LECTURES ON $\mathbb{F}_1$

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Abstract. This is a manuscript for four lectures held at IMPA in January 2014. The topics of these lectures are as follows.

In Lecture 1, we will review the early motivations of $\mathbb{F}_1$-geometry, which are *(a)* Jacques Tits' idea of Chevalley groups over $\mathbb{F}_1$ to explain combinatorial counterparts of incidence geometries over finite fields, and *(b)* the ambitious programme to prove the Riemann hypothesis with help of an interpretation of $\operatorname{Spec} \mathbb{Z}$ as a curve over $\mathbb{F}_1$. We give an overview of the different approaches towards $\mathbb{F}_1$-geometry until today, embedded into the historical development of ideas. Finally, we explain how toric geometry led to the definition of monoidal schemes, which form the core of $\mathbb{F}_1$-geometry.

In Lecture 2, we will introduce blueprints, which are objects that allow us to interpolate between monoids and rings. Blue schemes are spaces that are locally isomorphic to the prime spectrum of a blueprint. This class of objects contains usual schemes, monoidal schemes, $\mathbb{F}_1$-models of algebraic groups, and many more interesting examples.

In Lecture 3, we will explain why the functor $\operatorname{Hom}(\operatorname{Spec} \mathbb{F}_1, -)$ of $\mathbb{F}_1$-rational points cannot satisfy the expectations on $\mathbb{F}_1$-geometry. However, we will show that so-called Tits morphisms of blue schemes allow us to make heuristics on Euler characteristics and on Weyl groups of Chevalley groups precise. Furthermore, this class of morphisms contains group laws of Chevalley groups to $\mathbb{F}_1$.

In Lecture 4, we extend first results on Euler characteristics to a larger class of projective varieties with the help of quiver Grassmannians. Namely, all quiver Grassmannians of unramified tree modules admit an $\mathbb{F}_1$-model whose $\mathbb{F}_1$-rational points count the Euler characteristic. We demonstrate this in the example of a del Pezzo surface of degree 6.

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Lecture 1: Motivation, history, and the role of toric geometry

1. Tits' dream

The first mentioning of a "field of characteristic 1" can be found in Jacques Tits' lecture notes [Tit57]. Tits observed that incidence geometries over finite fields $\mathbb{F}_q$ have a combinatorial counterpart, which could be interpreted as the limit $q \to 1$. Tits wondered whether there would be a geometry over a field of characteristic 1 that would explain these analogies. The present-day name for this elusive field is $\mathbb{F}_1$, the *field with one element.*

To make Tits' idea plausible, consider the following examples. The projective space $\mathbb{P}^n$ over $\mathbb{F}_q$ has analogous properties to the set $P_n = \{[1:0:\ldots:0], \ldots, [0:\ldots:0:1]\}$ of coordinates axes of $\mathbb{A}^{n+1}$. Let $N_{\mathbb{P}^n}(q) = [n]_q$ be the number of $\mathbb{F}_q$-rational points of $\mathbb{P}^n$ where $[n]_q = \sum_{i=0}^n q^i$ is the *Gauß number.* Its limit $q \to 1$ is indeed

$$N_{\mathbb{P}^n}(q) \;=\; \sum_{i=0}^{n} q^i \quad \underset{q\to 1}{\longrightarrow} \quad n+1 \;=\; \#P_n.$$

For a general scheme, we call the function $N_X(q) = \#X(\mathbb{F}_q)$ the *counting function of* X or , in case it can be represented by a polynomial in q, the *counting polynomial of* X. A generalization of the above example are Grassmannians $\mathrm{Gr}(k,n)$ over $\mathbb{F}_q$ of linear k-subspaces in $\mathbb{F}_q^n$. Their combinatorial analogon is the set $M_{k,n}$ of k-subspaces that are spanned by standard basis vectors. Indeed, we have

$$N_{\mathrm{Gr}(k,n)}(q) \;=\; \begin{bmatrix} n \\ k \end{bmatrix}_q \quad \underset{q\to 1}{\longrightarrow} \quad \begin{pmatrix} n \\ k \end{pmatrix} \;=\; \#M_{k,n}$$

where $\left[\begin{smallmatrix} n \\ k \end{smallmatrix}\right]_q = \frac{[n]!}{[k]![n-k]!}$ is the *Gauß binom* and $[n]! = \prod_{i=0}^n [i]_q$ is the *Gauß factorial.*

We would like to think of P_n and $M_{k,n}$ as the sets of $\mathbb{F}_1$-rational points of $\mathbb{P}^n$ and $\mathrm{Gr}(k,n)$, respectively. Note that their respective cardinalities are equal to the Euler characteristics of $\mathbb{P}^n(\mathbb{C})$ and $\mathrm{Gr}(k,n)(\mathbb{C})$, which indicates a link between $\mathbb{F}_1$-rational points and the Euler characteristic of the space of $\mathbb{C}$-rational points.

The limits for $\mathbb{P}^n$ and $\mathrm{Gr}(k,n)$ fit with the fact that the combinatorial counterpart of the general linear group GL_n is its Weyl group W, which corresponds to the group of permutation matrices of GL_n. While $\mathrm{GL}_n(\mathbb{F}_q)$ acts transitively on $\mathrm{Gr}(k,n)(\mathbb{F}_q)$, the Weyl group W acts transitively on $M_{k,n}$. Using the Bruhat decomposition

$$\mathrm{GL}_n(\mathbb{F}_q) \;=\; \coprod_{w\in W} \underbrace{Bn_wB}_{\simeq T\times\mathbb{A}^{d_w}}(\mathbb{F}_q)$$

where B is the Borel subgroup of upper triangular matrices, n_w is the permutation matrix corresponding to w, $T \simeq \mathbb{G}_m^n$ is the diagonal torus and d_w equals the number of positive roots plus the length of w, we see that the counting polynomial

$$N_{\mathrm{GL}_n}(q) \;=\; (q-1)^n \cdot \sum_{w\in W} q^{d_w}$$

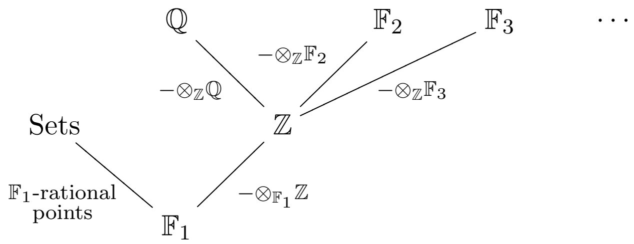
has limit 0 for $q \to 1$ (which equals the Euler characteristic of $\mathrm{GL}_m(\mathbb{C})$). If we, however, resolve the zero of $N_{\mathrm{GL}_n}(q)$ in $q = 1$, then we obtain the expected outcome

$$(q-1)^{-n} \cdot N_{\mathrm{GL}_n}(q) \;=\; \sum_{w\in W} q^{d_w} \quad \underset{q\to 1}{\longrightarrow} \quad \#W.$$

Note that these limits are compatible with the action of $\mathrm{GL}_n(\mathbb{F}_q)$ on $\mathrm{Gr}(k,n)(\mathbb{F}_q)$ and W on $M_{k,n}$ in the sense that the inclusions $W \subset \mathrm{GL}_n(\mathbb{F}_q)$ and $M_{k,n} \subset \mathrm{Gr}(k,n)(\mathbb{F}_q)$ commute with the respective group actions.

This example extends to all split reductive group schemes G and their actions on homogeneous spaces of the form G/P where P is a parabolic subgroup containing a maximal split torus T of G. The work of Chevalley and others (Demazure, Grothendieck, e.a.) clarified the theory of split reductive groups over fields of different characteristics by establishing split models over $\mathbb{Z}$, which are also called *Chevalley group schemes*. Such a model over $\mathbb{Z}$ inhabits the information for all characteristics, and the functor $\mathrm{Hom}(\mathrm{Spec}\,\mathbb{F}_q, -)$ associates with a Chevalley group scheme G the group $G(\mathbb{F}_q)$ for any prime power q.

To make Tits' idea precise, we are searching for a category of "schemes over $\mathbb{F}_1$" together with a base extension functor $-\otimes_{\mathbb{F}_1}\mathbb{Z}$ to Grothendieck schemes and a functor of $\mathbb{F}_1$-rational points such that we find models of Chevalley group schemes and homogeneous spaces over $\mathbb{F}_1$ with the expected sets of $\mathbb{F}_1$-rational points. This can be summarized in the picture



While Tits' dream was slumbering for many decades, it was other thoughts that drew the attention to $\mathbb{F}_1$ in the 80ies and 90ies. The single most-popular idea is arguably the programme to prove the Riemann hypothesis by developing methods from algebraic geometry for $\mathbb{F}_1$-schemes that allows us to interpret $\mathrm{Spec}\,\mathbb{Z}$ as a curve over $\mathbb{F}_1$ and to apply Weil's proof of the Riemann hypothesis for curves over finite fields to solve the classical conjecture. This idea first entered literature in Manin's lecture notes [Man95].

Around the same time other ideas appeared like Smirnov's programme to prove the ABC-conjecture with help of a Hurwitz equality for the morphism $\mathrm{Spec}\,\mathbb{Z} \to \mathbb{P}^1_{\mathbb{F}_1}$ (see [Smi92]) or Soulé's interpretation of the stable homotopy groups of spheres as the K-theory of $\mathbb{F}_1$ (see [Sou04]). An outline of the historical development of ideas around $\mathbb{F}_1$ and the introduction of $\mathbb{F}_1$-geometries can be found in Part 1 of [Lor13].

In the course of these lectures, we will concentrate on the original problem posted by Tits, and say little about other aspects of $\mathbb{F}_1$-geometry.

2. A dozen ways to define $\mathbb{F}_1$

The first suggestion of what a *variety over* $\mathbb{F}_1$ should be was given by Soulé in [Sou04]. One might speculate about the psychological impact of this first attempt to define $\mathbb{F}_1$-geometry: soon after, a dozen further approaches towards $\mathbb{F}_1$ (and more) appeared. Without keeping the chronological order, we mention them in the following.

Soulé's definition underwent a number of variations by himself in [Sou11] and by Connes and Consani in [CC11] and in [CC10]. The latter notion of an $\mathbb{F}_1$-scheme turns out to be in essence the same as a torified scheme as introduced by López Peña and the author in [LPL11] in order to produce examples for the $\mathbb{F}_1$-varieties considered in [Sou04] and [CC11].

Deitmar reinterprets in [Dei05] Kato's generalization ([Kat94]) of a toric variety as an $\mathbb{F}_1$-scheme. This approach is based on the notion of a prime ideal for a monoid, which yields a topological space of a combinatorial flavour together with a structure sheaf in monoids. This category of $\mathbb{F}_1$-schemes is minimalistic in the sense that it can be embedded into any other $\mathbb{F}_1$-theory. We will explain this approach in Section 3.

Toën and Vaquié generalize in [TV09] the functorial viewpoint on scheme theory to any closed complete and cocomplete symmetric monoidal category $\mathcal{C}$ that replaces the category of R-modules in the case of schemes over a ring R. They call the resulting objects *schemes relative to* $\mathcal{C}$. An $\mathbb{F}_1$-scheme is a scheme relative to the category of sets, together with the Cartesian product. It turns out that this approach is equivalent to Deitmar's.

Durov's view on Arakelov theory in terms of monads in his comprehensive thesis [Dur07] produces a notion of $\mathbb{F}_1$-schemes. Haran's idea is to study the categories of coherent sheaves instead of $\mathbb{F}_1$-schemes itself. His definition of a generalized scheme can be found in [Har07], and a modification in [Har09].

Borger's work on Λ-algebraic geometry yields a notion of an $\mathbb{F}_1$-scheme in [Bor09]. Lescot develops in [Les11] a geometry associated with idempotent semirings, which he gives the interpretation of an $\mathbb{F}_1$-geometry.

Berkovich describes a theory of congruence schemes for monoids (see [Ber11]), which can be seen as an enrichment of Deitmar's $\mathbb{F}_1$-geometry. Deitmar modifies this approach and applies it to sesquiads, which is a common generalization of rings and monoids.

The author introduces in [Lor12a] the theory of blueprints and blue schemes, which we will examine in more detail in the forthcoming lectures.

For more details about literature on $\mathbb{F}_1$-geometry, see Part 1 of [Lor13], which contains a tentatively exhaustive list of (pre-)publications until the end of 2012.

3. Toric geometry and monoidal schemes

> "Toric geometry is the very core of $\mathbb{F}_1$-geometry."

In this section, we like to explain this statement in terms of monoidal schemes, which form a category of $\mathbb{F}_1$-schemes that is intrinsically contained in all other approaches towards $\mathbb{F}_1$-geometry, possibly modulo some technical assumption like being of finite type.

To start with, we will like to answer the question

> "What is a toric variety?"

in a way that led to the generalization of a fan, considered by Kato in [Kat94], and that was later called $\mathbb{F}_1$-scheme by Deitmar in [Dei05], or $\mathcal{M}_0$-scheme, monoid scheme or monoidal scheme in other sources. We will start with some preliminary definitions and conventions for our exhibition. Note that we include a zero element from the beginning of our considerations though this was not done in [Kat94] and [Dei05], but only in later treatments of monoidal schemes.

