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Blueprints and tropical scheme theory

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Preface

These lecture notes accompany a course that I am giving in the term March–June 2018 at IMPA. I intend to add chapters accordingly to the progress of these lectures and to regularly put new versions of these notes online. To make the changes between the different version more visible, each version will carry a distinct date on the front page. To make it possible to print these notes chapter by chapter, chapters will start on odd pages and contain a partial bibliography. To make changes in older parts of the lectures visible, each chapter carries the date of the last changes on its initial page.

Aim of these notes

In these notes, we will introduce blueprints and blue schemes and explain how this theory can be used to endow the tropicalization of a classical variety with a schematic structure.

Once the basic constructions are explained, we discuss balancing conditions and connections to related theories as skeleta of Berkovich spaces, toroidal embeddings and log-structures. We put a particular emphasis on explaining open problems in this very young branch of tropical geometry.

Main references

The main aim of this course is to explain (parts of) the material of the papers [GG14] and [GG16] by Jeffrey and Noah Giansiracusa, [MR14] and [MR16] by Maclagan and Rincón, and [Lor15] by the author. There will be plenty of secondary references, which we will cite at the appropriate places.

A useful complementary source are the lecture notes [YALE17] of a series of lectures at YALE, which were given by various experts in the area and organized by Mincheva and Payne.

I am grateful for any kind of feedback that helps me to improve these notes!

chapter last edited on
March 12, 2018

Chapter 1

Why tropical scheme theory?

In this first chapter, we explain the purpose of tropical scheme theory, its main achievements as of today and some of the central question of this new branch of tropical geometry. At the end of this chapter, we give a brief outline of the previsioned structure of the rest of these notes.

1.1 Tropical varieties

In brevity, a tropical variety is a balanced polyhedral complex. In this section, we explain this definition, starting with the case of a tropical curve, which is easier to formulate than its higher dimensional analogue.

Definition 1.1.1. A *tropical curve (in $\mathbb{R}^n$)* is an embedded graph Γ in $\mathbb{R}^n$ with possibly unbounded edges together with a weight function

$$m: \operatorname{Edge}\Gamma \longrightarrow \mathbb{Z}_{>0}$$

such that all edges have rational slopes and such that the following so-called *balancing condition* is satisfied for every vertex p of Γ: for every edge e containing p, let $v_e \in \mathbb{Z}^n$ be the *primitive vector*, which is the smallest nonzero vector pointing from p in the direction of e; then

$$\sum_{p\in e} m(e)\cdot v_e \;=\; 0.$$

Example 1.1.2. In Figure 1.1, we depict a tropical curve in $\mathbb{R}^2$, explaining the balancing condition at the three vertices of the curve.

The generalization of the involved notions to higher dimensions requires some preparation and leads us to the following definitions.

Definition 1.1.3. A *halfspace in* $\mathbb{R}^n$ is a subset of $\mathbb{R}^n$ of the form

$$H \;=\; \big\{ (x_1,\dots,x_n)\in\mathbb{R}^n \,\big|\, a_1x_1+\cdots+a_nx_n \geqslant b \big\}$$

with $a_1,\dots,a_n,b\in\mathbb{R}$. The halfspace H is *rational* if $a_1,\dots,a_n\in\mathbb{Q}$.

Definition 1.1.4. A *(rational) polyhedron P (in $\mathbb{R}^n$)* is an intersection of finitely many (rational) halfspaces in $\mathbb{R}^n$. A *face* of a polyhedron P is a nonempty intersection of P with a halfspace H such that the boundary of H does not contain interior points of P.

