is not zero by assumption) and reorganise the terms, we find out that

$$\lambda^2 \;=\; (q+1-h)^2 \,.$$

8.4.4 In the case that ω is not trivial on $\mathrm{Cl}^0 X$, we conclude successively that

- $f_0 = b$ by equation (T).
- $a_1 = \frac{\lambda}{q+1}b = f_1$ by equations (x, t_D) and (x, c_0).
- $a_0 = b$ by equation (x, s_0).
- $f'_0 = b$ by equation (x, c_{z-x}).
- $\lambda a_1 = (q+1)b$ by equation (x, s_y).

All these equations can hold only if

$$\lambda^2 \;= (q+1)^2 \,.$$

We summarise what we found out in this section about the eigenvalues of a toroidal $\mathcal{H}_K$-eigenfunction. Let $X(\mathbf{F}_q) = \{x_1, \dots, x_h\}$ the set of $\mathbf{F}_q$-rational points.

8.4.5 Proposition. *Let $f \in \mathcal{A}^K_{\mathrm{tor}}$ be an $\mathcal{H}_K$-eigenfunction with eigencharacter λ_f. Then there are only the following possibilities for the eigenvalues $\lambda_{x_1}, \dots, \lambda_{x_h}$.*

(i) $\lambda_{x_1} = \cdots = \lambda_{x_h} = 0$.

(ii) *There is a $\lambda \in \mathbf{C}^\times$ with $\lambda^2 = (q+1-h)^2$ and $\lambda_z = \lambda$ for every place z of degree* 1.

(iii) *There is a character $\omega : \mathrm{Cl}\, X \to \{\pm 1\}$ that is not trivial on $\mathrm{Cl}^0 X$ and a $\lambda \in \mathbf{C}^\times$ with $\lambda^2 = (q+1)^2$ such that $\lambda_z = \omega(z)\lambda$ for every place z of degree* 1.

8.4.6 We discuss which $f \in \mathcal{A}^K_{\mathrm{tor}}$ can have the eigenvalues as described in the proposition. Recall from paragraph 3.7.18 that given $\lambda_{x_1}, \dots, \lambda_{x_h}$, there is up to constant multiple at most one $\mathcal{H}_K$-eigenfunction $f \in \widetilde{\mathcal{E}}$ such that $\Phi_{x_i} f = \lambda_{x_i} f$ for all $i = 1, \dots, h$. Recall from Lemmas 3.3.2 and 3.4.2 that $\lambda_z(\chi) = q^{1/2}\big(\chi^{-1}(\pi_z) + \chi(\pi_z)\big)$ is the eigenvalue of $\tilde{E}(\,\cdot\,, \chi)$ under Φ_z for every $z \in |X|$ and $\chi \in \Xi_0$.

If $\lambda_{x_1} = \cdots = \lambda_{x_h} = 0$, then f is a linear combination of $E(\,\cdot\,, |\;|^{\pi i/2\ln q})$ and a cusp form, cf. Theorem 8.2.1 (iii). We showed in Theorem 8.3.1 that there are no toroidal cusp forms, so f could only be $E(\,\cdot\,, |\;|^{\pi i/2\ln q})$ (Theorem 3.6.3). We come back to this case in a moment, but remark that this compatible with the Riemann hypothesis since $0 \in [-2q^{1/2}, 2q^{1/2}]$.

Let now $\lambda \neq 0$ and $\omega : \mathrm{Cl}\, X \to \{\pm 1\}$ be a character such that $\lambda_z = \omega(z)\lambda$ for every place z of degree 1. If ω is not trivial on $\mathrm{Cl}^0 X$, then $|\lambda| = (q+1) \notin [-2q^{1/2}, 2q^{1/2}]$. We determine the automorphic forms that admit such eigenvalues. Let $\chi = \omega |\ |^{1/2}$. The eigenvalues $\lambda_z(\chi)$ of $R(\,\cdot\,,\chi)$ satisfy

$$\lambda_z(\chi) \;=\; q^{1/2}\big(\omega(z)q^{1/2} + \omega(z)q^{-1/2}\big) \;=\; \omega(z)(q+1)$$

and if we put $\chi_{F'} = |\ |^{\pi \mathrm{i}/\ln q}$, then the eigenvalues $\lambda_z(\chi\chi_{F'})$ of $R(\,\cdot\,,\chi\chi_{F'})$ satisfy

$$\lambda_z(\chi\chi_{F'}) \;=\; q^{1/2}\big(\omega(z)\,|\pi_z|^{\pi \mathrm{i}/\ln q}\, q^{-1/2} + \omega(z)\,|\pi_z|^{\pi \mathrm{i}/\ln q}\, q^{-1/2}\big) \;=\; -\omega(z)(q+1)\,,$$

Since there are only two possibilities for a sign, the eigenvalues $\lambda_{x_1}, \dots, \lambda_{x_h}$ determine exactly those two functions (up to a multiple). These residues are indeed F'-toroidal (Theorem 8.3.6), but we excluded them to be toroidal (Theorem 8.3.11), so this case does not obstruct the condition that $\lambda \in [-2q^{1/2}, 2q^{1/2}]$ for all λ such that there is a $f \in \mathcal{A}^K_{\mathrm{tor}}$ with $\Phi_x(f) = \lambda f$.

If, however, ω is trivial, then $\lambda_{x_1} = \cdots = \lambda_{x_h} = \pm(q+1-h)$. Since residues of Eisenstein series have eigenvalues $\pm(q+1)$, we only have to look for Eisenstein series $E(\,\cdot\,,\chi)$ such that $\lambda_z(\chi) = \lambda_z$ for all $z \in X(\mathbf{F}_q)$. First let $h \neq q+1$. Then $\sum_{z \in X(\mathbf{F}_q)} \lambda_z = h(q+1-h) \neq 0$, and Proposition 3.7.9 implies that we only have to consider quasi-characters χ of the form $|\ |^s$. Hence we search for solutions of

$$q^{1/2}(q^s + q^{-s}) = \pm(q+1-h) \qquad \Longleftrightarrow \qquad q^{-2s} \pm \big(h-(q+1)\big)q^{-1/2-s} + 1 = 0\,.$$

With the substitution $T = q^{-(1/2+s)}$, this can be rewritten as

$$\big(qT^2 + \big(h-(q+1)\big)T + 1\big)\,\big(qT^2 - \big(h-(q+1)\big)T + 1\big) \;=\; 0\,.$$

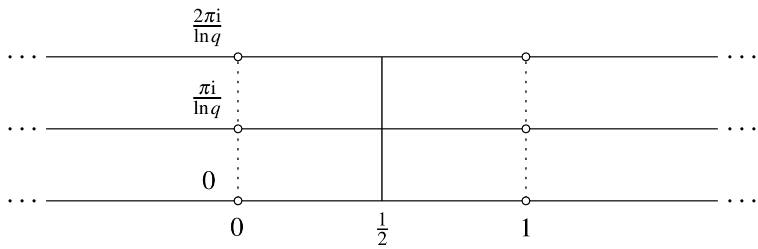
Note that by Theorem 6.2.3 we can conclude that the left hand side of the equation is a multiple of $\zeta_{F'}(1/2+s)$ without making use of the explicit form of $\zeta_{F'}$. Thus for every zero s_0 of $\zeta_{F'}$, the complex number $T^{-(1/2+s_0)}$ is a solution to that equation.