A *monoid* will always be a multiplicatively written commutative associative semigroup A with 0 and 1, i.e. elements that satisfy $0 \cdot a = 0$ and $1 \cdot a = a$ for all $a \in A$. A *morphism of monoids* is a multiplicative map $f : A_1 \to A_2$ that maps 0 to 0 and 1 to 1. This defines the category $\mathcal{M}_0$ of monoids (with zero). Note that this category is complete and cocomplete.

A *multiplicative subset of* A is a multiplicatively closed subset S of A that contains 1. We define the *localization of* A *at* S as the monoid

$$S^{-1}A \;=\; (S \times A)/\sim$$

where $(s, a) \sim (s', a')$ if and only if there is a $t \in S$ such that $ts'a = tsa'$. As usual, we denote the equivalence class of (s, a) by $\frac{a}{s}$. The multiplication is defined by $\frac{a}{s} \cdot \frac{a'}{s'} = \frac{aa'}{ss'}$ and $\frac{0}{1}$ and $\frac{1}{1}$ are the zero and the one of $S^{-1}A$, respectively. The localization $S^{-1}A$ comes together with a canonical monoid morphisms $A \to S^{-1}A$ that sends a to $\frac{a}{1}$. We say that a morphism $f : A_1 \to A_2$ of monoids is a *localization* if it is the canonical morphism of some localization of A_1.

Given a ring R, we define the *monoid ring of A over R* as

$$R[A] \;=\; \Big(\bigoplus_{a \in A} R.a \Big) \; / \; R.0_A$$

where 0_A denotes the zero in A, together with the multiplication that extends R-linearly the multiplication of A. We also write $A \otimes_{\mathbb{F}_1} R$ or A_R for $R[A]$, and consider $- \otimes_{\mathbb{F}_1} R$ as a base extension functor from monoids to R-algebras.

Exercise 3.1. Show that the ring homomorphism

$$S^{-1}(R[A]) \;\longrightarrow\; R[S^{-1}A]$$

that maps $\frac{r.a}{s}$ to $r.\big(\frac{a}{s}\big)$ is an isomorphism.

Let Δ be a *fan* in $\mathbb{Z}^n$, i.e. a collection of (strictly convex and rational) cones $\tau, \sigma, \ldots \in \mathbb{R}^n$ that contains all faces of cones such that every non-empty intersection $\tau \cap \sigma$ is face of both τ and σ. Let

$$\tau^\vee \;=\; \{ v \in \mathbb{R}^n | < v, w > \geq 0 \text{ for all } w \in \tau \}$$

be the dual cone of τ. Then $\tau^\vee \cap \mathbb{Z}^n$ is an additive semigroup, and we define the monoid A_τ as $\{0\} \cup \tau^\vee \cap \mathbb{Z}^n$, written multiplicatively. The collection of monoids A_τ come with monoid morphisms $A_\tau \to A_\sigma$ whenever $\sigma \subset \tau$. We define the *dual fan* Δ^{op} as the diagram of all A_τ with $\tau \in \Delta$ and monoid morphisms $A_\tau \to A_\sigma$ for $\sigma \subset \tau$.

Exercise 3.2. It is an interesting task to work out the conditions on diagrams $\mathcal{D}$ of monoids and monoid morphisms that characterize whether $\mathcal{D}$ is of the form Δ^{op} for a fan Δ. Some necessary conditions are:

(i) The diagram $\mathcal{D}$ is commutative.
(ii) The commutative category generated by $\mathcal{D}$ contains cofibre products.
(iii) The diagram $\mathcal{D}$ contains a terminal object A_0 such that $A_0 - \{0\}$ is a free abelian group of rank n.
(iv) All monoids A_τ in $\mathcal{D}$ are finitely generated and if $a^m = 1$ for $m \geq 1$, then $a = 1$.
(v) All morphisms in $\mathcal{D}$ are localizations.

This list is certainly not sufficient yet for $\mathcal{D}$ to be a dual fan.

We define the toric variety $X(\Delta)$ associated with Δ as follows. Let $U_A = \operatorname{Spec} R[A]$ for $A \in \Delta^{\mathrm{op}}$. Together with the morphisms $U_A \to U_{A'}$ for $A' \to A$ in Δ^{op}, this forms a system $\mathcal{U}$ of open immersions of R-schemes (by (v) in Exercise 3.2), which satisfies the cocycle condition for gluing (by (i)–(iii) in Exercise 3.2). Thus we can define $X(\Delta)$ as the colimit of $\mathcal{U}$ in the category of R-schemes, which is the *toric variety associated with* Δ.

Observation 1. Let $\mathcal{D}$ be a system of finitely many monoids A and monoid morphisms and $\mathcal{U}$ the associated system of affine R-schemes U_A and morphisms of R-schemes. All we need to glue the U_A to an R- scheme X such that the U_A define an open covering of X are the following two conditions on $\mathcal{D}$:

(i) All morphisms in $\mathcal{D}$ are localizations.

(ii) The diagram $\mathcal{D}$ satisfies the cocycle condition: given a diagram of localizations of monoids

$$\begin{tikzcd}
A_1 \arrow[r, "f^1_{12}"] \arrow[dr, "f^1_{13}"'] & A_{12} \arrow[dr, "g^{12}"] & \\
A_2 \arrow[ur, "f^2_{12}"] \arrow[dr, "f^2_{23}"'] & A_{13} \arrow[r, "g^{13}"] & A' \\
A_3 \arrow[r, "f^3_{23}"'] \arrow[ur, "f^3_{13}"] & A_{23} \arrow[ur, "g^{23}"'] &
\end{tikzcd}$$

with f^1_{12}, f^1_{13}, f^2_{12}, f^2_{23}, f^3_{13} and f^3_{23} in $\mathcal{D}$, then two squares of the above diagram commute only if the third square commutes as well, this is, if $g^{12} \circ f^1_{12} = g^{13} \circ f^1_{13}$ and $g^{12} \circ f^2_{12} = g^{23} \circ f^2_{23}$, then $g^{13} \circ f^3_{13} = g^{23} \circ f^3_{23}$.

Observation 2. We do not need the base ring R, but we can associate prime spectra to the monoids A in $\mathcal{D}$ itself. To do so, we introduce the following notions.

An *ideal of* A is a subset I of A such that $0 \in I$ and such that $IA \subset I$. A *prime ideal* of A is an ideal $\mathfrak{p}$ of A such that its complement $S = A - \mathfrak{p}$ is a multiplicative subset.

The *spectrum of* A is the set $X = \operatorname{Spec} A$ of all prime ideals $\mathfrak{p}$ of A together with the *Zariski topology*, which the open subsets of the form

$$U_f \quad = \quad \{\, \mathfrak{p} \in \operatorname{Spec} A \mid f \notin \mathfrak{p} \,\},$$

for $f \in A$ as a basis, and together with a *structure sheaf* $\mathcal{O}_X$ in $\mathcal{M}_0$ defined by

$$\mathcal{O}_X(U_f) \;=\; A[f^{-1}]$$

where $A[f^{-1}] = S^{-1}A$ with $S = \{f^i\}_{i \geq 0}$.

We describe some properties of the spectrum, which are in complete analogy to the case of spectra of rings. First of all note that there is indeed a (unique) sheaf $\mathcal{O}_X$ with $\mathcal{O}_X(U_f) = A[f^{-1}]$. This is much easier to see than in the case of rings since every monoid is *local*: the set $\mathfrak{m}$ of all non-invertible elements of A is its unique maximal ideal. Therefore every covering of U_f must contain U_f itself. This means that the spectrum $X = \operatorname{Spec} A$ is a *monoidal space*, i.e. a topological space together with a sheaf in $\mathcal{M}_0$, and its global sections $\mathcal{O}_X(X)$ are canonically isomorphic to A itself. Note that $\operatorname{Spec} A$ is compact.

Since $\mathcal{M}_0$ is cocomplete, we can define the *stalk* $\mathcal{O}_{X,\mathfrak{p}}$ of $\mathcal{O}_X$ in a point $\mathfrak{p} \in X$ as the colimit of $\mathcal{O}_X(U)$ over all open subsets U of X that contain $\mathfrak{p}$. It turns out that $\mathcal{O}_{X,\mathfrak{p}} \simeq A_\mathfrak{p}$ where $A_\mathfrak{p} = S^{-1}A$ with $S = A - \mathfrak{p}$. Since the monoids $A_\mathfrak{p}$ are local, $\operatorname{Spec} A$ is *local monoidal space*, which is a monoidal space whose stalks are local monoids. A local morphism of local monoidal spaces is a continuous map $\varphi : X \to Y$ together with a morphism of sheaves $\varphi^*\mathcal{O}_Y \to \mathcal{O}_X$ on X such that the induced morphisms between stalks map maximal ideals to maximal ideals.

We define a *monoidal scheme* as a local monoidal space that is covered by open subspaces that are isomorphic to spectra of monoids. Note that if we have given a diagram $\mathcal{D}$ of monoids that satisfies conditions (i) and (ii) of Observation 1, then the colimit of the spectra inside the category of local monoidal spaces defines a compact monoidal scheme, and every compact monoidal scheme is such a colimit[1].

The base extension $- \otimes_{\mathbb{F}_1} \mathbb{Z}$ from monoidal schemes to Grothendieck schemes is defined as follows. We set $(\operatorname{Spec} A) \otimes_{\mathbb{F}_1} \mathbb{Z} = \operatorname{Spec} \mathbb{Z}[A]$. Given a monoidal scheme X with an open affine covering $\{U_i\}$ with $U_i = \operatorname{Spec} A_i$, we glue the affine schemes $U_i \otimes_{\mathbb{F}_1} \mathbb{Z}$ along affine open subschemes

[1]This correspondence extends to a correspondence of possibly infinite diagrams $\mathcal{D}$ and possibly non-compact monoidal schemes. See [Lor12c] for a precise treatment.

$V \otimes_{\mathbb{F}_1} \mathbb{Z}$ for affine open monoidal subschemes V of the intersections $U_i \cap U_j$. This process is well-defined and independent of the chosen covering $\{U_i\}$. We also write $X_{\mathbb{Z}} = X \otimes_{\mathbb{F}_1} \mathbb{Z}$.

The construction of monoidal schemes and $-\otimes_{\mathbb{F}_1} \mathbb{Z}$ implies immediately that the dual Δ^{op} of a fan Δ of a toric variety provides the descend datum to a monoidal scheme X such that $X_{\mathbb{Z}}$ is intrinsically isomorphic to the toric variety associated with Δ.

Let us describe some examples of monoidal schemes.

Example 3.3 ($\operatorname{Spec}\mathbb{F}_1$)**.** We define $\mathbb{F}_1$ as the monoid $\{0,1\}$, which is the initial object in $\mathcal{M}_0$. Its spectrum $\operatorname{Spec}\mathbb{F}_1$ is the terminal object in the category of blue schemes. It has a unique point, which is the zero ideal (0). Similarly, if G is an abelian group and $G_0 = \{0\} \cup G$ is the associated monoid, then $\operatorname{Spec} G_0$ consists of a unique point, which is (0). Thus monoids A such that all elements $a \neq 0$ are invertible are the correct generalization of fields in the category $\mathcal{M}_0$.

Example 3.4 (Affine spaces)**.** Let $A = \mathbb{F}_1[T_1, \dots, T_n]$ be the monoid

$$\{0\} \cup \{\, T_1^{e_1} \cdots T_n^{e_n} \mid e_1, \dots, e_n \geq 0 \,\},$$

which is the free monoid in $T_1, \dots, T_n$ over $\mathbb{F}_1$. Since the associated ring $A_{\mathbb{Z}}^+$ is the polynomial ring $\mathbb{Z}[T_1, \dots, T_n]^+$, we define $\mathbb{A}^n_{\mathbb{F}_1} = \operatorname{Spec}\mathbb{F}_1[T_1, \dots, T_n]$.

We describe the underlying topological space of $\mathbb{A}^n_{\mathbb{F}_1}$ for $n = 1$ and $n = 2$. The underlying topological space of the *affine line* $\mathbb{A}^1_{\mathbb{F}_1} = \operatorname{Spec}\mathbb{F}_1[T]$ *over* $\mathbb{F}_1$ consists of the prime ideals $(0) = \{0\}$ and $(T) = \{0\} \cup \{T^i\}_{i>0}$, the latter one being a specialization of the former one. The *affine plane* $\mathbb{A}^2_{\mathbb{F}_1} = \operatorname{Spec}\mathbb{F}_1[S,T]$ *over* $\mathbb{F}_1$ has four points (0), (S), (T) and (S,T). More generally, the prime ideals of $\mathbb{F}_1[T_1, \dots, T_n]$ are of the form $\mathfrak{p}_I = (T_i)_{i \in I}$ where I ranges through the subsets of $\{1, \dots, n\}$. The affine line and the affine plane are illustrated in Figure 1. The lines between two points indicate that the point at the top is a specialization of the point at the bottom.