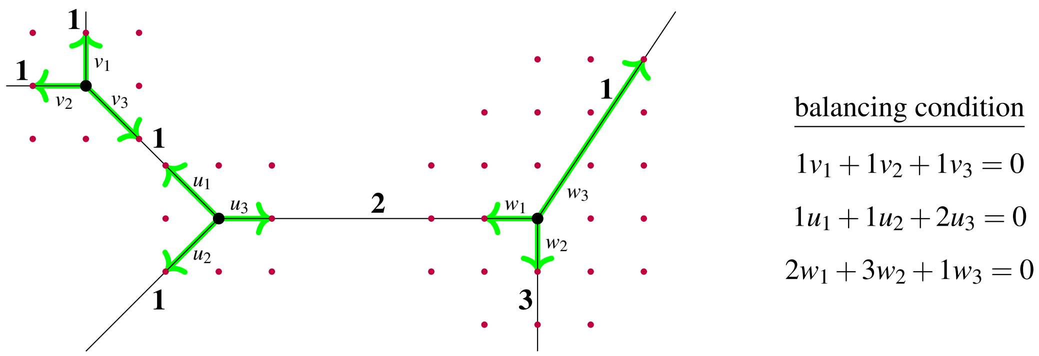

Figure 1.1: A tropical curve in $\mathbb{R}^2$ and the balancing condition

Note that the polyhedron P is a face of itself and that every face of a (rational) polyhedron is again a (rational) polyhedron.

Definition 1.1.5. A *polyhedral complex (in $\mathbb{R}^n$)* is a finite collection Δ of polyhedra in $\mathbb{R}^n$ such that the following two conditions are satisfied:

(1) each face of a polyhedron in Δ is in Δ;

(2) the intersection of two polyhedra in Δ is a face of both polyhedra or empty.

Definition 1.1.6. Let Δ be a polyhedral complex. The *support* of Δ is

$$|\Delta| \;=\; \bigcup_{P\in\Delta} P.$$

The *dimension* of Δ is $\dim\Delta \;=\; \max\{\dim P \,|\, P\in\Delta\}$. The polyhedral complex Δ is *equidimensional* if

$$|\Delta| \;=\; \bigcup_{\dim P=\dim\Delta} P$$

and Δ is *rational* if every polyhedron P in Δ is rational.

Exercise 1.1.7. Let H be a rational subvector space of $\mathbb{R}^n$, i.e. H has a basis in $\mathbb{Q}^n$. Show that the image of $\mathbb{Z}^n \subset \mathbb{R}^n$ under the quotient map $\pi : \mathbb{R}^n \to \mathbb{R}^n/H$ is a lattice, i.e. a discrete subgroup Λ that is isomorphic to $\mathbb{Z}^k$ where $k = n - \dim H$. The isomorphism $\Lambda \simeq \mathbb{Z}^k$ extends to an isomorphism $\mathbb{R}^n/H \simeq \mathbb{R}^k$ of vector spaces, i.e. we can identify π with a surjection $\pi' : \mathbb{R}^n \to \mathbb{R}^k$ that maps $\mathbb{Z}^n$ to $\mathbb{Z}^k$. Show that the image $\pi'(P)$ of a rational polyhedron P in $\mathbb{R}^n$ is a rational polyhedron in $\mathbb{R}^k$.

Let P be a rational polyhedron in $\mathbb{R}^n$ and $x_0 \in P$. Show that the subvector space H spanned by $\{x - x_0 | x \in P\}$ is rational and does not depend on the choice of x_0. Choose an isomorphism $\mathbb{R}^n/H \simeq \mathbb{R}^k$ as above. Conclude that the image $\overline{P}$ of P in $\mathbb{R}^k$ is a 0-dimensional rational polyhedron. More generally, let Q be rational polyhedron that contains P as a face. Show that the image $\overline{Q}$ of Q in $\mathbb{R}^k$ is a rational polyhedron of dimension $\dim Q - \dim P$.

We call the image $\overline{Q}$ under the quotient map $\pi' : \mathbb{R}^n \to \mathbb{R}^k$, as considered in Exercise 1.1.7, the *image of Q modulo the affine linear span of P*. If Q is a rational polyhedron of dimension $\dim Q = \dim P + 1$ that contains P as a face, then the image $\overline{Q}$ of Q in $\mathbb{R}^k$ is a one dimensional rational polyhedron that contains $\overline{P}$ as a boundary point. Thus we can speak of the *primitive vector* $v_{\overline{Q}}$ *of* $\overline{Q}$ *at* $\overline{P}$, which is the smallest nonzero vector in $\mathbb{R}^k$ with integral coefficients that is pointing from $\overline{P}$ in the direction of $\overline{Q}$.