If $h = q+1$, then we are in the exceptional case that $\lambda_{x_1} = \cdots = \lambda_{x_h} = 0$, and the only Eisenstein series with these eigenvalues is $E(\,\cdot\,, |\ |^{\pi \mathrm{i}/2\ln q})$ (up to a multiple). We saw in Theorem 8.3.6, that this is precisely the case where a derivative of an Eisenstein series occurs, which substitutes the missing second solution.

8.4.7 The equation $\lambda^2 = (q+1-h)^2$ implies that $\lambda \in \mathbf{R}$. By Corollary 6.6.6, the question if we can deduce the Riemann-hypothesis for curves of genus 1 over a finite field depends on whether $\lambda = \pm(q+1-h) \in [-2q^{1/2}, 2q^{1/2}]$. In explicit cases, this is easy to check, but the general statement is an immediate corollary of Hasse's theorem, i.e. the Riemann hypothesis for elliptic function fields ([55, Prop. 5.11]).

Since our method to study the toroidal conditions relies on the structure of the graphs of Hecke operators, and this in turn depends on the class number, it should come as no surprise that we find that eigenvalue estimates become equivalent to class number estimates. This does appear to imply that our method of computing $\mathcal{A}^K_{\mathrm{tor}}$ cannot be used to give an alternative proof of Hasse's theorem. Nevertheless, we show the connection between different estimations for h, unitarizability and the possible solutions to the equation $q^{1/2}(q^s + q^{-s}) = \pm(q+1-h)$.

- **(The trivial estimate $h > 0$)**
This in general does **not** imply that $\mathcal{A}_{\mathrm{tor}}^{\mathrm{nr}}$ is **unitarizable**, cf. Lemma 6.6.4. The solutions s for varying $h > 0$—considered as a complex number modulo $(\frac{2\pi \mathrm{i}}{\ln q})$—are drawn as the solid line in the picture:



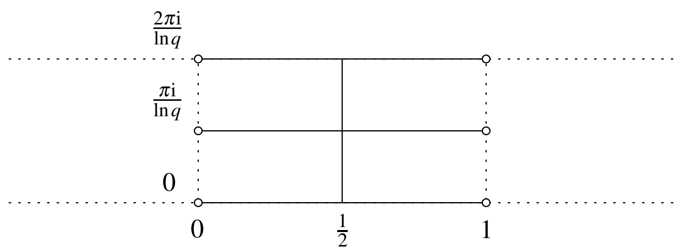
The circles on the solid lines indicate the values of those s such that $q^{1/2}(q^s + q^{-s})$ is the eigenvalue of a residuum of an Eisenstein series, which occur as a solution to the equation $q^{1/2}(q^s + q^{-s}) = \pm(q + 1 - h)$ if and only if $h = 0$ or $h = 2q + 2$.

- **(The estimate $0 < h < 2q + 2$ given by embedding X into $\mathbf{P}^2$)**
This estimate follows from the following. Every curve of genus 1 is given by a Weierstrass equation of the form

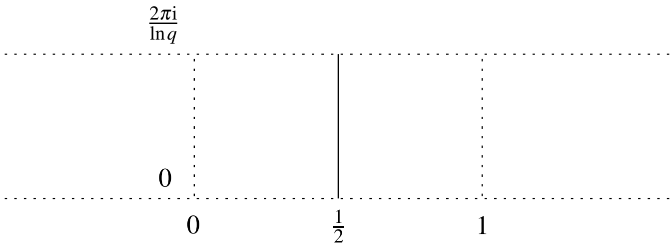
$$\underline{Y}^2 + a_1 \underline{X}\,\underline{Y} + a_3 \underline{Y} \;=\; \underline{X}^3 + a_2 \underline{X}^2 + a_4 \underline{X} + a_6 \,.$$

For every value for $\underline{X}$, there are at most two solutions in $\underline{Y}$ and there is an additional point at infinity. Hence the number h of $\mathbf{F}_q$-rational points of X satisfies the estimate $0 < h < 2q + 2$.

By Lemma 6.6.4 and paragraph 6.7.4, this estimate is equivalent to the fact that every irreducible subquotient of $\mathcal{A}_{\mathrm{tor}}^{\mathrm{nr}}$ is a **unitarizable** representation. The possible values for s are drawn as the solid line in the picture:

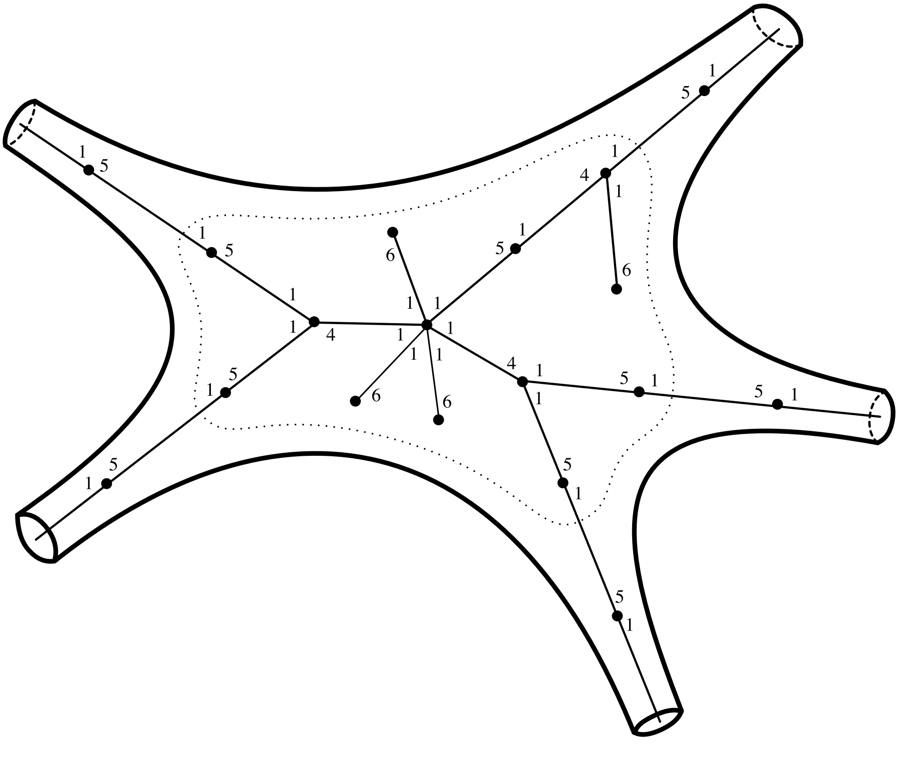


- **(The estimate $q + 1 - 2q^{1/2} \le h \le q + 1 + 2q^{1/2}$ from Hasse's theorem)**
The step from the previous estimate to this estimate is precisely what Hasse proved in [29]. The estimate is equivalent to the fact that every irreducible subquotient of $\mathcal{A}_{\mathrm{tor}}^{\mathrm{nr}}$ is a **tempered** representation (Lemma 6.6.3). The possible values for s are drawn as the solid line in the picture:



8.4.8 (Concluding remark) The calculations that lead to the conclusion that λ is a real number only involve the geometric interpretation of one toroidal condition and enough knowledge about the graphs of Hecke operators. We neither make use of the explicit form of the zeta function nor of our knowledge of the structure of the space of automorphic forms. This already shows a certain strength of the theory.