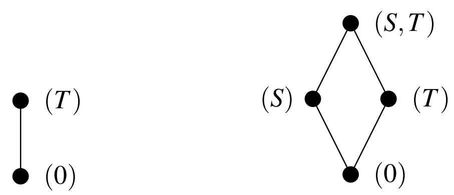


FIGURE 1. The affine line and the affine plane over $\mathbb{F}_1$

Example 3.5 (Tori)**.** Let G be a free multiplicatively written abelian group generated by $T_1, \dots, T_n$. We denote the monoid G_0 by $\mathbb{F}_1[T_1^{\pm 1}, \dots, T_n^{\pm 1}]$ in analogy to the notation $\mathbb{F}_1[T_1, \dots, T_n]$. Its base extension to $\mathbb{Z}$ is $\mathbb{Z}[T_1^{\pm 1}, \dots, T_n^{\pm 1}]$, which means that the base extension of $\operatorname{Spec} G_0$ is isomorphic to $\mathbb{G}^n_{m,\mathbb{Z}}$. Consequently we denote $\operatorname{Spec}\mathbb{F}_1[T_1^{\pm 1}, \dots, T_n^{\pm 1}]$ by $\mathbb{G}^n_{m,\mathbb{F}_1}$, and call it the (split) torus of rank n over $\mathbb{F}_1$. Since G is a group, the topological space of $\mathbb{G}^n_{m,\mathbb{F}_1}$ consists of one point, which is the zero ideal (0).

Since the group law of $\mathbb{G}^n_{m,\mathbb{Z}}$ is a toric morphism, it descends to $\mathbb{F}_1$, i.e. $\mathbb{G}^n_{m,\mathbb{F}_1}$ is a group object in the category of monoidal schemes. We will discuss group schemes over $\mathbb{F}_1$ in more detail in Lecture 3 where we will see that tori and constant group schemes are essentially the only group schemes that have models in the category of monoidal schemes.

Example 3.6 (Projective spaces)**.** Since the projective n-space $\mathbb{P}^n$ is a toric variety, the choice of a fan for $\mathbb{P}^n$ defines a monoidal scheme $\mathbb{P}^n_{\mathbb{F}_1}$ whose base extension to $\mathbb{Z}$ is $\mathbb{P}^n_{\mathbb{Z}}$. We describe the projective line in more detail. We can cover $\mathbb{P}^1$ by to affine lines, whose intersection is isomorphic to $\mathbb{G}_m$, i.e.

$$\mathbb{P}^1 \;=\; \mathbb{A}^1 \amalg_{\mathbb{G}_m} \mathbb{A}^1.$$

We can descend all the affine scheme on the right together with their inclusions to $\mathbb{F}_1$, which yields the identity

$$\mathbb{P}^1_{\mathbb{F}_1} \;=\; \mathbb{A}^1_{\mathbb{F}_1} \amalg_{\mathbb{G}_{m,\mathbb{F}_1}} \mathbb{A}^1_{\mathbb{F}_1}.$$

This means that $\mathbb{P}^1_{\mathbb{F}_1}$ has two closed points "[1 : 0]" and "[0 : 1]" and one generic point "[1 : 1]":

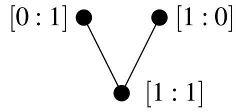


FIGURE 2. The projective line over $\mathbb{F}_1$

Note that the closed points correspond with the set P_1 as defined in Section 1. This holds true for all $\mathbb{F}_1$-models $\mathbb{P}^n_{\mathbb{F}_1}$ since the closed points correspond to the maximal cones in the fan of $\mathbb{P}^n$.

Exercise 3.7. Identify the underlying topological space of $\mathbb{P}^2_{\mathbb{F}_1}$ with the following illustration:

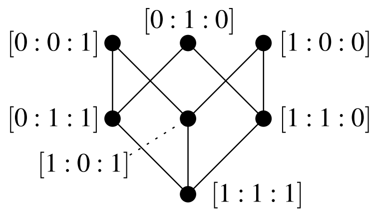


FIGURE 3. The projective plane over $\mathbb{F}_1$

All of our examples are toric varieties, which is not surprising from the way monoidal schemes were derived from toric geometry. The following theorem makes the connection between monoidal schemes and toric varieties precise.

Theorem 3.8 ([Dei08])**.** *Let X be a monoidal scheme. If $X_{\mathbb{Z}}$ is a connected separated integral scheme of finite type, then it is a toric variety.*

As a consequence, we see the limitations of $\mathbb{F}_1$-geometry if we work only with monoidal schemes. For instance, Grassmannians $\mathrm{Gr}(k,n)$ with $2 \leq k \leq n-2$ and reductive groups other than tori do not have $\mathbb{F}_1$-models as monoidal schemes. In the next lecture, we will extend the category of monoidal schemes to the category of blue schemes, which will be large enough to contain $\mathbb{F}_1$-models of these schemes.

Lecture 2: Blueprints and blue schemes

A blueprint can be seen as a multiplicatively closed subset A (with 0 and 1) of a semiring R. The addition of the semiring allows us to express relations $\sum a_i = \sum b_j$ between sums of elements a_i and b_j of A. In the extreme case $A = R$, the blueprint is a semiring, and in the other extreme case $A \subset R$ where $R = \mathbb{Z}[A]/(0)$ is the monoid ring of A, the blueprint behaves like a multiplicative monoid.

Its geometric counterpart, the notion of a blue scheme, includes monoidal schemes and usual schemes as subclasses. Beyond these well-known objects, it contains also a class of semiring schemes as well as $\mathbb{F}_1$-models for all schemes of finite type. This makes the category of blue schemes flexible enough to treat several problems in $\mathbb{F}_1$-geometry.

We will give precise definitions of blueprints and blue schemes in this lecture.

4. Blueprints

A *blueprint* B is a monoid A with zero together with a *pre-addition* $\mathcal{R}$, i.e. $\mathcal{R}$ is an equivalence relation on the semiring $\mathbb{N}[A] = \{\sum a_i | a_i \in A\}$ of finite formal sums of elements of A that satisfies the following axioms (where we write $\sum a_i \equiv \sum b_j$ whenever $(\sum a_i, \sum b_j) \in \mathcal{R}$):

(i) The relation $\mathcal{R}$ is additive and multiplicative, i.e. if $\sum a_i \equiv \sum b_j$ and $\sum c_k \equiv \sum d_l$, then $\sum a_i + \sum c_k \equiv \sum b_j + \sum d_l$ and $\sum a_i c_k \equiv \sum b_j d_l$.

(ii) The absorbing element 0 of A is in relation with the zero of $\mathbb{N}[A]$, i.e. $0 \equiv$ (empty sum).

(iii) If $a \equiv b$, then $a = b$ (as elements in A).

A *morphism* $f : B_1 \to B_2$ *of blueprints* is a multiplicative map $f : A_1 \to A_2$ between the underlying monoids of B_1 and B_2 with $f(0) = 0$ and $f(1) = 1$ such that for every relation $\sum a_i \equiv \sum b_j$ in the pre-addition $\mathcal{R}_1$ of B_1, the pre-addition $\mathcal{R}_2$ of B_2 contains the relation $\sum f(a_i) \equiv \sum f(b_j)$. Let $\mathcal{B}lpr$ be the category of blueprints.

In the following, we write $B = A /\!\!/ \mathcal{R}$ for a blueprint B with underlying monoid A and pre-addition $\mathcal{R}$. We adopt the conventions used for rings: we identify B with the underlying monoid A and write $a \in B$ or $S \subset B$ when we mean $a \in A$ or $S \subset A$, respectively. Further, we think of a relation $\sum a_i \equiv \sum b_j$ as an equality that holds in B (without the elements $\sum a_i$ and $\sum b_j$ being defined, in general).

Given a set S of relations, there is a smallest equivalence relation $\mathcal{R}$ on $\mathbb{N}[A]$ that contains S and satisfies (i) and (ii). If $\mathcal{R}$ satisfies also (iii), then we say that $\mathcal{R}$ is the pre-addition generated by S, and we write $\mathcal{R} = \langle S \rangle$. In particular, every monoid A with zero has a smallest pre-addition $\mathcal{R} = \langle \emptyset \rangle$.

4.1. **Relation to rings and semirings.** The idea behind the definition of a blueprint is that it is a blueprint of a ring (in the literal sense): given a blueprint $B = A /\!\!/ \mathcal{R}$, one can construct the ring $B^+_{\mathbb{Z}} = \mathbb{Z}[A]/I(\mathcal{R})$ where $I(\mathcal{R})$ is the ideal $\{\sum a_i - \sum b_j \in \mathbb{Z}[A] | \sum a_i \equiv \sum b_j \text{ in } \mathcal{R}\}$. We can extend a blueprint morphism $f : B_1 \to B_2$ linearly to a ring homomorphism $f^+_{\mathbb{Z}} : B^+_{1,\mathbb{Z}} \to B^+_{2,\mathbb{Z}}$ and obtain a functor $(-)^+_{\mathbb{Z}} : \mathcal{B}lpr \to \mathcal{R}ings$ from blueprints to rings.

Similarly, we can form the quotient $B^+ = \mathbb{N}[A]/\mathcal{R}$, which inherits the structure of a semiring by Axiom (i). This defines a functor $(-)^+ : \mathcal{B}lpr \to \mathcal{SR}ings$ from blueprints to semirings.

On the other hand, semirings can be seen as blueprints: given a (commutative and unital) semiring R, we can define the blueprint $B = A /\!\!/ \mathcal{R}$ where $A = R^\bullet$ is the underlying multiplicative monoid of R and $\mathcal{R} = \{\sum a_i \equiv \sum b_j | \sum a_i = \sum b_j \text{ in } R\}$. Under this identification, semiring homomorphisms are nothing else than blueprint morphisms, i.e. we obtain a full embedding $\iota_{\mathcal{S}} : \mathcal{SR}ings \to \mathcal{B}lpr$ from the category of semirings into the category of blueprints. This allows us to identify rings with a certain kind of blueprints, and we can view blueprints as a generalization of rings and semirings. Accordingly, we call blueprints in the essential image of $\iota_{\mathcal{S}}$ *semirings* and we call blueprints in the essential image of the restriction $\iota_{\mathcal{R}} : \mathcal{R}ings \to \mathcal{B}lpr$ of $\iota_{\mathcal{S}}$ *rings*.

The semiring B^+ and the ring $B^+_{\mathbb{Z}}$ come together with canonical maps $B \to B^+$ and $B \to B^+_{\mathbb{Z}}$, which are blueprint morphisms.

Remark 4.1. Note that we use the symbol "+" whenever we want to stress that the pre-addition is indeed an addition, i.e. the blueprint in question is a semiring. Since colimits of blueprints do not preserve addition, we are forced to invent notations like $\mathbb{Z}[T]^+$, $B \otimes^+_{\mathbb{Z}} C$, ${}^+\mathbb{A}^n_{\mathbb{Z}}$, ${}^+\mathbb{P}^n_{\mathbb{Z}}$, ${}^+\mathbb{G}^n_{m,\mathbb{Z}}$ to distinct the constructions in usual algebraic geometry from the analogous constructions for blueprints and blue schemes.

4.2. **Relation to monoids.** Another important class of blueprints are monoids. Namely, a monoid A with zero defines the blueprint $B = A /\!\!/ \langle\emptyset\rangle$. A morphism $f : A_1 \to A_2$ of monoids with zero is nothing else than a morphism of blueprints $f : A_1 /\!\!/ \langle\emptyset\rangle \to A_2 /\!\!/ \langle\emptyset\rangle$. This defines a full embedding $\iota_{\mathcal{M}_0} : \mathcal{M}_0 \to \mathcal{B}lpr$. This justifies that we may call blueprints in the essential image of $\iota_{\mathcal{M}_0}$ *monoids with zero* and that we identify in this case A with $B = A /\!\!/ \langle\emptyset\rangle$.

Note that the functor $- \otimes_{\mathbb{F}_1} \mathbb{Z}$ from monoids to rings is isomorphic to the composition $(-)^+_{\mathbb{Z}} \circ \iota_{\mathcal{M}_0}$.

4.3. **Cyclotomic field extensions.** We give some other examples, which are of interest for the purposes of this text. The initial object in $\mathcal{B}lpr$ is the monoid $\mathbb{F}_1 = \{0, 1\}$, the so-called *field with one element*. More general, we define the *cyclotomic field extension* $\mathbb{F}_{1^n}$ *of* $\mathbb{F}_1$ as the blueprint $B = A /\!\!/ \mathcal{R}$ where $A = \{0\} \cup \mu_n$ is the union of 0 with a cyclic group $\mu_n = \{\zeta_n^i | i = 1, \dots, n\}$ of order n with generator ζ_n and where $\mathcal{R}$ is generated by the relations $\sum_{i=0}^{n/d} \zeta_n^{di} \equiv 0$ for every proper divisor d of n. The associated ring of $\mathbb{F}_{1^n}$ is the ring $\mathbb{Z}[\zeta_n]$ of integers of the cyclotomic field extension $\mathbb{Q}[\zeta_n]$ of $\mathbb{Q}$ that is generated by the n-th roots of unity.

Of particular interest for Lecture 3 is the quadratic extension

$$\mathbb{F}_{1^2} \;=\; \{0, \pm 1\} /\!\!/ \langle 1 + (-1) \equiv 0 \rangle$$

of $\mathbb{F}_1$. We say that a blueprint is *with* -1 if there exists a morphism $\mathbb{F}_{1^2} \to B$. Such a morphism is necessarily unique. Note that a semiring R is a ring if and only if R is with -1.

4.4. **Special linear group.** We define the blueprint $\mathbb{F}_1[\mathrm{SL}_2]$ as

$$\mathbb{F}_1[T_1, T_2, T_3, T_4] /\!\!/ \langle T_1 T_4 \equiv T_2 T_3 + 1 \rangle.$$

Then its universal ring $\mathbb{F}_1[\mathrm{SL}_2]^+_{\mathbb{Z}}$ is isomorphic to the coordinate ring $\mathbb{Z}[\mathrm{SL}_2]$ of the special linear group $\mathrm{SL}_{2,\mathbb{Z}}$.