Definition 1.1.8. A *tropical variety (in* $\mathbb{R}^n$*)* is an equidimensional and rational polyhedral complex Δ together with a weight function

$$m : \{P \in \Delta \big| \dim P = \dim \Delta\} \longrightarrow \mathbb{Z}_{>0}$$

such that for every polyhedron $P \in \Delta$ with $\dim P = \dim \Delta - 1$, the top dimensional polyhedra in Δ containing P satisfy the balancing modulo the affine linear span of P, i.e.

$$\sum_{P \subsetneq Q} m(Q) v_{\overline{Q}} \; = \; 0$$

where $\overline{P}$ and $\overline{Q}$ are the images of P and Q modulo the affine linear span of P and where $v_{\overline{Q}}$ is the primitive vector of $\overline{Q}$ at $\overline{P}$.

1.2 Tropicalization of classical varieties

Let k be a field.

Definition 1.2.1. A *nonarchimedean absolute value of* k is a function $v : k \to \mathbb{R}_{\geqslant 0}$ such that for all $a, b \in k$,

(1) $v(0) = 0$ and $v(1) = 1$;

(2) $v(ab) = v(a)v(b)$;

(3) $v(a+b) \leqslant \max\{v(a), v(b)\}$.

An nonarchimedean absolute value is *trivial* if $v(a) = 1$ for all $a \in k^\times$. Otherwise it is called *nontrivial*. An nonarchimedean absolute value is *discrete* if $v(k^\times)$ is a discrete subset of $\mathbb{R}_{\geqslant 0}$.

A *nonarchimedean field* is an algebraically closed field k together with a nontrivial nonarchimedean absolute value v.

Exercise 1.2.2. Let v be a nonarchimedean absolute value on a field k. Show the following assertions.

(1) If v is trivial, then v is discrete. If k is algebraically closed and v is discrete, then v is trivial. Give an example of a discrete absolute value that is not trivial. If v is not discrete, then its image in $\mathbb{R}_{\geqslant 0}$ is dense.

(2) We have $v(k^\times) \subset \mathbb{R}_{>0}$ and $v(-1) = 1$. If $v(a) \neq v(b)$, then $v(a+b) = \max\{v(a), v(b)\}$. Conclude that if $\sum_{i=1}^n a_i = 0$ in k, then at least two terms $v(a_k)$ and $v(a_l)$ with $k \neq l$ assume the maximum $\max\{v(a_i)\}$.

For the rest of this chapter, we fix a nonarchimedean field (k, v). Let $X \subset (k^\times)^n$ be an algebraic variety, i.e. the zero set of Laurent polynomials $f_1, \ldots, f_r \in k[T_1^{\pm 1}, \ldots, T_n^{\pm 1}]$.

Definition 1.2.3. The *tropicalization of* X is defined as the topological closure $X^{\text{trop}} = \overline{\text{trop}(X)}$ of the image of X under the map

$$\text{trop} : (k^\times)^n \xrightarrow{(v,\ldots,v)} \mathbb{R}^n_{>0} \xrightarrow{(\log,\ldots,\log)} \mathbb{R}^n.$$

Example 1.2.4. In Figure 1.2, we illustrate the tropicalization of a genus 1 curve E, embedded sufficiently general in $(k^\times)^2$. More precisely, we illustrate the tropicalization of the compactification $\overline{E}$ of E, which embeds into the projective plane $\mathbb{P}^2$. This means that all unbounded edges of the tropicalization of E gain a second boundary point, which we illustrate by bullets in Figure 1.2. Note that this picture suggests that tropicalizations preserve certain geometric invariants like the genus.