An estimate of the class number, which only uses the fact that every elliptic curve has a Weierstrass equation, restricts the zeros of the zeta function to the possibilities that correspond to unitarizable representations. This is the result whose analogue for **Q** would imply the validity of the Riemann hypothesis.



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Samenvatting

Klassieke automorfe vormen

Alvorens het begrip automorfe vorm voor functielichamen over een eindig lichaam in te voeren, zullen we in de eerste paragraaf eerst de klassieke notie van automorfe vorm herhalen. Voor de experts: in deze inleiding beperken we ons tot onvertakte automorfe vormen, en zullen dat niet steeds herhalen. Tenslotte merken we op dat de notatie in deze samenvatting niet volledig in overeenstemming is met die in de hoofdtekst.

Zij $\mathbf{H} = \{x + iy \in \mathbf{C} \mid x, y \in \mathbf{R}, y > 0\}$ het complexe bovenhalfvlak van Poincaré en $\mathrm{SL}_2\,\mathbf{Z}$ de group van geheeltallige twee-bij-twee matrices met determinant 1. Deze group werkt als groep van isometrieën voor de Poincaré-metriek op $\mathbf{H}$ door Möbiustransformaties als volgt:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}.z \;=\; \frac{az+b}{cz+d}.$$

Afbeelding 1 laat een fundamentaaldomein zien voor deze actie, alsook de bijbehorende quotiëntafbeelding. Op de gladde complexwaardige functies op $\mathbf{H}$ werkt de Laplace-Beltrami-operator $\Delta = -y^2\big(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\big)$.

Een gladde functie $f : \mathbf{H} \to \mathbf{C}$ wordt *automorfe vorm* genoemt als f voldoet aan

- f is invariant onder de actie van $\mathrm{SL}_2\,\mathbf{Z}$: $f(\gamma.z) = f(z)$ voor alle $\gamma \in \mathrm{SL}_2\,\mathbf{Z}$;
- f is van polynomiale groei: er bestaat een $n \in \mathbf{N}$ zodat $f(\mathrm{i}y) \in O(|y|^n)$;
- f is Δ-eindig: $\{\Delta^i f\}_{i\geq 0}$ brengt een eindig-dimensionale vectorruimte voort.

De ruimte van automorfe vormen wordt genoteerd als $\mathcal{A}_{\mathbf{Q}}$.

De Poincaré-metriek staat in een nauw verband met de Laplace-Beltrami-operator. Zij $z \in \mathbf{H}$ en $\mathbf{S}_\epsilon(z)$ de richtingsruimte van z. Zij z_s het unieke element op afstand 1 van z op een geodetische halflijn met beginpunt z en aanvangsrichting $s \in \mathbf{S}_\epsilon(z)$. Dan is er een meromorfe functie $c : \mathbf{C} \to \mathbf{C}$ zodat voor elke gladde eigenfunctie $f : \mathbf{H} \to \mathbf{C}$ van Δ (d.w.z. $\Delta f = \lambda f$ voor een $\lambda \in \mathbf{C}$) geldt dat

$$\lambda f(z) \;=\; c(\lambda) \int\limits_{\mathbf{S}_\epsilon(z)} f(z_s)\, ds\ .$$

Voor het begrip van de volgende paragrafen is het niet strikt noodzakelijk om te weten wat adeles zijn, maar we beschrijven wel even kort de omformulering van het quotiënt $\mathrm{SL}_2\,\mathbf{Z}\backslash\mathbf{H}$ in de taal van adeles $\mathbf{A_Q}$ over $\mathbf{Q}$ (voor een definitie zie 1.1.3). Zij Z het centrum van GL_2 en $K_{\mathbf{Q}} = O_2 \times \prod \mathrm{GL}_2\,\mathbf{Z}_p$, waarbij O_2 de orthogonale group in $\mathrm{GL}_2(\mathbf{R})$ is, $\mathbf{Z}_p$ de p-adische getallen zijn en het product over alle priemgetallen loopt. Als gevolg van sterke approximatie voor SL_2 bestaat er dan een homeomorfisme

$$\mathrm{SL}_2\,\mathbf{Z}\backslash\mathbf{H} \;\cong\; \mathrm{GL}_2\,\mathbf{Q}\,Z(\mathbf{A_Q})\backslash\,\mathrm{GL}_2\,\mathbf{A_Q}\,/\,K_{\mathbf{Q}}.$$

Automorfe vormen voor functielichamen

Zij $\mathbf{F}_q$ het lichaam met q elementen. Er bestaat een sterke analogie tussen $\mathbf{Q}$ en het rationale functielichaam $F = \mathbf{F}_q(t)$, d.w.z. het breukenlichaam van de ring van veeltermen over $\mathbf{F}_q$. Met name bestaat er het volgende woordenboek:

$\mathbf{Q}$	$F = \mathbf{F}_q(t)$
$\mathbf{Z}$	$\mathcal{O}_F = \mathbf{F}_q[t]$
$\lvert\,\rvert$	$\lvert\,\rvert_\infty : P/Q \mapsto q^{\deg Q - \deg P}$
$\mathbf{R}$	$F_\infty = \mathbf{F}_q((t^{-1}))$
n.v.t.	$\mathcal{O}_\infty = \mathbf{F}_q[[t^{-1}]]$
$\mathbf{H}$	$\mathcal{T}^{(0)} = \mathrm{GL}_2\,F_\infty\,/\,Z(F_\infty)\,\mathrm{GL}_2\,\mathcal{O}_\infty$
$\mathrm{SL}_2\,\mathbf{Z}$	$\mathrm{GL}_2\,\mathcal{O}_F$
$\mathbf{A_Q}$	$\mathbf{A} = \mathbf{A}_F$
$K_{\mathbf{Q}}$	K (zie 1.3.1)

Ook in het geval van een functielichaam impliceert sterke approximatie voor SL_2 het bestaan van een homeomorfisme