4.5. **Localization.** Let $B = A /\!\!/ \mathcal{R}$ be a blueprint and S a multiplicative subset. Then the *localization of* B *at* S is the blueprint $S^{-1}B = A_S /\!\!/ \mathcal{R}_S$ where $A_S = S^{-1}A$ is the localization of the monoid A at S and where

$$\mathcal{R}_S \;=\; \Big\langle \sum \frac{a_i}{1} \equiv \sum \frac{b_j}{1} \;\Big|\; \sum a_i \equiv \sum b_j \text{ in } B \Big\rangle.$$

This comes together with the canonical morphism $B \to S^{-1}B$ that sends a to $\frac{a}{1}$.

4.6. **Blue B-modules.** Let M be a pointed set. We denote the base point of M by 0. A *pre-addition on* M is an equivalence relation $\mathcal{P}$ on the semigroup $\mathbb{N}[M] = \{\sum a_i | a_i \in M\}$ of finite formal sums in M with the following properties (as usual, we write $\sum m_i \equiv \sum n_j$ if $\sum m_i$ stays in relation to $\sum n_j$):

(i) $\sum m_i \equiv \sum n_j$ and $\sum p_k \equiv \sum q_l$ implies $\sum m_i + \sum p_k \equiv \sum n_j + \sum q_l$,

(ii) $0 \equiv$ (empty sum), and
(iii) if $m \equiv n$, then $m = n$ (in M).

Let $B = A /\!\!/ \mathcal{R}$ be a blueprint. A *blue B-module* is a set M together with a pre-addition $\mathcal{P}$ and a *B-action* $B \times M \to M$, which is a map $(b, m) \mapsto b.m$ that satisfies the following properties:

(i) $1.m = m$, $0.m = 0$ and $a.0 = 0$,
(ii) $(ab).m = a.(b.m)$, and
(iii) $\sum a_i \equiv \sum b_j$ and $\sum m_k \equiv \sum n_l$ implies $\sum a_i.m_k \equiv \sum b_j.n_l$.

A *morphism of blue B-modules M and N* is a map $f : M \to N$ such that

(i) $f(a.m) = a.f(m)$ for all $a \in B$ and $m \in M$ and
(ii) whenever $\sum m_i \equiv \sum n_j$ in M, then $\sum f(m_i) \equiv \sum f(n_j)$ in N.

This implies in particular that $f(0) = 0$. We denote the category of blue B-modules by ${}^{\mathrm{bl}}\mathrm{Mod}\, B$. Note that in case of a ring B, every B-module is a blue B-module, but not vice versa. Note further that the category ${}^{\mathrm{bl}}\mathrm{Mod}\, B$ is closed, complete and cocomplete. The trivial blue module $\mathbf{0} = \{0\}$ is an initial and terminal object of ${}^{\mathrm{bl}}\mathrm{Mod}\, B$.

Given a multiplicative subset S of B, we define the *localization of M at S* as the blue B-module $S^{-1}M$ whose underlying set is

$$S^{-1}M \;=\; (S \times M) /\sim$$

where $(s, m) \sim (s', m')$ if and only if there is a $t \in S$ such that $ts.m' = ts'.m$, and whose pre-addition is

$$\mathcal{P}_S \;=\; \left\langle \sum \frac{m_i}{1} \equiv \sum \frac{n_j}{1} \;\middle|\; \sum m_i \equiv \sum n_j \text{ in } M \right\rangle$$

where we write $\frac{m}{s}$ for the class of (s, m). The localization $S^{-1}M$ comes with the natural structure of a blue $S^{-1}B$-module that extends the action of B, and the map $M \to S^{-1}M$ that maps m to $\frac{m}{1}$ is a morphism of blue B-modules.

5. The prime spectrum and blue schemes

In this section, we introduce the spectrum of a blueprint B and embed the spectra in the category of locally blueprinted spaces, which allows to glue affine pieces to blue schemes.

5.1. Ideals. An *ideal of B* is a subset I such that for all $\sum a_i \equiv \sum b_j + c$ with $a_i, B_j \in I$ also $c \in I$ and such that $IB \subset I$. This definition is reasoned by the following fact.

Proposition 5.1 ([Lor12a]). *A subset I of B is an ideal if and only if there is a blueprint morphism $f : B \to C$ such that $I = f^{-1}(0)$. In fact, there is a morphism $f_0 : B \to C_0$ with $I = f^{-1}(0)$ that is universal among all morphisms $f : B \to C$ with $f(I) \subset \{0\}$. We denote C_0 by B/I and call it the* quotient of B by I.

A *prime ideal of B* is an ideal $\mathfrak{p}$ whose complement $S = B - \mathfrak{p}$ is a multiplicative subset of B. A *maximal ideal* is a proper ideal $\mathfrak{m}$ that is not contained in any other proper ideal. We say that B is *integral* if the multiplication by any non-zero element defines an injective map $B \to B$. We say that B is a *blue field* if every non-zero element is invertible.

The following facts extend from usual ring theory to all blueprints. An ideal I of B is prime if and only if B/I is integral, and I is maximal if and only if B/I is a blue field.

5.2. **Locally blueprinted spaces.** A *local blueprint* is a blueprint with a unique maximal ideal. Consequently, the localization $B_\mathfrak{p} = S^{-1}B$ with $S = B - \mathfrak{p}$ at a prime ideal $\mathfrak{p}$ of B is local since its unique maximal ideal is $\mathfrak{m}_\mathfrak{p} = \mathfrak{p}B_\mathfrak{p}$. Consequently, the quotient $\kappa(\mathfrak{p}) = B_\mathfrak{p}/\mathfrak{m}_\mathfrak{p}$ is a blue field.

A *locally blueprinted space* is a topological space X together and a class of covering families that form a site together with a sheaf $\mathcal{O}_X$ in $\mathcal{B}lpr$ on this site such that the stalk

$$\mathcal{O}_{X,x} \;=\; \operatorname*{colim}_{x\in U} \mathcal{O}_X(U)$$

is a local blueprint for every $x \in X$. We denote the maximal ideal of $\mathcal{O}_{X,x}$ by $\mathfrak{m}_x$ and call $\mathcal{O}_X$ the *structure sheaf of* X. We call a covering family of X an *admissible covering*.

A *local morphism of locally blueprinted spaces* is a continuous map $\varphi : X \to Y$ that pulls back covering families to covering families together with a morphism $\varphi^*\mathcal{O}_Y \to \mathcal{O}_X$ of sheaves on X such that the induced morphisms $\varphi_x : \mathcal{O}_{Y,\varphi(x)} \to \mathcal{O}_{X,x}$ of stalks maps $\mathfrak{m}_{\varphi(x)}$ to $\mathfrak{m}_x$. This defines the category $\mathcal{L}oc\mathcal{B}lpr\mathcal{S}p$ of locally blueprinted spaces.

5.3. **The spectrum.** The *spectrum of* B is the following locally blueprinted space. Its underlying topological space is the set $\operatorname{Spec} B$ endowed with the Zariski topology that is generated by the open subset

$$U_h \;=\; \{\, \mathfrak{p} \in \operatorname{Spec} B \mid h \notin \mathfrak{p} \,\}$$

with $h \in B$. The class of covering families is generated by families of the form $\{U_h\}_{h\in I}$ with $I \subset B$ that admit a finite subset $J \subset I$ such that the functor

$$\prod_{h\in J} (-)[h^{-1}] : \quad {}^{\mathrm{bl}}\mathrm{Mod}\, B \quad \longrightarrow \quad \prod_{h\in J} {}^{\mathrm{bl}}\mathrm{Mod}\, B[h^{-1}]$$

is conservative, i.e. if $\prod_{h\in J}(f)[h^{-1}]$ is an isomorphism, then $f : M_1 \to M_2$ is an isomorphism of blue B-modules. The structure sheaf $\mathcal{O}_X$ on X is defined on open subsets U of X as the set of functions

$$s :\; U \;\longrightarrow\; \coprod_{\mathfrak{p}\in U} B_\mathfrak{p}$$

with $s(\mathfrak{p}) \in B_\mathfrak{p}$ such that there is covering family $\{U_h\}_{h\in I}$ and elements $s_h \in B[h^{-1}]$ for every $h \in I$ such that $s(\mathfrak{p}) = \frac{s_h}{1}$ in $B_\mathfrak{q}$ for all $\mathfrak{q} \in U_h$ and all $h \in I$.

Note that the conservativeness of $\prod_{h\in J}(-)[h^{-1}]$ implies that $X = \bigcup_{h\in J} U_h$. The converse is, however, not true as the following example shows.

Example 5.2. Let k be a blue field. Consider the blueprint $B = k \times k /\!\!/ \mathcal{R}$ where $\mathcal{R}$ is the pre-addition that consists of relations

$$\textstyle\sum(a_i, 0) \equiv \sum(b_j, 0) \qquad \text{and} \qquad \sum(0, a_i) \equiv \sum(0, b_j)$$

where $\sum a_i \equiv \sum b_j$ are relations in k. Then $\operatorname{Spec} B$ consists of the two prime ideals $k \times \{0\}$ and $\{0\} \times k$, endowed with the discrete topology since $U_{(1,0)} = \{k \times \{0\}\}$ and $U_{(0,1)} = \{\{0\} \times k\}$ are open subsets. We have $\operatorname{Spec} B = U_{(1,0)} \cup U_{(0,1)}$, but the functor

$${}^{\mathrm{bl}}\mathrm{Mod}\, B \quad \longrightarrow \quad {}^{\mathrm{bl}}\mathrm{Mod}\, B[(1,0)^{-1}] \times {}^{\mathrm{bl}}\mathrm{Mod}\, B[(0,1)^{-1}]$$

is not conservative as can be seen as follows. Let $B' = k \times k /\!\!/ \mathcal{R}'$ be the blueprint where $\mathcal{R}'$ is the pre-addition

$$\big\{\, \textstyle\sum(a_i, c_i) \equiv \sum(b_j, d_j) \,\big|\, \sum a_i \equiv \sum b_j \text{ and } \sum c_i \equiv \sum d_j \text{ in } k \,\big\},$$

which contains the pre-addition $\mathcal{R}$ properly. Then the identity map $f : B \to B'$ is a morphism of blue B-modules that is not an isomorphism. However, the localizations $B[(1,0)^{-1}]$ and $B'[(1,0)^{-1}]$ are both isomorphic to k and

$$f[(1,0)^{-1}] : \ B[(1,0)^{-1}] \ \longrightarrow \ B'[(1,0)^{-1}]$$

is an isomorphism of blue $B[(1,0)^{-1}]$-modules. Similarly, $f[(0,1)^{-1}]$ is an isomorphism. Therefore $\{U_{(1,0)}, U_{(0,1)}\}$ is not an admissible covering of $\operatorname{Spec} B$.

An important consequence is of the definition of the spectrum of a blueprint is the following.

Theorem 5.3. *Let $X = \operatorname{Spec} B$ and $h \in B$. Then there is a canonical isomorphism $B[h^{-1}] \simeq \mathcal{O}_X(U_h)$. In particular, $B \simeq \mathcal{O}_X(X)$.*

This theorem follows essentially from the comparison of blue schemes with relative schemes in [Lor12c] and the machinery of relative schemes in Toën and Vaquié's paper [TV09].[2]

Exercise 5.4. Consider the monoid $\mathbb{F}_1[a_1, a_2, h_1, h_2]/\sim$ with the relation $a_1h_2 \sim a_2h_1$ and the blueprint $B = A /\!\!/ \langle h_1 + h_2 \equiv 1\rangle$. Show that $\operatorname{Spec} B = U_{h_1} \cup U_{h_2}$, but that $\{U_{h_1}, U_{h_2}\}$ is not an admissible covering of $\operatorname{Spec} B$. *Hint: consider the section that sends a prime ideal $\mathfrak{p}$ to either $\frac{a_1}{h_1}$ or $\frac{a_2}{h_2}$.*

5.4. **Blue schemes.** An *affine blue scheme* is a locally blueprinted space that is isomorphic to the spectrum of a blueprint. A *blue scheme* is a locally blueprinted space that has an admissible covering by affine blue schemes. A morphism of blue schemes is a local morphism of locally blueprinted spaces. We denote the category of blue schemes by $\mathrm{Sch}_{\mathbb{F}_1}$.

It is easy to see that a morphism of blueprints defines a local morphism between their spectra by pulling back prime ideals. By the techniques from [Lor12a], we derive the converse statement.

Theorem 5.5. *The contravariant functor* $\operatorname{Spec} : \mathcal{B}lpr \to \mathrm{Sch}_{\mathbb{F}_1}$ *is a full embedding of categories. Consequently, every morphism $\varphi : X \to Y$ of blue schemes is locally given by morphisms of blueprints.*

This statement justifies the name *locally algebraic morphisms* for morphisms of blue schemes. We will use this name to distinct this class of morphisms from Tits morphisms of blue schemes, which we will introduce in Lecture 3.

5.5. **Monoidal, semiring and Grothendieck schemes.** The category of monoidal schemes embeds naturally as a full subcategory into the category of blue schemes by considering a local monoidal space as a locally blueprinted space. This defines a full embedding of the category of monoidal schemes into the category of blueprints. In the following, we will identify the category of monoidal schemes with the essential image of this embedding. In other words, a blue scheme is monoidal if it has an affine open covering by spectra of monoids. In particular, the examples at the end of Lecture 1 are blue schemes.