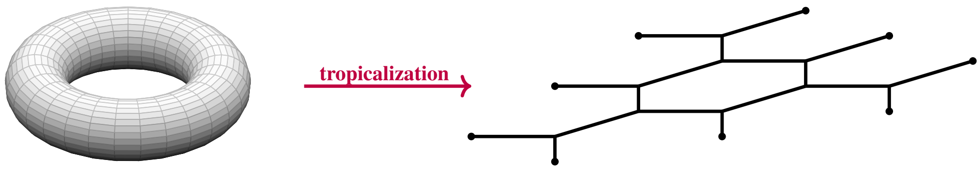

Figure 1.2: Tropicalization of an elliptic curve, including its points at infinity

Theorem 1.2.5 (Structure theorem for tropicalizations)**.** *Let (k,v) be a nonarchimedean field and $X \subset (k^\times)^n$ an equidimensional algebraic variety. Then*

(1) $X^{\text{trop}} = |\Delta|$ for a rational and equidimensional polyhedral complex Δ;

(2) $X \subset (k^\times)^n$ determines a weight function

$$m : \{P \in \Delta \,|\, \dim P = \dim \Delta\} \longrightarrow \mathbb{Z}_{>0}$$

such that (Δ, m) is a tropical variety.

The first part of the structure theorem has been proven by Bieri and Groves in their 1984 paper [BG84], which precedes tropical geometry by around 15 years and uses a slightly different setup than we do in our statement. The second part has been proven by Speyer in his thesis [Spe05]. A formulation of the structure theorem that is very close to ours appears in Maclagan and Sturmfels' book [MS15] as Theorem 3.3.6.

1.3 Two problems with the concept of a tropical variety

There are two oddities with the concept of a tropical variety that create difficulties for the development of algebro-geometric tools for tropical geometry and their application to tropicalizations of classical varieties.

The first problem is that the polyhedral complex Δ with $|\Delta| = X^{\text{trop}}$ is not determined by the classical variety $X \subset (k^\times)^n$. In other words,

the tropicalization of a classical variety is *not* a tropical variety.

The second problem relates to the functions of a tropical variety. The explanation of this issue requires some preliminary definitions.

Definition 1.3.1. The *tropical semifield* is the set $\mathbb{T} = \mathbb{R}_{\geqslant 0}$ together with the addition

$$a+b = \max\{a,b\}$$

and the usual multiplication

$$a \cdot b = ab$$

of nonnegative real numbers a,b.

Together with these operations $\mathbb{T}$ is indeed a semifield, i.e. it satisfies all of the axioms of a field except for the existence of additive inverses. The tropical semifield allows for the following reformulation of Definition 1.2.1: a nonarchimedean absolute value is a multiplicative map $v : k \to \mathbb{T}$ that is *subadditive*, i.e. $v(a+b) \leqslant v(a)+v(b)$ where the latter sum is taken with respect to the addition in $\mathbb{T}$.

Remark 1.3.2. In these lecture notes, we adopt the "max-times"-convention for the tropical numbers, which is less common than the "max-plus" or the "min-plus"-convention. To explain, the map $\log : \mathbb{T} \to \overline{\mathfrak{R}}$ defines an isomorphism of semirings between the tropical semifield $\mathbb{T}$ and the *max-plus-algebra* $\overline{\mathfrak{R}} = (\mathbb{R} \cup \{-\infty\}, \max, +)$. Multiplication of with (-1) defines an isomorphism $\overline{\mathfrak{R}} \to \overline{\mathbb{R}}$ between the max-plus-algebra with the *min-plus-algebra* $\overline{\mathbb{R}} = (\mathbb{R} \cup \{\infty\}, \min, +)$.

A priori, it is a matter of choice, with which semifield one works. But depending on the situation, some choices are more natural than others. When considering tropical varieties as polyhedral complexes, then the piecewise linear structure of the tropical variety is only visible in the logarithmic picture, i.e. one is led to work with the max-plus or the min-plus-algebra.

When working with tropical polynomials and tropical functions, in particular when compared to classical polynomials and functions, then it is more natural and less confusing to work with the max-times-convention.

Definition 1.3.3. The *tropical polynomial algebra in $T_1,\dots,T_n$* is the set

$$\mathbb{T}[T_1,\dots,T_n] \;=\; \Big\{ \sum_{J=(e_1,\dots,e_n)} a_J T_1^{e_1}\cdots T_n^{e_n} \,\Big|\, a_J \in \mathbb{T} \text{ and } a_J = 0 \text{ for almost all } J \Big\},$$

which is a semiring with respect to the usual addition and multiplication rules for polynomials where we apply the tropical addition $a_I + a_J = \max\{a_i, a_J\}$ to add coefficients.