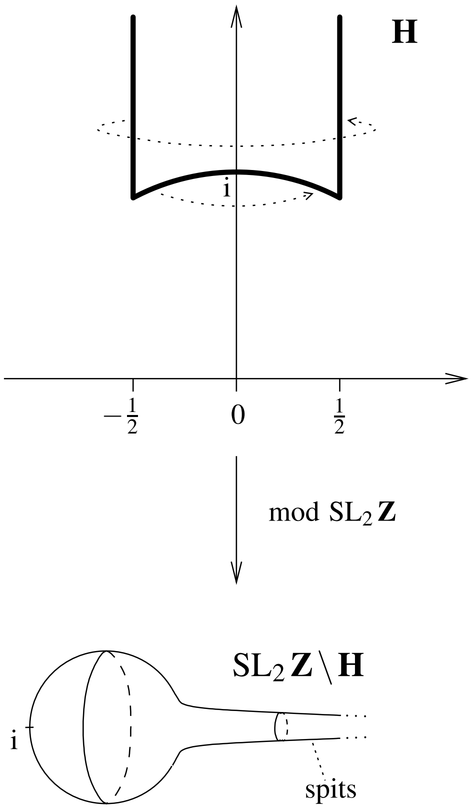
$$\mathrm{GL}_2\,\mathcal{O}_F\backslash\mathcal{T}^{(0)} \;\cong\; \mathrm{GL}_2\,F\,Z(\mathbf{A})\backslash\,\mathrm{GL}_2\,\mathbf{A}\,/\,K,$$

waarbij $\mathcal{T}^{(0)}$ als discrete verzameling bekeken wordt. Omdat $\mathcal{T}^{(0)}$ uit rechts-nevenklassen van matrices bestaat, is er een natuurlijke werking van $\mathrm{GL}_2\,\mathcal{O}_F$ door links-vermenigvuldigen van matrices.

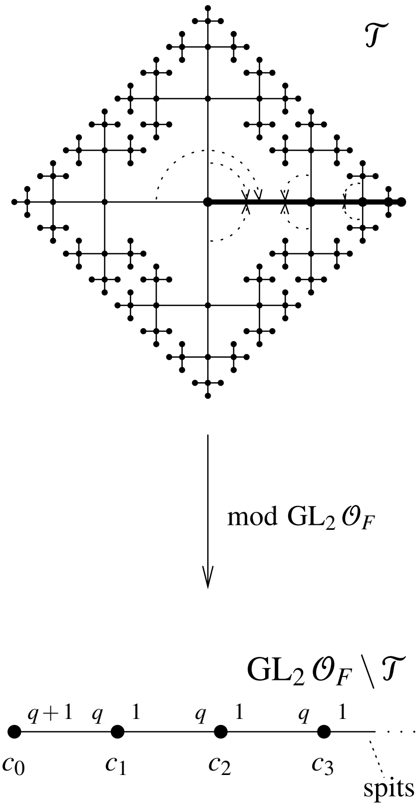
Een functie $f : \mathcal{T}^{(0)} \to \mathbf{C}$ wordt automorfe vorm genoemd als $f(\gamma.g) = f(g)$ voor alle $\gamma \in \mathrm{GL}_2\,\mathcal{O}_F$ en er een $n \in \mathbf{N}$ is zodat $f\Big(\big(\begin{smallmatrix} t^i & 0 \\ 0 & 1\end{smallmatrix}\big)\Big) \in O(\lvert t^i\rvert_\infty^n)$. De ruimte van automorfe vormen wordt met $\mathcal{A}$ genoteerd.

De rol van de Laplace-Beltrami-operator Δ wordt in de wereld van functielichamen overgenomen door de Hecke-operator Φ gedefineerd door

$$\Phi(f)(g) \;=\; f\Big(\underbrace{\big(\begin{smallmatrix} 1 & 0 \\ 0 & t^{-1}\end{smallmatrix}\big).g}_{=:g_\infty}\Big) + \sum_{b\in\mathbf{F}_q} f\Big(\underbrace{\big(\begin{smallmatrix} t^{-1} & b \\ 0 & 1\end{smallmatrix}\big).g}_{=:g_b}\Big)$$



Afbeelding 1: Het bovenhalfvlak van Poincaré



Afbeelding 2: De Bruhat-Tits-boom

voor $f \in \mathcal{A}$ en $g \in \mathcal{T}^{(0)}$. Geldt $\Phi f = \lambda f$ voor een $\lambda \in \mathbf{C}$, dan volgt uit de definitie van Φ dat

$$\lambda f(g) \;=\; \int_{\mathbf{P}^1(\mathbf{F}_q)} f(g_s)\,ds$$

als $\mathbf{P}^1(\mathbf{F}_q) = \mathbf{F}_q \cup \{\infty\}$ van de discrete maat voorzien wordt. Door deze formule te vergelijken met de corresponderende klassieke integraal wordt duidelijk dat g_s de punten 'op afstand 1' van g moeten zijn. Dit wordt gerealiseerd door een graaf $\mathcal{T}$ die $\mathcal{T}^{(0)}$ als knooppunten heeft, waarin g precies met de knooppunten g_s door een tak is verbonden. Deze graaf wordt ook de Bruhat-Tits-boom van $\mathrm{PGL}_2\,F_\infty$ genoemd en is inderdaad een boom. De actie van $\mathrm{GL}_2\,\mathcal{O}_F$ heeft een natuurlijke voortzetting op de boom.

Afbeelding 2 laat de analogie met het bovenhalfvlak van Poincaré zien. In de quotiëntafbeelding van de boom geven de getallen naast een knooppunt aan hoeveel takken uit de boom erop worden afgebeeld. Verder is $c_i \in \mathrm{GL}_2\,\mathcal{O}_F \backslash \mathcal{T}^{(0)}$ de klasse van de matrix $\left(\begin{smallmatrix} t^i & 0 \\ 0 & 1 \end{smallmatrix}\right)$.

Bewijs van de Riemannhypothese voor F

De zetafunctie voor F wordt gedefineerd door de formule

$$\zeta_F(s) \;=\; \frac{1}{1-q^{-s}} \quad\cdot \prod_{\substack{P\in\mathcal{O}_F\\ \text{monisch and irreducibel}}} \frac{1}{1-q^{-s\cdot\deg P}}$$

als $\operatorname{Re} s > 1$. De eerste faktor in het product komt overeen met de gamma-factor voor de Riemann zetafunctie. Ook voor eindige lichaamuitbreidingen E van F, zogenoemde *globale functielichamen*, is een zetafunctie ζ_E op een vergelijkbare manier gedefinieerd. Voor deze zetafuncties is de Riemannhypothese bewezen, d.w.z. dat alle nulpunten van ζ_E reëel deel $1/2$ hebben. Voor F een rationaal functielichaam is zelfs ζ_F de constante functie 1, maar de zetafunctie van een algemeen globaal functielichaam heeft wel nulpunten en voor deze is de geldigheid van de Riemannhypothese een diepe stelling van Hasse en Weil.

Een stelling van Erich Hecke geeft een verband tussen zetafuncties en een integraal over Eisensteinreeksen als functies op het bovenhalfvlak van Poincaré. De adelische vertaling geldt ook voor globale functielichamen:

Stelling (Hecke, 1959). *Voor elke $g \in \mathrm{GL}_2\,\mathbf{A}_E$ en elke over E gedefinieerde maximale anisotrope torus $T \subset \mathrm{GL}_2$ bestaat een holomorfe functie $e_{T,g} : \mathbf{C}-\{0,1\} \to \mathbf{C}$ zodat voor elke $s \in \mathbf{C}-\{0,1\}$,*

$$\int\limits_{T(F)Z(\mathbf{A})\backslash T(\mathbf{A})} E(s)(tg)\,dt \;=\; e_{T,g}(s)\cdot\zeta_F(s+1/2)$$

geldt waarbij $E(s)$ de Eisensteinreeks van gewicht s is.