Similarly, the category of Grothendieck schemes is a full subcategory of the category of blue schemes by considering locally ringed spaces as locally blueprinted spaces. The essential image of this embedding are blue schemes that have an affine open covering by spectra of rings. We denote the category of Grothendieck schemes by $\mathrm{Sch}_{\mathbb{Z}}^{+}$.

The class of Grothendieck schemes is a subclass of semiring schemes, which are blue schemes with an admissable affine open covering by spectra of semirings. We denote the full subcategory of semiring schemes by $\mathrm{Sch}_{\mathbb{N}}^{+}$.

[2] Note that the definition of $\operatorname{Spec} B$ differs from the definitions made in previous texts on blueprints. With respect to the former definition, Theorem 5.3 is not true.

The functors $(-)^+$ from blueprints to semirings and $(-)^+_{\mathbb{Z}}$ from blueprints to rings extend to functors

$$(-)^+ : \operatorname{Sch}_{\mathbb{F}_1} \longrightarrow \operatorname{Sch}^+_{\mathbb{N}} \qquad \text{and} \qquad (-)^+_{\mathbb{Z}} : \operatorname{Sch}_{\mathbb{F}_1} \longrightarrow \operatorname{Sch}^+_{\mathbb{Z}}.$$

The base extensions to semirings and rings come with canonical morphisms $X^+ \to X$ and $\beta_X : X^+_{\mathbb{Z}} \to X$ of blue schemes.

Note that the restriction of $(-)^+_{\mathbb{Z}}$ to monoidal schemes is isomorphic to the functor $-\otimes_{\mathbb{F}_1}\mathbb{Z}$ that we considered in Lecture 1.

5.6. **Descending closed subschemes.** We know already that toric varieties have models as monoidal schemes. Indeed, every closed subscheme of a toric variety descends to $\mathbb{F}_1$ as we will explain in the following.

Let X be a monoidal scheme and $X^+_{\mathbb{Z}}$ be the universal Grothendieck scheme of X, which is a toric variety under certain assumption (see Theorem 3.8). Let Z be a closed subscheme of $X^+_{\mathbb{Z}}$.

If $X = \operatorname{Spec} B$ is affine, then $X^+_{\mathbb{Z}} = \operatorname{Spec} B^+_{\mathbb{Z}}$ is affine and we have $B^+_{\mathbb{Z}} = \mathbb{Z}[A]/I(\mathcal{R})$ if $B = A/\!\!/\mathcal{R}$ and

$$I(\mathcal{R}) \;=\; \{\textstyle\sum a_i - \sum b_j \mid \sum a_i \equiv \sum b_j \text{ in } B\,\}.$$

In this case Z is affine as a closed subscheme of $X^+_{\mathbb{Z}}$, i.e. $Z = \operatorname{Spec}\mathbb{Z}[A]/J$ for an ideal J of $\mathbb{Z}[A]$ that contains $I(\mathcal{R})$. If we define

$$\mathcal{R}_Z \;=\; \{\textstyle\sum a_i \equiv \sum b_j \mid \sum a_i - \sum b_j \in J\,\},$$

then $Y = \operatorname{Spec}\big(A/\!\!/\mathcal{R}_Z\big)^3$ is a subscheme of X such that $Y^+_{\mathbb{Z}} \simeq Z$.

This process is compatible with patching affine pieces, so that we obtain a subscheme Y of X with $Y^+_{\mathbb{Z}} \simeq Z$ for any closed subscheme Z of $X^+_{\mathbb{Z}}$. In particular, the topological space of Y is a subspace of the topological space of X. We call a scheme Y that is derived from a closed subscheme of $X^+_{\mathbb{Z}}$ a *closed subscheme of* X, and the inclusion $Y \to X$ a *closed immersion*.

We apply this mechanism to the following examples.

Example 5.6 (Special linear group)**.** We considered already in paragraph 4.4 the blueprint $\mathbb{F}_1[\mathrm{SL}_2] = \mathbb{F}_1[T_1,T_2,T_3,T_4]/\!\!/\langle T_1T_4 \equiv T_2T_3 + 1\rangle$, which is a quotient of the free blueprint $\mathbb{F}_1[T_1,T_2,T_3,T_4]$. Therefore the blue scheme $\mathrm{SL}_{2,\mathbb{F}_1} = \operatorname{Spec}\mathbb{F}_1[\mathrm{SL}_2]$ is a subscheme of $\mathbb{A}^4_{\mathbb{F}_1}$.

In Example 3.4, we saw that the points of $\mathbb{A}^4_{\mathbb{F}_1}$ are the ideals $\mathfrak{p}_I = (T_i)_{i\in I}$ for a subset I of $\{1,2,3,4\}$. Since 1 cannot be contained in any prime ideal and $T_1T_4 \equiv T_2T_3 + 1$, either T_1T_4 or T_2T_3 is not contained in a prime ideal $\mathfrak{p}_I$ of $\mathrm{SL}_{2,\mathbb{F}_1}$. This is the only restriction for a point of $\mathbb{A}^4_{\mathbb{F}_1}$ to be contained in $\mathrm{SL}_{2,\mathbb{F}_1}$. Thus we can illustrate $\mathrm{SL}_{2,\mathbb{F}_1}$ as follows.

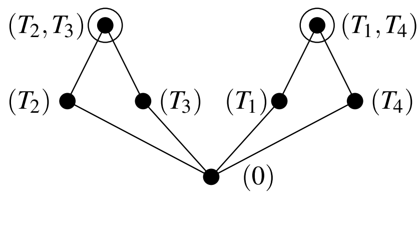


FIGURE 4. The special linear group $\mathrm{SL}_{2,\mathbb{F}_1}$

We like to draw the readers attention to the closed points, which are encircled in the illustration. If we write the coordinates in matrix form $\left(\begin{smallmatrix} T_1 & T_2 \\ T_3 & T_4\end{smallmatrix}\right)$, then the point (T_2,T_3) corresponds

[3] Note that $\mathcal{R}_Z$ might not satisfy axiom (iii) of a pre-addition. In this case, we have to take the quotient A' of A that identifies all elements a and b with $a \equiv b$. Then we obtain a pre-addition $\mathcal{R}'_Z$ on A', and it is the blueprint $A'/\!\!/\mathcal{R}'_Z$ that we mean when we write $A/\!\!/\mathcal{R}_Z$.

to the matrices $\left(\begin{smallmatrix} T_1 & 0 \\ 0 & T_4 \end{smallmatrix}\right)$ with $T_2 = T_3 = 0$. These form the diagonal torus T of SL_2. The other closed point (T_1, T_4) corresponds to the matrices $\left(\begin{smallmatrix} 0 & T_2 \\ T_3 & 0 \end{smallmatrix}\right)$, which form the antidiagonal torus $\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)T$ of SL_2. The two subschemes T and $\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)T$ coincide with the elements of the Weyl group W of SL_2.

This indicates a link between the closed points of $\mathrm{SL}_{2,\mathbb{F}_1}$ with the Weyl group of SL_2, which we would like to interpret as the group of $\mathbb{F}_1$-rational points of $\mathrm{SL}_{2,\mathbb{F}_1}$. Note, however, that $\mathrm{Hom}(\mathrm{Spec}\,\mathbb{F}_1, \mathrm{SL}_{2,\mathbb{F}_1})$ consists of three morphisms with respective images (T_2, T_3), (T_2) and (T_3).

Note that in a similar manner, we can define blue schemes that are $\mathbb{F}_1$-models of other split Chevalley groups. In [Lor12b], one finds a list of explicit constructions for SL_n, GL_n, ${}^+\mathbb{P}(2n)$ and $\mathrm{SO}(n)$ (both odd and even n). All these $\mathbb{F}_1$-models have in common that the set of closed points corresponds to the respective Weyl group. We will explain this connection in more detail in the upcoming lecture.

To give one explicit example: the blue scheme

$$\mathrm{GL}_{2,\mathbb{F}_1} = \mathrm{Spec}\big(\mathbb{F}_1[T_1, T_2, T_3, T_4, \delta] /\!\!/ \langle \delta T_1 T_4 \equiv \delta T_2 T_3 + 1 \rangle\big)$$

is an $\mathbb{F}_1$-model of GL_2, whose underlying topological space is homeomorphic to the topological space of $\mathrm{SL}_{2,\mathbb{F}_1}$. Note that δ plays the role of $\det\left(\begin{smallmatrix} T_1 & T_2 \\ T_3 & T_4 \end{smallmatrix}\right)^{-1}$.

Example 5.7 (Grassmannians)**.** By the strategy of paragraph 5.6, we can descend every closed subscheme of ${}^+\mathbb{P}^n_{\mathbb{Z}}$ to a subscheme of $P^n_{\mathbb{F}_1}$. This allows us to define Grassmannians over $\mathbb{F}_1$ as blue schemes with help of the defining Plücker relations. For instance $\mathrm{Gr}(2,4)_{\mathbb{Z}} = \mathrm{Proj}\big(\mathbb{Z}[T_0, \dots, T_5]/(T_0T_5 - T_1T_4 + T_2T_3)\big)$ descends to $\mathrm{Gr}(2,4)_{\mathbb{F}_1}$ whose underlying topological space can be illustrated as follows.

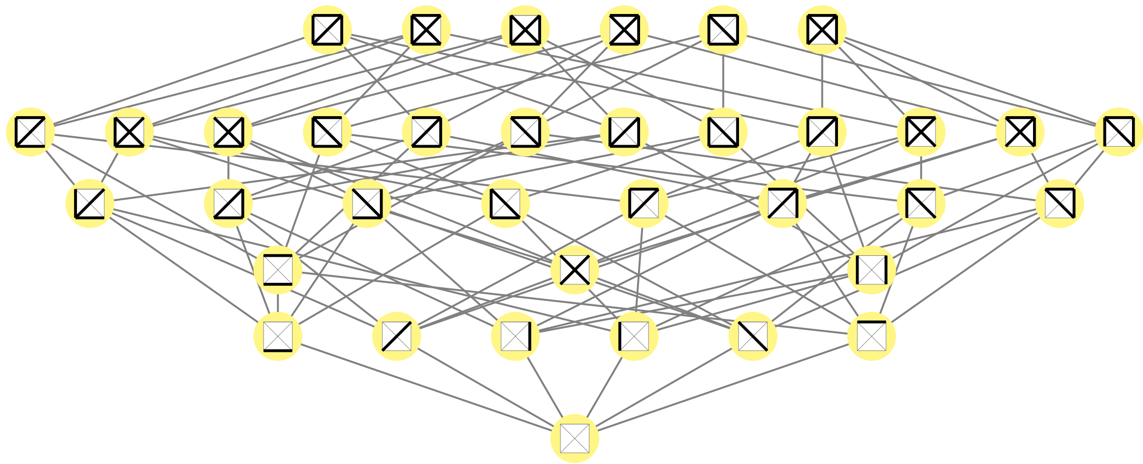


FIGURE 5. The Grassmannian $\mathrm{Gr}(2,4)_{\mathbb{F}_1}$

The bold lines indicate the generators of an ideal $\mathfrak{p} = (T_i)_{i\in I}$ of the "homogeneous coordinate blueprint" $\mathbb{F}_1[T_0, \dots, T_5] /\!\!/ \langle T_0T_5 + T_2T_3 \equiv T_1T_4 \rangle$ of $\mathrm{Gr}(2,4)_{\mathbb{F}_1}$. See [LPL12] for more details.

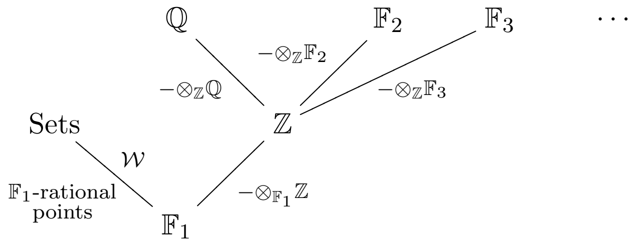
Note that, once more, the six closed points correspond to the expected set $M_{2,4}$ of $\mathbb{F}_1$-rational points of $\mathrm{Gr}(2,4)$, which has little in common with set morphism set $\mathrm{Hom}(\mathrm{Spec}\,\mathbb{F}_1, \mathrm{Gr}(2,4)_{\mathbb{F}_1})$. As we will see in Lecture 4, this behaviour occurs for all Grassmannians defined by Plücker relations: the closed points of $Gr(k,n)_{\mathbb{F}_1}$ correspond to the elements of the set $M_{k,n}$.

It will be the task of the next lecture to resolve the discrepancy between the expected value of the set of $\mathbb{F}_1$-rational points and the morphism set $\mathrm{Hom}(\mathrm{Spec}\,\mathbb{F}_1, -)$.

Lecture 3: $\mathbb{F}_1$-rational points and Chevalley groups over $\mathbb{F}_1$

6. The Problem with the concept of $\mathbb{F}_1$-rational points

To start with, we like to collect what we have achieved so far. Recall the diagram from the introduction in Lecture 1.



In the first two lectures, we have constructed $\mathbb{F}_1$-models of projective spaces, Grassmannians, SL_2, GL_2, and other schemes in terms of data that one can almost consider as natural givens: the fan of projective space; the Plücker relations for Grassmannians; the standard representations of SL_2 and GL_2. We saw that for these models the set of closed points corresponds to the sets that we expect as the sets of $\mathbb{F}_1$-rational points. This indicates that the theory of blue schemes contains $\mathbb{F}_1$-models of schemes that we are interested in, and gives a reasonable meaning to them.