A tropical polynomial $f = \sum a_J T_1^{e_1}\cdots T_n^{e_n}$ defines the function

$$\begin{array}{cccc} f(-): & \mathbb{T}^n & \longrightarrow & \mathbb{T}. \\ & x=(x_1,\dots,x_n) & \longmapsto & f(x) = \max\left\{a_J x_1^{e_1}\cdots x_n^{e_n}\right\} \end{array}$$

We are prepared to explain the second problem with tropical varieties. Namely, different polynomials can define the same function, as demonstrated in the following example.

Example 1.3.4. Consider $f_1 = T^2+1$ and $f_2 = T^2+T+1$. Then

$$f_1(x) = x^2+1 = \max\{x^2,1\} = \max\{x^2,x,1\} = f_2(x)$$

for all $x \in \mathbb{T}$.

In other words,

tropical functions are *not* the same as tropical polynomials.

To understand why tropical scheme theory promises to resolve these digressions, let us have a look at classical algebraic geometry.

For varieties over an algebraically closed field, Hilbert's Nullstellensatz guarantees that functions are the same as polynomials. However, if one tries to generalize the concept of a variety to arbitrary field or even rings, one faces the same problem: different polynomials can define the same function.

Grothendieck surpassed this problem with the invention of schemes. Since the functions of a tropical variety do not form a ring, but merely a semiring, it is clear that Grothendieck's concept of a scheme does not find applications in tropical geometry.

However, $\mathbb{F}_1$-geometry has provided a theory of so-called semiring schemes, cf. the papers [Dur07] of Durov, [TV09] of Toën-Vaquié and [Lor12] of the author. This theory and its refinement in terms of blueprints provides an appropriate framework for tropical scheme theory.

1.4 Semiring schemes

In this section, we give an idea of the definition of a semiring scheme. Similar to a scheme, it is built from the spectra of semirings. In order to understand this relation between tropical varieties and semiring schemes that we have in mind, we explain this concept in analogy to classical varieties and schemes, concentrating on the affine situation. More details about the construction of semiring schemes will be explained in later parts of these notes.

Let k be an algebraically closed field and $X \subset k^n$ a variety, i.e. the zero set of polynomials $f_1,\dots,f_r \in k[T_1,\dots,T_n]$. Let

$$I \;=\; \left\{ f \in k[T_1,\dots,T_n] \,\middle|\, f(x_1,\dots,x_n) = 0 \text{ for all } (x_1,\dots,x_n) \in X \right\}.$$

be its ideal of definition and $A \;=\; k[T_1,\dots,T_n]/I$ its ring of regular functions.

The associated scheme is the spectrum of A, which is the set $\operatorname{Spec} A$ of all prime ideals of A together with the topology generated by the principal open subsets

$$U_h \;=\; \{ \mathfrak{p} \subset A \,|\, h \notin \mathfrak{p} \}$$

for $h \in A$ and with the structure sheaf

$$\begin{array}{rccc} \mathcal{O}: & \{\text{open subsets of } \operatorname{Spec} A\} & \longrightarrow & \text{Rings.} \\ & U_h & \longmapsto & A[h^{-1}] \end{array}$$

We can recover the variety X from $\operatorname{Spec} A$ as follows. The ring of regular functions $A = k[T_1,\dots,T_n]/I$ equals the ring of global sections

$$\mathcal{O}(\operatorname{Spec} A) \;=\; A[1^{-1}] \;=\; A.$$

The variety X is equal to the set of k-rational points of $\operatorname{Spec} A$, i.e. we have a canonical bijection

$$X \longrightarrow \operatorname{Hom}_k(A,k) \;=\; \operatorname{Hom}_k(\operatorname{Spec} k, \operatorname{Spec} A)$$

that sends a point $x = (x_1,\dots,x_n)$ of X to the evaluation map

$$\operatorname{ev}_x: \; h \;\mapsto\; h(x).$$

Its inverse sends a homomorphism $f: A \to k$ to the point $\big(f(T_1),\dots,f(T_n)\big)$ of X.

The definition of $\operatorname{Spec} A$ extends to any semiring A as follows. There are natural extensions of the notions of prime ideals and localizations from rings to semirings.