Deze formulering stamt uit een artikel van Don Zagier uit 1979, die verder opmerkte dat voor elk nulpunt $s+1/2$ van de zetafunctie bovenstaande integraal van de Eisensteinreeks van gewicht s onafhankelijk van T en g verdwijnt. Hij concludeerde dat de Riemannhypothese voor E volgt als de ruimte $\mathcal{A}_{\mathrm{tor}}$ van automorfe vormen waarvoor bovenstaande integraal voor elke T en g verdwijnt een getemperde voorstelling is. Hij noemde de automorfe vormen in deze ruimte *toroïdaal*.

Hiermee komen we nu aan bij de inhoud van dit proefschrift. We bestuderen ruimten van toroïdale automorfe vormen voor functie lichamen.

We illustreren eerst onze methode met het het ‘triviale’ geval van een rationaal functielichaam $F = \mathbf{F}_q(t)$.

Allereerst merken we op dat $\mathcal{A}_{\mathrm{tor}}$ invariant is onder de actie van Hecke-operatoren.

Door de interpretatie van $\mathcal{T}^{(0)}$ als verzameling van isomorfieklassen van rang-twee vectorbundels over de projectieve lijn over $\mathbf{F}_q$ en door de interpretatie van een van de tori als sporen van lijnbundels over de projectieve lijn over $\mathbf{F}_{q^2}$ kan de integraal in dit geval berekend worden als som van functiewaarden in de hoekpunten van het quotiënt van de boom. Op die manier kunnen we aantonen dat als $f \in \mathcal{A}_{\mathrm{tor}}$, dan $f(c_0) = 0$, bekeken als functie op $\mathrm{GL}_2\,\mathcal{O}_F \backslash \mathcal{T}^{(0)}$.

Nu is genoeg bekend om een nieuwe bewijs voor de Riemannhypothese voor ζ_F te geven. Alhoewel het hier een soort “met een kanon op een mug schieten” betreft, is het opmerkelijke van dit bewijs dat het geen gebruik maakt van de expliciete vorm van de zetafunctie.

Stelling. *De ruimte van toroïdale automorfe vormen voor een rationaal functielichaam is triviaal:* $\mathcal{A}_{\mathrm{tor}} = \{0\}$. *Bijgevolg heeft* ζ_F *geen nulpunten.*

Bewijs. Het tweede deel van de stelling volgt uit het eerste deel door gebruik te maken van de stelling van Hecke. Voor het bewijs van het eerste deel: zij $f \in \mathcal{A}_{\mathrm{tor}}$, dus $f(c_0) = 0$. Dan zijn ook alle $\Phi^i(f)$ toroïdaal, dus $\Phi^i(f)(c_0) = 0$. Uit de gewichten van $\mathrm{GL}_2\,\mathcal{O}_F \setminus \mathcal{T}$ kan men deze termen berekenen en inductief de conclusie trekken dat

$$\begin{array}{lllll} 0 &=& \Phi(f)(c_0) &=& (q+1)f(c_1) & \Rightarrow & f(c_1)=0 \\ 0 &=& \Phi^2(f)(c_0) &=& (q+1)f(c_2)+q(q+1)f(c_0) & \Rightarrow & f(c_2)=0 \\ && \vdots &&&& \vdots \\ 0 &=& \Phi^i(f)(c_0) &=& (q+1)f(c_i)+\text{“lagere termen”} & \Rightarrow & f(c_i)=0 \end{array}$$

Dus moet f de nulfunctie zijn. □

Bovenstaande theorie werkt niet alleen voor F, maar voor elk globaal functielichaam E. In het hoofddeel van dit proefschrift wordt de ruimte van toroïdale automorfe vormen voor een dergelijk algemeen functielichaam E onderzocht. Als g het geslacht en h het klassengetal van E zijn en k de dimensie van de ruimte van spitsenvormen is, dan bewijzen we de afschatting

$$g+(h-1)(g-1) \;\leq\; \dim \mathcal{A}_{\mathrm{tor}} \;\leq\; 2g+2(h-1)(g-1)+k\ .$$

Voor het bewijs wordt eerst een algemene theorie van ‘grafen van Hecke-operatoren’ ontwikkeld, die de theorie van Serre over het quotiënt van de Bruhat-Tits-boom veralgemeniseert. Vervolgens wordt gebruik gemaakt van de interpretatie van de hoekpunten van de boom als isomorfiekassen van rang-twee vectorbundels op de kromme die bij E hoort, om een structuurtheorie voor deze grafen op te stellen.

In het geval van een elliptisch functielichaam E (d.w.z. $g = 1$ en E heeft een plaats van graad 1) tonen we aan dat de ruimte van toroïdale automorfe vormen precies van dimensie 1 is, en wordt opgespannen door een Eisensteinreeks van gewicht een nulpunt van de zetafunctie van E (het bewijs werkt niet als $q = 2$ en $h = q+1$).

We leiden ook uit werk van Hasse, Weil en Drinfeld af dat de irreducibele quotiënten van de voorstellingsruimte van toroïdale vormen getemperd zijn. Momenteel is niet duidelijk hoe omgekeerd de stelling van Hasse en Weil (equivalent van de Riemannhypothese voor E) kan worden afgeleid uit de hier ontwikkelde algemene theorie van toroïdale automorfe vormen.

Zusammenfassung

„Die ganzen Zahlen hat der liebe Gott gemacht, alles andere ist Menschenwerk.“

Leopold Kronecker, 1886

Zahlentheorie

Kroneckers Ausspruch war ein Bekenntnis zum mathematischen Konstruktivismus ([75, S. 19]) – er wollte die Mathematik auf die Arithmetik der ganzen Zahlen zurückführen, alle mathematischen Aussagen sollten in endlich vielen logischen Schlüssen nachvollziehbar sein. Obwohl heutzutage der Beweis durch Widerspruch in der Mathematik weitgehend akzeptiert ist und die Frage, ob ein solcher Beweis durch eine Konstruktion ersetzt werden kann, als eine philosophische betrachtet wird, nähren die Probleme um die Arithmetik der ganzen Zahlen einen blühenden Zweig der Mathematik: die Zahlentheorie.