We saw, however, that in none of the examples of a blue scheme X, the closed points of correspond to the morphisms in $\mathrm{Hom}(\operatorname{Spec}\mathbb{F}_1, X)$. If we like to stick with the theory of monoidal schemes and blue schemes, then this means that the functor $\mathcal{W}$ in the above diagram cannot be $\mathrm{Hom}(\operatorname{Spec}\mathbb{F}_1, -)$, which would be the set of $\mathbb{F}_1$-rational points in the literal sense. Another problem is that the group laws of Chevalley groups other than split tori do not descend to locally algebraic morphisms between the described $\mathbb{F}_1$-models.

There is another, more profound reason why $\mathcal{W}$ cannot be $\mathrm{Hom}(\operatorname{Spec}\mathbb{F}_1, -)$ if we want Tits' dream to become true. Let T be the diagonal torus of SL_2 and N its normalizer, which is $T\cup wT$ where $w = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$. Then the Weyl group of SL_2 (w.r.t. the diagonal torus) is per definition the quotient $W = N(\mathbb{Z})/T(\mathbb{Z})$. If we had an identification $W = \mathrm{Hom}(\operatorname{Spec}\mathbb{F}_1, \mathrm{SL}_{2,\mathbb{F}_1})$, then we obtain a map

$$\sigma:\ W \ =\ \mathrm{Hom}(\operatorname{Spec}\mathbb{F}_1, \mathrm{SL}_{2,\mathbb{F}_1}) \ \xrightarrow{-\otimes_{\mathbb{F}_1}\mathbb{Z}}\ \mathrm{Hom}(\operatorname{Spec}\mathbb{Z}, \mathrm{SL}_{2,\mathbb{Z}}) \ =\ \mathrm{SL}_2(\mathbb{Z}).$$

As explained in the introduction in Lecture 1, σ should map T to an element of $T(\mathbb{Z})$ and wT to an element of $wT(\mathbb{Z})$. If we furthermore expect that $-\otimes_{\mathbb{F}_1}\mathbb{Z}$ commutes with products, which is necessary to base extend group schemes over $\mathbb{F}_1$ to group schemes over $\mathbb{Z}$, then it follows that $\sigma: W \to \mathrm{SL}_2(\mathbb{Z})$ is a group homomorphism. This is, however, not possible since $wT(\mathbb{Z})$ does not contain an element of order 2.

In the course of this lecture, we will define the rank space of a blue scheme (under some simplifying assumptions), which is closely connected to the set of closed points of a blue scheme. The rank space is the key to define the functor $\mathcal{W}$, called the Weyl extension functor, and a class of so-called Tits morphisms that allow us to descend group laws to $\mathbb{F}_1$.

7. The rank space

A blueprint B is *cancellative* if $\sum a_i + c \equiv \sum b_j + c$ implies $\sum a_i \equiv \sum b_j$. Note that a blueprint B is cancellative if and only if the canonical morphism $B \to B_{\mathbb{Z}}^+$ is injective. A blue scheme is *cancellative* if it is covered by spectra of cancellative blueprints. Recall that $\mathbb{F}_1$ is the monoid $\{0,1\}$ and that $\mathbb{F}_{1^2}$ is the blueprint $\{0,\pm 1\}/\!\!/\langle 1 + (-1) \equiv 0\rangle$.

Let X be a blue scheme and $x \in X$ a point. As in usual scheme theory, every closed subset Z of X comes with a natural structure of a *(reduced) closed subscheme of* X, cf. [Lor12b, Section 1.4]. The *rank* $\operatorname{rk} x$ *of* x is the dimension of the $\mathbb{Q}$-scheme $\overline{x}^+_{\mathbb{Q}}$ where $\overline{x}$ denotes the closure of x in X together with its natural structure as a closed subscheme. Define

$$r \quad = \quad \min \{ \operatorname{rk} x \mid x \in X \}.$$

For the sake of simplicity, we will make the following general hypothesis on X.

(H) The blue scheme X is connected and cancellative. For all $x \in X$ with $\operatorname{rk} x = r$, the closed subscheme $\overline{x}$ of X is isomorphic to either $\mathbb{G}^r_{m,\mathbb{F}_1}$ or $\mathbb{G}^r_{m,\mathbb{F}_{1^2}}$.

This hypothesis allows us to surpass certain technical aspects in the definition of the rank space.

Assume that X satisfies (H). Then the number r is denoted by $\operatorname{rk} X$ and is called the *rank of* X. The *rank space of* X is the blue scheme

$$X^{\mathrm{rk}} \quad = \quad \coprod_{\operatorname{rk} x = r} \overline{x},$$

and it comes together with a closed immersion $\rho_X : X^{\mathrm{rk}} \to X$.

Note that Hypothesis (H) implies that X^{rk} is the disjoint union of tori $\mathbb{G}^r_{m,\mathbb{F}_1}$ and $\mathbb{G}^r_{m,\mathbb{F}_{1^2}}$. Since the underlying set of both $\mathbb{G}^r_{m,\mathbb{F}_1}$ and $\mathbb{G}^r_{m,\mathbb{F}_{1^2}}$ is the one-point set, the underlying set of X^{rk} is $\mathcal{W}(X) = \{x \in X | \operatorname{rk} x = r\}$. Note further that $X^{\mathrm{rk},+}_{\mathbb{Z}} \simeq \coprod_{\operatorname{rk} x = r} {}^+\mathbb{G}^r_{m,\mathbb{Z}}$.

8. The Tits category and the Weyl extension

A *Tits morphism* $\varphi : X \to Y$ between two blue schemes X and Y is a pair $\varphi = (\varphi^{\mathrm{rk}}, \varphi^+)$ of a locally algebraic morphism $\varphi^{\mathrm{rk}} : X^{\mathrm{rk}} \to Y^{\mathrm{rk}}$ and a locally algebraic morphism $\varphi^+ : X^+ \to Y^+$ such that the diagram

$$\begin{tikzcd}
X^{\mathrm{rk},+}_{\mathbb{Z}} \arrow[r, "\varphi^{\mathrm{rk},+}_{\mathbb{Z}}"] \arrow[d, "\rho^+_{X,\mathbb{Z}}"'] & Y^{\mathrm{rk},+}_{\mathbb{Z}} \arrow[d, "\rho^+_{Y,\mathbb{Z}}"] \\
X^+_{\mathbb{Z}} \arrow[r, "\varphi^+_{\mathbb{Z}}"] & Y^+_{\mathbb{Z}}
\end{tikzcd}$$

commutes where $\varphi^+_{\mathbb{Z}}$ is the base extension of φ^+ to Grothendieck schemes and $\varphi^{\mathrm{rk},+}_{\mathbb{Z}}$ is the base extension of φ^{rk} to Grothendieck schemes. We denote the category of blue schemes together with Tits morphisms by $\mathrm{Sch}_{\mathcal{T}}$ and call it the *Tits category*.

The Tits category comes together with two important functors. The *Weyl extension* $\mathcal{W}$: $\mathrm{Sch}_{\mathcal{T}} \to \mathrm{Sets}$ sends a blue scheme X to the underlying set $\mathcal{W}(X) = \{x \in X | \operatorname{rk} x = r\}$ of X^{rk} and a Tits morphism $\varphi : X \to Y$ to the underlying map $\mathcal{W}(\varphi) : \mathcal{W}(X) \to \mathcal{W}(Y)$ of the morphism $\varphi^{\mathrm{rk}} : X^{\mathrm{rk}} \to Y^{\mathrm{rk}}$. The base extension $(-)^+ : \mathrm{Sch}_{\mathcal{T}} \to \mathrm{Sch}^+$ sends a blue scheme X to its universal semiring scheme X^+ and a Tits morphism $\varphi : X \to Y$ to $\varphi^+ : X^+ \to Y^+$. We obtain the following diagram of "base extension functors"

$$\begin{tikzcd}
\mathrm{Sets} \arrow[ddrr, dash, "\mathcal{W}"'] & & \mathrm{Sch}^+_{\mathbb{Z}} \arrow[dr, dash, "(-)^+_{\mathbb{Z}}"'] & & \mathrm{Sch}^+_R \arrow[dl, dash, "(-)^+_R"] \\
& & & \mathrm{Sch}^+_{\mathbb{N}} \arrow[dl, dash, "(-)^+"] & \\
& & \mathrm{Sch}_{\mathcal{T}} & &
\end{tikzcd}$$

from the Tits category $\mathrm{Sch}_{\mathcal{T}}$ to the category Sets of sets, to the category $\mathrm{Sch}_{\mathbb{Z}}^{+}$ of Grothendieck schemes and to the category Sch_{R}^{+} of semiring schemes over any semiring R.

Theorem 8.1 ([Lor12b, Thm. 3.8]). *All functors appearing in the above diagram commute with finite products. Consequently, all functors send (semi)group objects to (semi)group objects.*

9. Tits-Weyl models

A *Tits monoid* is a (not necessarily commutative) monoid in $\mathrm{Sch}_{\mathcal{T}}$, i.e. a blue scheme G together with an associative multiplication $\mu : G \times G \to G$ in $\mathrm{Sch}_{\mathcal{T}}$ that has an identity $\epsilon : \operatorname{Spec}\mathbb{F}_1 \to G$. We often understand the multiplication μ implicitly and refer to a Tits monoid simply by G. The Weyl extension $\mathcal{W}(G)$ of a Tits monoid G is a unital associative semigroup. The base extension G^{+} is a (not necessarily commutative) monoid in $\mathrm{Sch}_{\mathbb{N}}^{+}$.

Given a Tits monoid G satisfying (H) with multiplication μ and identity ϵ, then the image of $\epsilon : \operatorname{Spec}\mathbb{F}_1 \to G$ consists of a closed point e of X. The closed reduced subscheme $\mathfrak{e} = \{e\}$ of G is called the *Weyl kernel of G*.

Lemma 9.1 ([Lor12b, Lemma 3.11]). *The multiplication μ restricts to $\mathfrak{e}$, and with this, $\mathfrak{e}$ is isomorphic to the torus $\mathbb{G}_{m,\mathbb{F}_1}^{r}$ as a group scheme.*

This means that $\mathfrak{e}_{\mathbb{Z}}^{+}$ is a split torus $T \simeq {}^{+}\mathbb{G}_{m,\mathbb{Z}}^{r}$ of $\mathcal{G} = G_{\mathbb{Z}}^{+}$, which we call the *canonical torus of $\mathcal{G}$ (w.r.t. G)*.

If $\mathcal{G}$ is an affine smooth group scheme of finite type, then we obtain a canonical morphism

$$\Psi_{\mathfrak{e}} : \; G_{\mathbb{Z}}^{\mathrm{rk},+}/\mathfrak{e}_{\mathbb{Z}}^{+} \quad \longrightarrow \quad W(T)$$

where $W(T) = \mathrm{Norm}_{\mathcal{G}}(T)/\,\mathrm{Cent}_{\mathcal{G}}(T)$ is the *Weyl group of $\mathcal{G}$ w.r.t. T*. We say that G is a *Tits-Weyl model of $\mathcal{G}$* if T is a maximal torus of $\mathcal{G}$ and $\Psi_{\mathfrak{e}}$ is an isomorphism.

This definition has some immediate consequences. We review some definitions, before we state Theorem 9.2. The *ordinary Weyl group of $\mathcal{G}$* is the underlying group W of $W(T)$. The *reductive rank of $\mathcal{G}$* is the rank of a maximal torus of $\mathcal{G}$. For a split reductive group scheme, we denote the *extended Weyl group* or *Tits group* $\mathrm{Norm}_{\mathcal{G}}(T)(\mathbb{Z})$ by $\widetilde{W}$ (cf. [Lor12b, Section 3.3]).

For a blueprint B, the set $G^{\mathcal{T}}(B)$ of Tits morphisms from $\operatorname{Spec} B$ to G inherits the structure of an associative unital semigroup. In case, G has several connected components, we define the *rank of G* as the rank of the connected component of G that contains the image of the unit $\epsilon : \operatorname{Spec}\mathbb{F}_1 \to G$.

Theorem 9.2 ([Lor12b, Thm. 3.14]). *Let $\mathcal{G}$ be an affine smooth group scheme of finite type. If $\mathcal{G}$ has a Tits-Weyl model G, then the following properties hold true.*

(i) *The Weyl group $\mathcal{W}(G)$ is canonically isomorphic to the ordinary Weyl group W of $\mathcal{G}$.*
(ii) *The rank of G is equal to the reductive rank of $\mathcal{G}$.*
(iii) *The semigroup $G^{\mathcal{T}}(\mathbb{F}_1)$ is canonically a subgroup of $\mathcal{W}(G)$.*
(iv) *If $\mathcal{G}$ is a split reductive group scheme, then $G^{\mathcal{T}}(\mathbb{F}_{1^2})$ is canonically isomorphic to the extended Weyl group $\widetilde{W}$ of $\mathcal{G}$.*

The following theorem is proven in [Lor12b] for a large class of split reductive group schemes $\mathcal{G}$ and their Levi- and parabolic subgroups. Markus Reineke added an idea, which helped to extend it to all split reductive group schemes.