Definition 1.4.1. The *spectrum of* A is the set $\operatorname{Spec} A$ of all prime ideals of A together with the topology generated by the principal open subsets

$$U_h = \{\mathfrak{p} \subset A \,|\, h \notin \mathfrak{p}\}$$

for $h \in A$ and with the structure sheaf

$$\begin{array}{rccc} \mathcal{O}: & \{\text{open subsets of } \operatorname{Spec} A\} & \longrightarrow & \text{Semirings} \\ & U_h & \longmapsto & A[h^{-1}] \end{array}$$

A semiring scheme is a topological space together with a sheaf in the category of semiring that is locally isomorphic to the spectra of semirings. A detailed definition of all this terminology will be given in later chapters.

1.5 Scheme theoretic tropicalization

In this section, we give an outline of the Giansiracusa tropicalization, which associates with a classical variety a semiring scheme whose $\mathbb{T}$-rational points correspond to the set theoretic tropicalization as considered in section 1.2. For the sake of simplicity, we explain this for subvarieties of affine space opposed to suvarieties of a torus, which is the context of section 1.2.

We require some notation. For a multi-index $J = (e_1, \ldots, e_n)$, we write $T^J = T_1^{e_1} \cdots T_n^{e_n}$ and $x^J = x_1^{e_1} \cdots x_n^{e_n}$. Let $f = \sum a_J T^J \in k[T_1, \ldots, T_n]$. We define

$$f^{\text{trop}} = \sum v(a_J) T^J \qquad \in \mathbb{T}[T_1, \ldots, T_n].$$

Let $X \subset k^n$ a variety with ideal of definition I.

Definition 1.5.1. The *Giansiracusa tropicalization* of X is the semiring scheme

$$\operatorname{Trop}_v(X) = \operatorname{Spec}\Big(\mathbb{T}[T_1, \ldots, T_n] \,/\, \operatorname{bend}_v(I)\Big)$$

where the *bend relations* $\operatorname{bend}_v(I)$ are defined as

$$\operatorname{bend}_v(I) = \Big(f^{\text{trop}} \sim f^{\text{trop}} + v(b_J) T^J \,\Big|\, f + b_J T^J \in I \Big).$$

The main result of Jeffrey and Noah Giansiracusa in [GG16] is the following connection to the set theoretic tropicalization X^{trop} of X, which stays in analogy to the corresponding result for schemes and varieties over an algebraically closed field.

Theorem 1.5.2 (Jeffrey and Noah Giansiracusa '13)**.** *We can recover the tropical variety X^{trop} as a set via a natural bijection*

$$X^{\text{trop}} \xrightarrow{\sim} \operatorname{Hom}_{\mathbb{T}}(\operatorname{Spec} \mathbb{T}, \operatorname{Trop}_v(X)).$$

Moreover, in case of a projective variety X, the Giansiracusa brothers associate with $\operatorname{Trop}_v(X)$ a Hilbert polynomial and show that it coincides with the Hilbert polynomial of X. This might be seen as the first striking result of tropical scheme theory.

Diane Maclagan and Felipe Rincón have shown in [MR14] that the embedding of $\operatorname{Trop}_v(X)$ into the n-dimensional tropical torus remembers the weights of the tropical variety X^{trop}, provided one has chosen the structure of a polyhedral complex. To wit, the embedding of a variety X into $(k^\times)^n$ yields an embedding of $\operatorname{Trop}_v(X)$ into the n-dimensional tropical torus $\operatorname{Spec} \mathbb{T}[T_1^{\pm 1}, \ldots, T_n^{\pm 1}]$.

Theorem 1.5.3 (Maclagan-Rincón '14). *Assume that $X \subset (k^\times)^n$ is equidimensional. Then the weight function m of any realization of X^{trop} as a tropical variety (Δ, m) is determined by the embedding of $\text{Trop}_v(X)$ into $\text{Spec}\,\mathbb{T}[T_1^{\pm 1}, \dots, T_n^{\pm 1}]$.*

In the author's paper [Lor15], the above results are refined and generalized by using blueprints and blue schemes. We mention two applications of this refined approach: the Giansiracusa tropicalization can be applied to more general situations than tropicalizations of subvarieties of toric varieties; for instance, it is possible to endow skeleta of Berkovich spaces with a schematic structure under certain additional hypotheses. Another feature is that the weight function of the tropical variety is already encoded into the structure sheaf of the "blue tropical scheme", which opens the possibility for a theory of abstract tropical schemes, opposed to embedded tropical schemes.