Um den Keim der Komplexität in der Arithmetik aufzuspüren, lohnt es sich einen Schritt zurück zu tun und nur die positiven ganzen Zahlen

$$1,\ 2,\ 3,\ 4,\ 5,\ 6,\ 7,\ \ldots$$

zu betrachten. Diese sind mit zwei natürlichen Operationen ausgestattet: Der Addition und der Multiplikation. Bezüglich der Addition hat jede positive ganze Zahl n eine eindeutige Darstellung als Summe von Einsen:

$$n \;=\; \underbrace{1+\cdots+1}_{n\text{-mal}}.$$

Die Bausteine der Multiplikation sind Primzahlen

$$2,\ 3,\ 5,\ 7,\ 11,\ 13,\ 17,\ 19,\ 23,\ 29,\ 31,\ 37,\ 41,\ 43,\ 47,\ 53,\ldots,$$

die dadurch charakterisiert sind, daß sie ungleich 1 sind und nur durch 1 und sich selbst teilbar sind. Jede positive ganze Zahl n besitzt eine eindeutige Primfaktorzerlegung, was bedeutet, daß es eine Darstellung von n als Produkt

$$n \;=\; p_1 \cdot \cdots \cdot p_r$$

von Primzahlen $p_1, \dots, p_r$ gibt, welche bis auf Reihenfolge eindeutig bestimmt sind. Die Menge aller erdenklichen Kombinationen von Produkten von Primzahlen

$$2, \quad 3, \quad 2\cdot 2, \quad 5, \quad 2\cdot 3, \quad 7, \quad 2\cdot 2\cdot 2, \quad 3\cdot 3, \quad 2\cdot 5, \quad 11, \quad \dots$$

entspricht genau der Menge der positiven ganzen Zahlen. Multiplikation ist somit in nur einem Aspekt komplizierter als Addition: Anstatt eines Bausteins gibt es unendlich viele.

Der Ursprung der Arithmetik liegt in der Kombination von Addition und Multiplikation, die es erlaubt, Fragen unbegrenzter Schwierigkeit zu stellen, und deren Gesetzmäßigkeiten von beliebiger Tiefe scheinen. Soweit die ältesten Aufzeichnungen zurückreichen, gibt es Zahlenmystiker, die arithmetische Gleichungen wie

$$2^3 + 1 = 3^2, \qquad 3^2 + 4^2 = 5^2 \qquad \text{oder} \qquad 12^3 + 1^3 = 10^3 + 9^3,$$

finden und Zahlentheoretiker, die die Strukturen solcher Gleichungen untersuchen. Ein Beispiel bilden die Gleichungen der Form

$$1 + 2 + \ldots + m = 1^2 + 2^2 + \ldots + n^2$$

für positive ganze Zahlen m und n. Ausprobieren (im Zweifel mit Hilfe eines Computers) liefert die Lösungen

$$m = 1 \text{ und } n = 1, \quad m = 10 \text{ und } n = 5, \quad m = 13 \text{ und } n = 6, \quad m = 645 \text{ und } n = 85.$$

Tatsächlich sind dies bereits alle Lösungen. Der Beweis ([40]) führt allerdings über den Rahmen dieser Zusammenfassung hinaus. Für eine populärwissenschaftliche Darstellung siehe [14].

Die Riemannschen Vermutung

Sei s größer als 1, dann nähern sich die unendliche Summe

$$\frac{1}{1^s} + \frac{1}{2^s} + \frac{1}{3^s} + \cdots + \frac{1}{n^s} + \cdots$$

(n durchläuft hier alle positiven ganzen Zahlen) und das unendliche Produkt

$$\frac{1}{1-\frac{1}{2^s}} \cdot \frac{1}{1-\frac{1}{3^s}} \cdot \frac{1}{1-\frac{1}{5^s}} \cdot \cdots \cdot \frac{1}{1-\frac{1}{p^s}} \cdot \cdots$$

(p durchläuft hier alle Primzahlen) einem wohlbestimmten Wert an in dem Sinne, wie sich $0,99999\ldots$ der Zahl 1 annähert, und dieser Wert ist für beide Ausdrücke derselbe. Dies folgt im Wesentlichen aus der eindeutigen Primfaktorzerlegung.

Die Riemannsche Zetafunktion ζ ist eine „meromorphe“ Funktion von den „komplexen“ Zahlen in die komplexen Zahlen, welche für komplexe Zahlen s, deren „Realteil“ größer als 1 ist, den soeben beschriebenen Wert annimmt. Die sogenannten trivialen Nullstellen sind Nullstellen von ζ in allen geraden negativen ganzen Zahlen. Die Riemannsche Vermutung besagt, daß alle weiteren Nullstellen Realteil $\frac{1}{2}$ haben.

Die Riemannsche Vermutung wurde von Bernard Riemann ([54]) im Jahre 1859 formuliert. Ihre Gültigkeit würde die erstaunlich regelmäßige Verteilung von Primzahlen in den natürlichen Zahlen erklären (siehe Abschnitt 6.5). Sie hat zahlreiche Umformulierungen, Folgerungen und implizierende Bedingungen in verschiedensten Gebieten der Mathematik gefunden, ihre Gültigkeit ist aber bis heute eine offene Frage.

Die ganzen Zahlen bilden nicht den einzigen Zahlenbereich, der für die Zahlentheorie interessant ist, sondern es gibt eine Vielfalt anderer Zahlenbereiche, die mit einer Addition und einer Multiplikation ausgestattet sind und deren arithmetische Eigenschaften eine weitreichende Ähnlichkeit mit denen der ganzen Zahlen aufweisen. So lassen sich auch für gewisse Zahlenbereiche Zetafunktionen definieren und eine Riemannsche Vermutung formulieren.

So tief die Gültigkeit der Riemannschen Vermutung mit der Arithmetik des Zahlenbereichs verwoben ist, so vielfältig sind die Herangehensweisen an einen Beweis. Wie oben erwähnt, hatte allerdings noch keine Methode Erfolg im Fall der Riemannschen Zetafunktion ζ. In dieser Doktorarbeit wird ein Ansatz verfolgt, der Ende der siebziger Jahre durch Don Zagier formuliert wurde. Im Falle einiger Zahlenbereiche, für die die Riemannsche Vermutung schon gezeigt wurde, gelingt ein neuer Beweis.

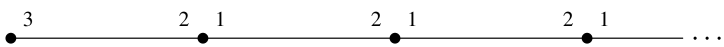
Graphen

Die mathematischen Konzepte und Methoden, die sich um arithmetische Fragen ranken, sind oft von hoher Abstraktheit, da sich gewisse Gesetze der Arithmetik einer naiven Betrachtungsweise verschließen. Zur besseren Handhabung abstrakter Begriffe werden ihnen oft einfachere Objekte zugeordnet, die sich darauf beschränken, die entscheidenden Eigenschaften für den jeweiligen Zweck widerzuspiegeln. In einigen Situationen reicht es aus, dem abstrakten Begriff eine Zahl oder eine Reihe von Zahlen zuzuordnen, in anderen ist es nötig, mehr Charakteristika zu wahren.

Beliebte Kandidaten mit mehr Möglichkeiten sind sogenannte Graphen. Ein Graph besteht im Wesentlichen aus einer Menge von Punkten, sogenannten Knoten, und einer Menge von Verbindungslinien zwischen diesen Knoten, sogenannten Kanten. Ein Grund für ihre Beliebtheit ist die Möglichkeit, sie auf ein Stück Papier zu zeichnen.