Theorem 9.3. (i) *Every split reductive group scheme $\mathcal{G}$ has a Tits-Weyl model G.*
(ii) *Let T be the canonical torus of $\mathcal{G}$ and $\mathcal{M}$ a Levi subgroup of $\mathcal{G}$ containing T. Then $\mathcal{M}$ has a Tits-Weyl model M that comes together with a locally closed embedding $M \to G$ of Tits-monoids that is a Tits morphism.*

(iii) *Let $\mathcal{P}$ a parabolic subgroup of $\mathcal{G}$ containing T. Then $\mathcal{P}$ has a Tits-Weyl model P that comes together with a locally closed embedding $P \to G$ of Tits-monoids that is a Tits morphism.*

(iv) *Let $\mathcal{U}$ be the unipotent radical of a parabolic subgroup $\mathcal{P}$ of $\mathrm{GL}_{n,\mathbb{Z}}$ that contains the diagonal torus T. Then $\mathcal{U}$, $\mathcal{P}$ and $\mathrm{GL}_{n,\mathbb{Z}}$ have respective Tits-Weyl models U, P and $\mathrm{GL}_{n,\mathbb{F}_1}$, together with locally closed embeddings $U \to P \to \mathrm{GL}_{n,\mathbb{F}_1}$ of Tits-monoids that are Tits morphisms and such that T is the canonical torus of $\mathcal{P}$ and $\mathrm{GL}_{n,\mathbb{Z}}$.*

10. Total positivity

Matrix coefficients of particular importance are the generalized minors of Fomin and Zelevinsky and elements of Lustzig's canonical basis. There is a link to Tits-Weyl models, but this is far from being well-understood. In the example of SL_n, however, we can make the connection to generalized minors precise.

Let I and J be subsets of $\{1,\dots,n\}$ of the same cardinality k. Then we can consider the matrix minors

$$\Delta_{I,J}(A) \quad = \quad \det(a_{i,j})_{i\in I, j\in J}$$

of an $n\times n$-matrix $A=(a_{i,j})_{i,j=1,\dots,n}$. Following Fomin and Zelevinsky, we say that a real matrix A of determinant 1 is *totally non-negative* if $\Delta_{I,J}(A)\geq 0$ for all matrix minors $\Delta_{I,J}$.

Let $\mathbb{F}_1[\Delta_{I,J}]$ be the free monoid with zero that is generated by all matrix minors $\Delta_{I,J}$ for varying $k\in\{1,\dots,n-1\}$ and subsets $I,J\subset\{1,\dots,n\}$ of cardinality k. Let $\mathcal{R}$ be the pre-addition of $\mathbb{F}_1[\Delta_{I,J}]$ that consists of all additive relations $\sum a_i \equiv \sum b_j$ between generalized minors that hold in the coordinate ring $\mathbb{Z}[\mathrm{SL}_n]$ of SL_n over $\mathbb{Z}$. Define the blueprint

$$\mathbb{F}_1[SL_n] \quad = \quad \mathbb{F}_1[\Delta_{I,J}] /\!\!/ \mathcal{R}.$$

and the blue scheme $\mathrm{SL}_{n,\mathbb{F}_1}=\operatorname{Spec}\mathbb{F}_1[SL_n]$. The following theorem is due to Javier López Peña, Markus Reineke and the author.

Theorem 10.1. *The blue scheme $\mathrm{SL}_{n,\mathbb{F}_1}$ has a unique structure of a Tits-Weyl model of $\mathrm{SL}_{n,\mathbb{Z}}$. It satisfies that $\mathrm{SL}_{n,\mathbb{F}_1}(\mathbb{R}_{\geq 0})$ is the semigroup of all totally non-negative matrices.*

Remark 10.2. The Tits-Weyl model $\mathrm{SL}_{n,\mathbb{F}_1}$ yields the notion of matrices $A\in\mathrm{SL}_n(R)$ with coefficients in any semiring R. This might be of particular interest in the case of idempotent semirings as the tropical real numbers.

Lecture 4: Quiver Grassmannians and Euler characteristics

11. Results on Euler characteristics, so far

In the previous lectures, we have seen that there is a natural correspondence

$$\mathcal{W}(\mathbb{P}^n_{\mathbb{F}_1}) \;=\; \big\{ [1:0:\dots:0], \dots, [0:\dots:0:1] \big\},$$

which shows that the cardinality of $\mathcal{W}(\mathbb{P}^n_{\mathbb{F}_1})$ equals the Euler characteristic of the complex projective n-space, i.e.

$$\#\,\mathcal{W}(\mathbb{P}^n_{\mathbb{F}_1}) \;=\; n+1 \;=\; \chi\big(\mathbb{P}^n(\mathbb{C})\big).$$

This phenomenon extends to the Grassmannians $\mathrm{Gr}(k,n)_{\mathbb{F}_1}$ that are defined by Plücker relations.

Proposition 11.1. *The blue scheme* $\mathrm{Gr}(k,n)_{\mathbb{F}_1}$ *satisfies Hypothesis (H) and*

$$\#\,\mathcal{W}\big(\mathrm{Gr}(k,n)_{\mathbb{F}_1}\big) \;=\; \binom{n}{k} \;=\; \chi\big(\mathrm{Gr}(k,n)(\mathbb{C})\big).$$

We would like to extend this connection between the cardinality of the Weyl extension of a projective blue scheme and the Euler characteristic of the space of complex points to a larger class of varieties. A useful fact is the following.

Lemma 11.2. *The rank of any non-empty closed connected subscheme of* $\mathbb{P}^n_{\mathbb{F}_1}$ *with Hypothesis (H) is* 0.

Proof. Let X be a closed subscheme of $\mathbb{P}^n_{\mathbb{F}_1}$ with Hypothesis (H). Let $x \in X$ be a point. If $\mathrm{rk}\, x = 0$, then we have nothing to prove. If $\mathrm{rk}\, x > 0$, then the $\{x\}^+_{\mathbb{Q}}$ is affine and a locally closed subscheme of ${}^+\mathbb{P}^n_{\mathbb{Q}}$. Its closure in ${}^+\mathbb{P}^n_{\mathbb{Q}}$ contains a point that maps to some point $y \neq x$ of X, and $\{y\}^+_{\mathbb{Q}}$ is of a smaller dimension than $\{x\}^+_{\mathbb{Q}}$. Thus $\mathrm{rk}\, y < \mathrm{rk}\, x$, which shows that X always contains a point of rank 0. □

The following examples show some limitations of the interpretation of the Euler characteristic of the complex scheme as the cardinality of the Weyl extension.

Example 11.3 (Elliptic curve)**.** Let X be the closed subscheme of $\mathbb{P}^2_{\mathbb{F}_1}$ that is associated to the elliptic curve with equation $T_2^2T_3 = T_1^3 + T_1T_3^2$ in $\mathbb{P}^2_{\mathbb{Z}}$. Then we see that $X^+_{\mathbb{Z}}$ contains two k-rational points with coordinates $[0:1:0]$ and $[0:0:1]$, and all other k-rational points are of the form $[t_1:t_2:t_3]$ with $t_1t_2t_3 \neq 0$. This means that the image of the immersion $X \to \mathbb{P}^2_{\mathbb{F}_1}$ consists of the three points $[0:1:0]$, $[0:0:1]$ and $[1:1:1]$, cf. 3.7. The former two points are of rank 0 while $[1:1:1]$ is of rank 1. Since the residue fields of the two rank 0-points is $\mathbb{F}_1$, i.e. the subscheme supported at these points is isomorphic to $\mathbb{G}^0_{m,\mathbb{F}_1}$, the cubic curve X over $\mathbb{F}_1$ satisfies Hypothesis (H).

From this we see that $\mathcal{W}(X) = \{[0:1:0],[0:0:1]\}$. However, the Euler characteristic of the complex spaces $X(\mathbb{C})$, which is a topological torus, is 0.

Example 11.4 (Skew projective line)**.** Consider the closed subscheme X of $\mathbb{P}^2_{\mathbb{F}_1}$ that is associated with the closed subscheme of $\mathbb{P}^2_{\mathbb{Z}}$ that is defined by the equation $T_1+T_2+T_3=0$. Then we see that X has three closed points "$[1:-1:0]$", "$[-1:0:1]$" and "$[0:1:-1]$", which are of rank 0 and whose residue field is $\mathbb{F}_{1^2}$, i.e. the subscheme supported at these points is isomorphic to $\mathbb{G}^0_{m,\mathbb{F}_{1^2}}$. Thus X satisfies Hypothesis (H) and its Weyl extension is

$$\mathcal{W}(X) \;=\; \{[1:-1:0],[-1:0:1],[0:1:-1]\}.$$

However, $X^+_{\mathbb{Z}}$ is isomorphic to $\mathbb{P}^1_{\mathbb{Z}}$, and thus $\mathcal{W}(X) = 3 \neq 2 = \chi\big(X(\mathbb{C})\big)$.

12. Quiver Grassmannians

Fomin and Zelevinsky introduced cluster algebras as a tool to better understand Lusztig's canonical bases for quantum groups. A priori, cluster algebras are defined in terms of a recursive process, which makes it difficult to get hold of them. In 2006, however, Caldero and Chapoton published a formula that expresses cluster variables as elementary functions in the Euler characteristics of quiver Grassmannians. This formula made it possible to describe cluster algebras explicitly in terms of generators and relation and caused an active interest in the Euler characteristics of quiver Grassmannians in recent years.

This creates a mutual interest between quiver Grassmannians and $\mathbb{F}_1$-geometry. On the one side, $\mathbb{F}_1$-geometry searches for structure that rigidifies schemes in order to descend them to $\mathbb{F}_1$. It turns out that quiver Grassmannians come with such a rigidifying structure. On the other side, a successful interpretation of the Euler characteristic of a complex scheme in terms of its $\mathbb{F}_1$-rational points might give access to new methods to compute Euler characteristics.

Note that every projective scheme can be realized as a quiver Grassmannian (see [Rei13]). Therefore quiver Grassmannians are suited as a device to define $\mathbb{F}_1$-models of projective schemes. We will explain how this works in the following. We refer to Section 4 of [Lor12b] for references on the notions in the rest of this lecture.

A *quiver* is a finite directed graph Q. We denote the vertex set by Q_0, the set of arrows by Q_1, the source map by $s : Q_1 \to Q_0$ and the target map by $t : Q_1 \to Q_0$. A *(complex) representation of* Q is a collection of complex vector spaces M_i for each vertex $i \in Q_0$ together with a collection of $\mathbb{C}$-linear maps $M_\alpha : M_i \to M_j$ for each arrow $\alpha : i \to j$ in Q_1. We write $M = \big((M_i), (M_\alpha)\big)$ for a representation of Q. The *dimension vector* $\underline{\dim}\, M$ *of* M is the tuple $\underline{d} = (d_i)$ where d_i is the dimension of M_i. A *subrepresentation of* M is a collection of $\mathbb{C}$-linear subspaces N_i of M_i for each vertex $i \in Q_0$ such that the linear maps $M_\alpha : M_i \to M_j$ restrict to linear maps $N_i \to N_j$ for all arrows $\alpha : i \to j$.

Let $\underline{e} = (e_i)$ be a tuple of non-negative integers with $e_i \leq d_i$ for all $i \in Q_0$. Then there is a complex scheme $\mathrm{Gr}_{\underline{e}}(M)_{\mathbb{C}}$ whose set of $\mathbb{C}$-rational points corresponds to the set

$$\mathrm{Gr}_{\underline{e}}(M, \mathbb{C}) \quad = \quad \big\{ \, N \subset M \text{ subrepresentation} \, \big| \, \underline{\dim}\, N = \underline{e} \, \big\}.$$

The choice of a basis $\mathcal{B}_i$ for each vector space M_i defines a closed embedding

$$\mathrm{Gr}_{\underline{e}}(M)_{\mathbb{C}} \quad \longrightarrow \quad \prod_{i \in Q_0} \mathrm{Gr}(e_i, d_i)_{\mathbb{C}}$$

of the quiver Grassmannian into a product of usual Grassmannians. This endows $\mathrm{Gr}_{\underline{e}}(M)$ with the structure of a complex variety.

Remark 12.1. In fact, $\mathrm{Gr}_{\underline{e}}(M)_{\mathbb{C}}$ comes with the structure of a complex scheme. In words, the scheme theoretic definition of $\mathrm{Gr}_{\underline{e}}(M)$ is as the fibre at M of the universal Q-Grassmannian of $\underline{e}$-dimensional subrepresentations that fibres over the moduli space of representations of Q. For the following considerations, however, it suffices to consider $\mathrm{Gr}_{\underline{e}}(M)_{\mathbb{C}}$ as a complex variety.

12.1. **$\mathbb{F}_1$-models of quiver Grassmannians.** Let Q be a quiver, M be a representation of Q with basis $\mathcal{B} = \bigcup \mathcal{B}_i$ and $\underline{e}$ a dimension vector for Q. We assume that the $\mathbb{C}$-linear maps M_α correspond to matrices with integral coefficients w.r.t. the basis $\mathcal{B}$. We call such a basis an *integral basis of* M. An integral basis yields an integral model $\mathrm{Gr}_{\underline{e}}(M)$ of the complex quiver Grassmannian $\mathrm{Gr}_{\underline{e}}(M)_{\mathbb{C}}$.