1.6 A central problem in tropical scheme theory

The aforementioned results give hope that the replacement of tropical varieties by tropical schemes will allow for new tools in tropical geometry, such as sheaf cohomology or a cohomological interpretation of intersection theory. However, it is not at all clear what a good notion of a "tropical scheme" might be.

The theory of semiring schemes comes with the notion of a $\mathbb{T}$-scheme, which is a morphism $X \to \text{Spec}\,\mathbb{T}$ of semiring schemes. However, there are too many $\mathbb{T}$-schemes to make this a useful class. For example, every hyperplane in $\mathbb{R}^n$ can be realized as a $\mathbb{T}$-scheme, and such subsets of $\mathbb{R}^n$ cannot satisfy the balancing condition with respect to any polyhedral subdivision and any choice of weight function. Even worse, every intersection of hyperplanes can be realized as $\mathbb{T}$-schemes, and such intersections include all bounded convex subsets of $\mathbb{R}^n$, e.g. the unit ball.

This makes clear that we have to restrict our attention to a subclass of $\mathbb{T}$-schemes in order to obtain a useful class that could replace the class of tropical varieties. Maclagan and Rincon make a suggestion for such a class, which is based on the observation that the ideal of definition of the tropicalization of a classical variety is a valuated matroid. In [MR14] and [MR16], they investigate the class of $\mathbb{T}$-schemes whose ideal of definition is a valuated matroid and show certain desirable properties like chain conditions for "tropical ideals" and the preservation of Hilbert functions.

Unfortunately, this theory encounters some serious difficulties since the class of tropical ideals is, a priori, too restrictive. For instance, the ideals of definition of some prominent spaces in tropical geometry, like linear tropical spaces and Grassmannians, are not tropical ideals. Moreover both the intersection and the sum of two tropical ideals fail to be a tropical ideal in general, which provides obstacles for primary decompositions and intersection theory of schemes, respectively.

It might be the case that there is natural way to associate a "generically generated" tropical ideal with ideals occuring in the situations explained above, but this seems to be a difficult problem. It might be the case that the class of tropical ideals, as considered in [MR14], is too restrictive for a useful theory of "tropical schemes".

In so far, we formulate the central problem of tropical scheme theory in the following way. We would like to find a class $\mathcal{C}$ of $\mathbb{T}$-schemes that satisfies the following criteria:

- $\mathcal{C}$ contains the tropicalizations of all classical varieties and for every tropical variety, $\mathcal{C}$ contains a $\mathbb{T}$-scheme representing it;

- $\mathcal{C}$ contains "universally constructable $\mathbb{T}$-schemes" such as tropical linear spaces and tropical Grassmannians;
- the $\mathbb{T}$-rational points of every $\mathbb{T}$-scheme in $\mathcal{C}$ yields a tropical variety; in particular, this involves a theory of balancing conditions for $\mathbb{T}$-schemes;
- defining ideals of schemes in $\mathcal{C}$ are closed under intersections and sums;
- $\mathcal{C}$ allows for a dimension theory by considering chains of irreducible reduced $\mathbb{T}$-schemes in $\mathcal{C}$; in particular, this involves the notion of an irreducible $\mathbb{T}$-scheme.

A more comprehensive list of open problems in tropical scheme theory was compiled at a workshop in April 2017 at the American Institute of Mathematics, see [AIM17] for a link to the problem list.

1.7 Outline of the previsioned contents of these notes

The central goal of these notes is to explain the material of the previous sections in detail. This includes reviewing some parts of "classical" tropical geometry and introducing semiring schemes, monoid schemes and blue schemes. We intend to discuss the Giansiracusa tropicalization and subsequent results from the papers [GG14] and [GG16] by Jeffrey and Noah Giansiracusa, [MR14] and [MR16] by Maclagan and Rincón, and [Lor15] by the author.