Je nach Bedarf werden die Knoten und Kanten mit gewissen Dekorationen versehen: Sie werden eingefärbt, gerichtet und gewichtet oder es werden ihnen selbst wieder abstrakte Objekte zugeordnet. In der Literatur finden sich Cayleygraphen, Kinderzeichnungen, Feynmangraphen und Bruhat-Tits-Bäume als Vereinfachung komplizierter Objekte.

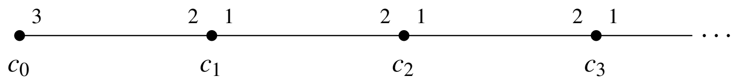
In Kapitel 4 dieser Arbeit wird der Graph eines Heckeoperators eingeführt. Dieser ist im „unverzweigten“ Fall, der in dieser Arbeit fast ausschließlich betrachtet wird, ein Graph, dessen Kanten zwei Zahlen dekorieren, eine an jedem Ende der Kante. Der einfachste Graph eines Heckeoperatoren, der vorkommt, sieht wie folgt aus:



Der Graph hat unendlich viele Knoten und Kanten, setzt sich aber nach rechts vollkommen regelmäßig fort. Die Zahlen kodieren die Wirkung des betrachteten Heckeoperators auf eine Weise, die im folgenden Abschnitt demonstriert werden soll.

Beweis der Riemannschen Vermutung in einem Beispiel

In diesem Abschnitt wird der Beweis der Riemannschen Vermutung nach der in dieser Arbeit benutzten Methode am Beispiel des Zahlenbereichs „$\mathbf{F}_2[t]$“ demonstriert. Der Graph eines gewissen Heckeoperators, der mit Φ bezeichnet werden soll, ist der im vorherigen Abschnitt vorgestellte. Der Handhabbarkeit halber indizieren wir die Knoten mit c_0, c_1, c_2 und so weiter:



Es werden ein paar mathematische Begriffe benötigt, die auf möglichst einfache Weise eingeführt werden.

Eine *automorphe Form* ist eine Vorschrift f, die jedem Knoten c_i des Graphen eine komplexe Zahl $f(c_i)$ zuordnet und die eine gewisse Wachstumsbedingung erfüllt, die jedoch für den weiteren Beweis ohne Bedeutung ist und auf die deswegen nicht weiter eingegangen werden soll.

Der Heckeoperator Φ bildet eine automorphe Form f auf die automorphe Form f^* ab, die sich dadurch definiert, daß

$$f^*(c_0) = 3 \cdot f(c_1), \tag{1}$$

$$f^*(c_1) = 1 \cdot f(c_2) + 2 \cdot f(c_0), \tag{2}$$

$$f^*(c_2) = 1 \cdot f(c_3) + 2 \cdot f(c_1) \quad \text{und so weiter} \tag{3}$$

gilt. Genau diese Gleichungen werden durch die Kanten und ihre Zahlen im obigen Graphen dargestellt.

Don Zagier definierte in einer Arbeit von 1979, unter welchen Bedingungen eine automorphe Form *toroidal* heißt. Für diesen Beweis ist interessant, daß eine toroidale automorphe Form f zwei Eigenschaften hat: $f(c_0) = 0$, und f^* ist toroidal.

Der Zusammenhang mit der Zetafunktion „$\zeta_{\mathbf{F}_2[t]}$“ von $\mathbf{F}_2[t]$ wird durch eine Formel von Erich Hecke von 1959 gegeben. Diese Formel impliziert, daß es zu jeder Nullstelle von $\zeta_{\mathbf{F}_2[t]}$ eine toroidale automorphe Form f gibt, die mindestens einem Knoten c_n eine Zahl $f(c_n)$ ungleich 0 zuordnet.

Die folgende Überlegung zeigt, daß es keine solche toroidale automorphe Form geben kann. Daraus folgt, daß $\zeta_{\mathbf{F}_2[t]}$ keine Nullstelle haben kann und somit die Riemannsche Vermutung für $\mathbf{F}_2[t]$ gilt.

Sei also f eine toroidale automorphe Form. Dann ist $f(c_0) = 0$, und f^*, definiert wie oben, ist toroidal. Folglich gilt auch $f^*(c_0) = 0$ und – zufolge Gleichung (1) – $f(c_1) = 0$.

Zusammenfassend wurde im letzten Absatz für eine toroidale automorphe Form f bewiesen, daß neben $f(c_0) = 0$ auch $f(c_1) = 0$ gilt. Da auch f^* toroidal ist, gilt ebenso $f^*(c_1) = 0$. Aus Gleichung (2) folgt nun, daß $f(c_2) = 0$ ist.

Das Argument aus dem vorherigen Absatz läßt sich beliebig oft wiederholen, und somit gilt für jede positive Zahl n, daß $f(c_n) = 0$. Damit ist die Riemansche Vermutung für $\mathbf{F}_2[t]$ bewiesen.

Acknowledgement

Mathematical

A thesis, as a book in general, gathers the work of many people, but only the person that puts down the words on paper is honored as author. Here I want to thank all those that took part in my study journey during the last four years that finally led to the present thesis.

I would like to thank my advisor Gunther Cornelissen who guided and helped me in all aspects of my studies: mathematical contents, organisation of research, culture of the mathematical society up to bureaucratic and computer problems.

I would like to thank Don Zagier for asking the questions that formed the starting point for my research and I thank him for taking the time to share his thoughts with me. I would like to thank Günter Harder for explaining his ideas on the subject to me. I would like to thank Roelof Bruggeman and Frits Beukers for their comments on many lectures and drafts that formed the blueprint of my thesis. I would like to thank Bas Edixhoven for organising various seminars that were important for my learning of the theory of automorphic forms.

I would like to thank Gerard Laumon, Laurent Clozel and Jean-Loup Waldspurger for their hospitality and mathematical help during a fruitful month in Paris.

I would like to thank my fellow PhD students in arithmetic geometry for many discussions, namely, Sander Dahmen, Jakub Byszewski and Jeroen Sijsling. I would like to thank Johnny Edwards for correcting my English, Arthur van Dam for his help with LaTeX, Tammo-Jan Dijkema for the realisation of the cover illustration and Robert Kremser for his ideas on future projects.