Define $|\underline{e}| = \sum e_i$, $|\underline{d}| = \sum d_i$ and $m = \binom{|\underline{d}|}{|\underline{e}|} - 1$. Consider the series of closed embeddings

$$\mathrm{Gr}_{\underline{e}}(M) \quad \longrightarrow \quad \prod_{i \in Q_0} \mathrm{Gr}(e_i, d_i) \quad \longrightarrow \quad \mathrm{Gr}(|\underline{e}|\,, |\underline{d}|) \quad \longrightarrow \quad {}^{+}\mathbb{P}^m_{\mathbb{Z}}.$$

As explained in Paragraph 5.6, this defines the $\mathbb{F}_1$-model $\mathrm{Gr}_{\underline{e}}(M)_{\mathbb{F}_1}$ of $\mathrm{Gr}_{\underline{e}}(M)$.

12.2. **Tree modules.** Let M be a representation of Q with basis $\mathcal{B}$. Let $\alpha : s \to t$ be an arrow of Q and $b \in \mathcal{B}_s$. Then we have the equations

$$M_\alpha(b) \quad = \quad \sum_{b' \in \mathcal{B}_t} \lambda_{\alpha,b,c}\, c$$

with uniquely determined coefficients $\lambda_{\alpha,b,c} \in \mathbb{C}$. The *coefficient quiver of M w.r.t. $\mathcal{B}$* is the quiver $\Gamma = \Gamma(M,\mathcal{B})$ with vertex set $\Gamma_0 = \mathcal{B}$ and with arrow set

$$\Gamma_1 \quad = \quad \left\{ \ (\alpha,b,c) \in Q_1 \times \mathcal{B} \times \mathcal{B} \ \middle| \ b \in \mathcal{B}_{s(\alpha)}, c \in \mathcal{B}_{t(\alpha)} \text{ and } \lambda_{\alpha,b,c} \neq 0 \ \right\}.$$

It comes together with a morphism $F : \Gamma \to Q$ that sends $b \in \mathcal{B}_p$ to p and (α,b,c) to α.

The representation M is called a *tree module* if there exists a basis $\mathcal{B}$ of M such that the coefficient quiver $\Gamma = \Gamma(M,\mathcal{B})$ is a disjoint union of trees. We call such a basis a *tree basis for* M.[4]

Note that if Γ is a disjoint union of trees, then we can replace the basis elements b by certain non-zero multiples b' such that all $\lambda_{\alpha,b,c}$ equal 1. Thus we do not restrict the class of quiver Grassmannians of tree modules if we make the assumption that all coefficients of the linear maps M_α are equal to 0 or 1.

12.3. **Euler characteristics.** Let M be a representation of Q with basis $\mathcal{B}$ and coefficient quiver Γ. We say that the canonical morphism $F : \Gamma \to Q$ is *unramified* if it a locally injective map of the underlying CW-complexes.

Let X be a blue scheme and $X = \coprod_{i \in I} X_i$ a decomposition into connected blue schemes X_i that satisfy all Hypothesis (H). Then we define

$$\mathcal{W}(X) \ = \ \coprod_{i \in I} \mathcal{W}(X_i).$$

Theorem 12.2. *Assume that $\mathcal{B}$ is a tree basis for M such that for all $\alpha \in Q_1$ the matrix coefficients of M_α are 0 or 1. Assume further that $F : \Gamma \to Q$ is unramified. Then every connected component of $\mathrm{Gr}_{\underline{e}}(M)$ satisfies Hypothesis (H) and*

$$\#\, \mathcal{W}\big(\mathrm{Gr}_{\underline{e}}(M)_{\mathbb{F}_1}\big) \ = \ \chi\big(\mathrm{Gr}_{\underline{e}}(M,\mathbb{C})\big).$$

Proof. This proof is based on *Cerulli's trick* (see [CI11]). We can define a torus action on $X = \mathrm{Gr}_{\underline{e}}(M)$ as follows. We define a weight function $w : \mathcal{B} \to \mathbb{N}$ such that whenever there are two arrows

$$b_1 \xrightarrow{\alpha} b_1' \qquad \text{and} \qquad b_2 \xrightarrow{\alpha} b_2'$$

in Γ with the same image α in Q, then $w(b_2) - w(b_1) = w(b_2') - w(b_1')$. Consequently, the number $w(\alpha) = w(b_1') - w(b_1)$ does not depend on the choice of inverse image of α in Γ. We define the action of $T = \mathbb{G}_m$ on $\mathrm{Gr}_{\underline{e}}(M)$ by the rule

$$t.b_i \ = \ t^{w(b_i)} \cdot b_i.$$

Then we have for all vectors $v = \sum a_l b_l$ in M_i and $\alpha : i \to j$ in Q that

$$M_\alpha t.v \ = \ M_\alpha \cdot \sum t^{w(b_l)} a_l b_l \ = \ t^{-w(\alpha)} \sum t^{w(b_l')} a_l b_l' \ = \ t^{-w(\alpha)} M_\alpha v,$$

which is a scalar multiple of $M_\alpha v$. Thus for a subrepresentation N of M, the collection $t.N$ of subspaces of M is still a subrepresentation.

[4]Usually, one requires Γ to be connected if one refers to tree bases. Since the techniques that we intend to use in the following do work also for disconnected Γ, we weaken the axioms on a tree basis for our purpose.

The fix point set X^T of this torus action is a closed subvariety of $\mathrm{Gr}_{\underline{e}}(M)$. There is a choice of weight function with sufficiently many different weights such that X^T is a proper subvariety of $\mathrm{Gr}_{\underline{e}}(M)$. Since the Euler characteristic is additive in decompositions into locally closed subsets and multiplicative in fibrations, we have

$$\chi\big(\mathrm{Gr}_{\underline{e}}(M,\mathbb{C})\big) \;=\; \chi(X^T) \;+\; \chi((X-X^T)/T)\cdot\underbrace{\chi(T)}_{=0} \;=\; \chi(X^T).$$

Note that there is not necessarily a choice of weight function such that X^T is a finite set of points. However, we can choose a second, refined weight function for X^T that defines a new torus action on X^T with a proper subvariety of fix points. With help of this procedure, we end up with a finite number of points X_0 and the equality

$$\chi\big(\mathrm{Gr}_{\underline{e}}(M,\mathbb{C})\big) \;=\; \#\, X_0.$$

Since we are scaling basis vectors, subrepresentations N that are spanned by basis vectors will always be contained in the fix points sets. On the other hand, we have to choose the weights sufficiently different to exclude all other subrepresentation if we want to end up with a finite set X_0. Therefore X_0 corresponds to the subrepresentations N of M with $\underline{\dim}\, N = \underline{e}$ that are spanned by basis vectors.

To see the connection to the Weyl extension $\mathcal{W}(\mathrm{Gr}_{\underline{e}}(M)_{\mathbb{F}_1})$, consider the base extension morphism $\beta_X : \mathrm{Gr}_{\underline{e}}(M)_{\mathbb{Z}} \to \mathrm{Gr}_{\underline{e}}(M)_{\mathbb{F}_1}$. Since for a vector $v = \sum a_l b_l$ of M_i with $i \in Q_0$

$$a_l \;= < v, b_l > \neq\; 0 \qquad \text{if and only if} \qquad t^{w(b_l)} \cdot a_l \;= < t.v, b_l > \neq\; 0,$$

we have that the image $\beta_X(T.N) = \{\beta_X(N)\}$ of the orbit $T.N$ in $\mathrm{Gr}_{\underline{e}}(M)_{\mathbb{F}_1}$ consists of one point. Thus the rank 0-points of $\mathrm{Gr}_{\underline{e}}(M)_{\mathbb{F}_1}$ correspond precisely to the fix points of the torus action of T.

By considering the Plücker coordinates of $\mathrm{Gr}_{\underline{e}}(M)$, it is easy to see that $\overline{x}$ is isomorphic to $\mathbb{G}^0_{m,\mathbb{F}_1}$ for every rank 0-point x. This shows that every connected component of $\mathrm{Gr}_{\underline{e}}(M)$ satisfies Hypothesis (H) and that the Weyl extension stays in bijection to X_0, i.e. $\#\mathcal{W}(\mathrm{Gr}_{\underline{e}}(M)_{\mathbb{F}_1}) = \chi(\mathrm{Gr}_{\underline{e}}(M,\mathbb{C}))$. $\square$

Example 12.3 (A del Pezzo surface of degree 6)**.** Let Q be the quiver

$$\begin{tikzcd}
x \arrow[dr, "\alpha"] & & \\
& t & y \arrow[l, "\gamma"'] \\
z \arrow[ur, "\eta"'] & &
\end{tikzcd}$$

of type D_4 and M the representation

$$\begin{tikzcd}
\mathbb{C}^2 \arrow[dr, "\left(\begin{smallmatrix}1&0\\0&1\\0&0\end{smallmatrix}\right)"] & & \\
& \mathbb{C}^3 & \mathbb{C}^2 \arrow[l, "\left(\begin{smallmatrix}1&0\\0&0\\0&1\end{smallmatrix}\right)"'] \\
\mathbb{C}^2 \arrow[ur, "\left(\begin{smallmatrix}0&0\\1&0\\0&1\end{smallmatrix}\right)"'] & &
\end{tikzcd}$$

of Q. With the obvious choice of basis $\mathcal{B}$, the coefficient quiver Γ is

$$\begin{tikzcd}
 & 4 \arrow[drr, "\alpha"] & & & \\
5 \arrow[dr, "\alpha"] & & & 1 & 6 \arrow[l, "\gamma"'] \\
 & & 2 & & \\
8 \arrow[ur, "\eta"'] & & & 3 & 7 \arrow[l, "\gamma"'] \\
 & 9 \arrow[urr, "\eta"'] & & &
\end{tikzcd}$$

where we use numbers $1, \dots, 9$ for the elements of $\mathcal{B}$ and where we label the arrows by its image under F.

We consider X as a closed subvariety of $\mathrm{Gr}(2,3) \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$. Note that for a subrepresentation N of M with dimension vector $\underline{e}$, the 1-dimensional subspaces N_x, N_y and N_z of M_x, M_y and M_z, respectively, determine the 2-dimensional subspace N_t of M_t uniquely. The images of $N_x = \langle \binom{x_0}{x_1} \rangle$, $N_y = \langle \binom{y_0}{y_1} \rangle$ and $N_z = \langle \binom{z_0}{z_1} \rangle$ in M_t lie in a plane if and only if

$$\det \begin{bmatrix} x_0 & y_0 & 0 \\ x_1 & 0 & z_0 \\ 0 & y_1 & z_1 \end{bmatrix} \quad = \quad -x_0\, y_1\, z_0 - x_1\, y_0\, z_1 \quad = \quad 0.$$

Therefore the projection $\mathrm{Gr}(2,3) \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ yields an isomorphism

$$\mathrm{Gr}_{\underline{e}}(M) \quad \xrightarrow{\sim} \quad \big\{ \, [\, x_0 : x_1 \,|\, y_0 : y_1 \,|\, z_0 : z_1 \,] \in \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \; \big| \; x_0\, y_1\, z_0 + x_1\, y_0\, z_1 = 0 \, \big\}.$$

Since there is no point in $\mathrm{Gr}_{\underline{e}}(M)$ for that all derivatives of the defining equation vanish, $\mathrm{Gr}_{\underline{e}}(M)$ is smooth.

The projection $\pi_{1,3} : \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1 \times \mathbb{P}^1$ to the first and third coordinate restricts to a surjective morphism $\pi_{1,3} : \mathrm{Gr}_{\underline{e}}(M) \to \mathbb{P}^1 \times \mathbb{P}^1$. It is bijective outside the fibres of $[1:0|0:1]$ and $[0:1|1:0]$, and these two fibres are

$$\pi_{1,3}^{-1}\big([\,1:0\,|\,0:1\,]\big) \quad = \quad \big\{ [\,1:0\,|\,y_0:y_1\,|\,0:1\,] \,\big\} \quad \simeq \quad \mathbb{P}^1$$

and

$$\pi_{1,3}^{-1}\big([\,0:1\,|\,1:0\,]\big) \quad = \quad \big\{ [\,0:1\,|\,y_0:y_1\,|\,1:0\,] \,\big\} \quad \simeq \quad \mathbb{P}^1.$$

This hsows that $\mathrm{Gr}_{\underline{e}}(M)_{\mathbb{C}}$ is a del-Pezzo surface of degree 6.

By the method of the proof of Theorem 12.2, we see that the elements of $\mathcal{W}(\mathrm{Gr}_{\underline{e}}(M)_{\mathbb{F}_1}$ correspond to the six points

$$\begin{array}{ccc} [\,1:0\,|\,1:0\,|\,1:0\,], & [\,0:1\,|\,1:0\,|\,1:0\,], & [\,0:1\,|\,0:1\,|\,1:0\,], \\ [\,0:1\,|\,0:1\,|\,0:1\,], & [\,1:0\,|\,0:1\,|\,0:1\,], & [\,1:0\,|\,1:0\,|\,0:1\,]. \end{array}$$

in $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$. These points correspond to the elements of the Weyl extension of $\mathrm{Gr}_{\underline{e}}(M)_{\mathbb{F}_1}$ and we have

$$\#\, \mathcal{W}\big(\mathrm{Gr}_{\underline{e}}(M)_{\mathbb{F}_1}\big) \;=\; \chi\big(\mathrm{Gr}_{\underline{e}}(M, \mathbb{C})\big).$$

These six points show some more particular behaviours. Namely, they are precisely the intersection points of pairs of the six -1-curves on the del Pezzo surface of degree six. The subrepresentations of M corresponding to these six points, considered as $\mathbb{C}$-rational points of $\mathrm{Gr}_{\underline{e}}(M)$, are precisely those subrepresentations of M that decompose into a direct sum of two non-trivial smaller representations.

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