Privat

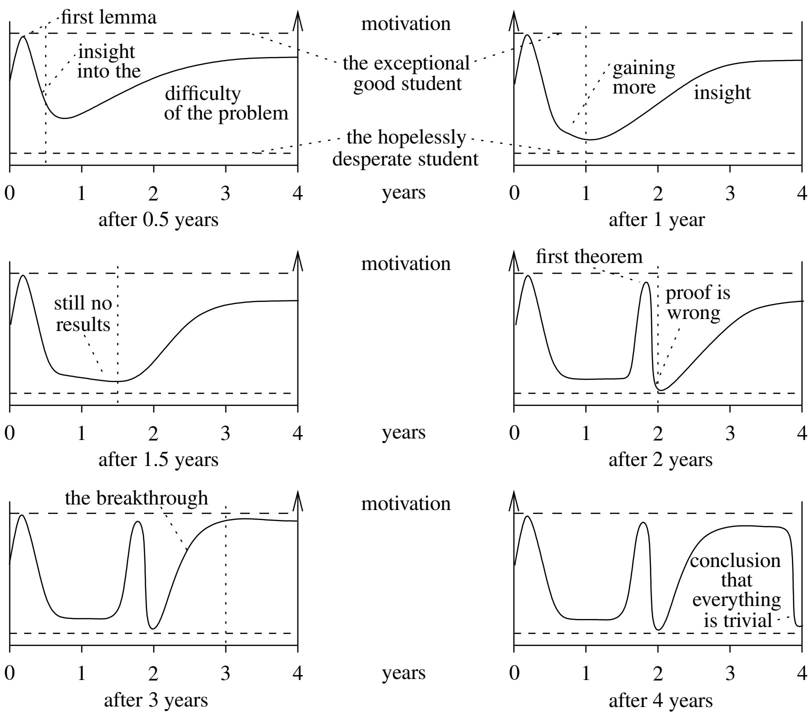
Ebensowenig wie eine Doktorarbeit das Werk eines einzelnen Mathematikers ist, ist sie das alleinige Werk von Mathematikern. Besonders in den letzten Wochen des intensiven Aufschreibens habe ich die Tragkraft der Menschen um mich gespürt, die mich beim Durchleben eigentümlicher geistiger Zustände mathematischer Konzentration an die Existenz menschlicher Grundwerte erinnerten.

Am wirkungsvollsten konnte dies stets Martha Klotzek, der ich für ihren seelischen Beistand herzlich danken möchte.

Meinen Eltern und meiner Schwester danke ich dafür, daß sie meine Familie sind, und ich keinen Zweifel hege, daß sie es bleiben.

Ich danke den liebgewonnenen Personen, die mit mir die schönen Momente des Lebens teilten und teilen: Lars Dittmar, Susanne Röhrig, Mark Broich, Marek Hebest, Yip Katerbau und Florian Hinz. Ebenso danke ich allen meinen weiteren Verwandten und Freunden.

Nicht unerwähnt darf bleiben, daß es auch unter Mathematikern ein soziales Leben gibt. In diesem Sinne möchte ich allen meinen Kollegen in Utrecht und anderswo danken und nicht zuletzt Gunther Cornelissen für die prognostizierten Motivationskurven, die zeitweilige Stimmungstiefs relativierten:



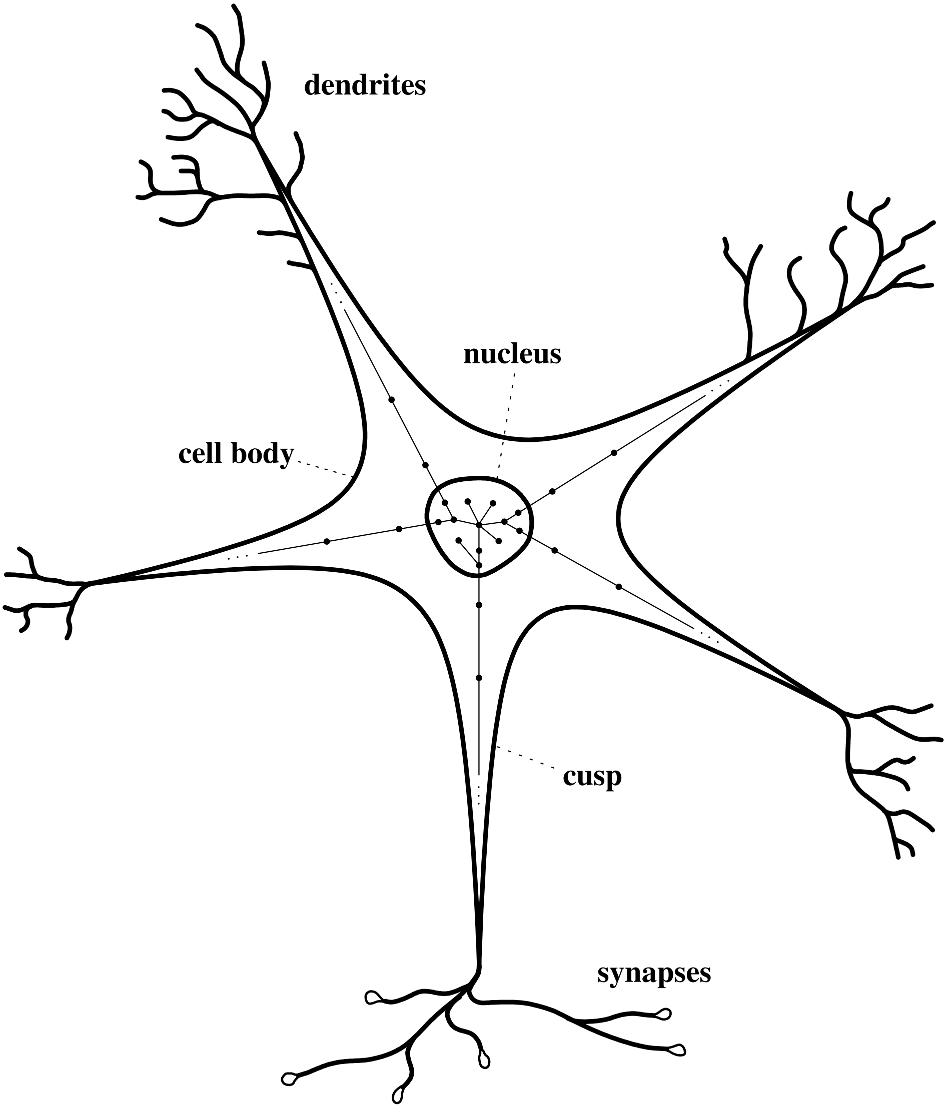
Curriculum Vitae

Oliver Lorscheid was born in Cologne, Germany, on 8 February 1978. From the age of three till six, he visited the kindergarden St. Quirinus and made first arithmetical discoveries such as $10 \cdot 10 = 100$. In 1997 he obtained his highschool degree at the Drei-Königs-Gymnasium in Cologne.

After fulfilling his civil service in a hospital, he moved to Bonn in 1998 and began his studies in mathematics and physics. Soon he concentrated on the former subject. In 2000 he obtained his bachelor degree in mathematics, in 2001 his bachelor degree in physics. He wrote his undergraduate thesis under the guidance of Florian Pop and received his degree in mathematics in 2004.

In June 2004, he started his PhD studies in Utrecht with Gunther Cornelissen as part of the NWO-VIDI Innovational Research Project "Non-archimedean Geometry and Automorphic Forms". In June 2007, he held a NWO Van Gogh grant for a visit to Paris, which allowed him to talk to several experts in the field of automorphic forms.

From September 2008 on, Oliver Lorscheid will visit the Max-Planck Institut für Mathematik in Bonn.


dendrites
nucleus
cell body
cusp
synapses

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List of Notation

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