If we achieve this central goal in time, then we intend to treat more advanced topics like scheme theoretic skeleta of Berkovich spaces, schemes over the tropical hyperfield or families of matroids.

The chapters of these notes will be grouped into parts. The first part reviews the algebraic foundations, which are (ordered) semirings, monoids, blueprints, localizations, ideals and congruences. The second part is dedicated to generalized scheme theory and contains the constructions of semiring schemes, monoid schemes and blue schemes. The third part enters the central the theme of these notes, which is scheme theoretic tropicalization.

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Part I

Algebraic foundations

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Chapter 2

Semirings

In this chapter, we will provide the necessary background on semirings for our purposes. A standard source for the theory of semirings is Golan's book [Gol99], which the reader might want to confer as a secondary reference.

We illustrate the basic definitions and facts in numerous examples. Certain basic facts, which are either easy to prove or allow for a proof analogous to the case of rings, will be left as exercises.

2.1 The category of semirings

Definition 2.1.1. A *(commutative) semiring (with* 0 *and* 1*)* is a set R together with an addition $+ : R \times R \to T$, a multiplication $\cdot : R \times R \to R$ and two constants 0 and 1 such that the following axioms are satisfied:

(1) $(R,+)$ is an associative and commutative semigroup with neutral element 0;

(2) $(R,\cdot)$ is an associative and commutative semigroup with neutral element 1;

(3) $(a+b)c = ac+bc$ for all $a,b,c \in R$;

(4) $0 \cdot a = 0$ for all $a \in R$.

A *morphism between semirings* R_1 *and* R_2 is a map $f : R_1 \to R_2$ such that $f(0) = 0$, $f(1) = 1$, $f(a+b) = f(a)+f(b)$ and $f(ab) = f(a)\cdot f(b)$ for all $a,b \in R$. We denote the category of semirings by SRings.

Let R be a semiring. A *subsemiring of* R is a subset S that contains 0 and 1 and is closed under sums and products. The *unit group* or *units of* R is the subset $R^\times$ of multiplicatively invertible elements together with the restriction of the multiplication of R to $R^\times$. A *semifield* is a semiring R such that $R^\times = R - \{0\}$.

Note that the constants 0 and 1 of a semiring R are uniquely determined as the neutral elements of addition and multiplication, respectively. In some examples, we take the liberty to omit an explicit description of these constants. Note further that the multiplication of R does indeed restrict to a multiplication $R^\times \times R^\times \to R^\times$, which turns $R^\times$ into a multiplicative group.

Remark 2.1.2. Similar to the notion of a ring, the notion of a semiring is not standardized in the literature. In other texts, the reader will find noncommutative semirings and semirings without 0 or 1. Similarly, semiring morphism might not required to preserve 0 or 1, which are properties that do not follow automatically from the other axioms. We will not encounter such weaker notions of semirings in these notes.

Example 2.1.3. Every ring is tautologically a semiring. Examples of semirings that are not rings are the following: the natural numbers $\mathbb{N}$ with respect to the usual addition and multiplication; the nonnegative real numbers $\mathbb{R}_{\geqslant 0}$ with respect to the usual addition and multiplication; and the tropical numbers $\mathbb{T}$.

Note that a subsemiring S of R is a semiring with respect to the restrictions of the addition and multiplication of R. This includes the subsemiring of *tropical integers* $\mathcal{O}_{\mathbb{T}} = \{a \in \mathbb{T} | a \leqslant 1\}$ of $\mathbb{T}$ and the subsemiring of Boolean numbers $\mathbb{B} = \{0,1\}$ of $\mathcal{O}_{\mathbb{T}}$.

Examples of morphisms of semirings are inclusions $S \hookrightarrow R$ of subsemirings into the ambient semiring. Other examples are the following maps: $f : \mathbb{T} \to \mathbb{B}$ with $f(a) = 1$ for all $a \neq 0$; $g : \mathbb{N} \to \mathbb{B}$ with $g(a) = 1$ for all $a \neq 0$; $h : \mathcal{O}_{\mathbb{T}} \to \mathbb{B}$ with $h(a) = 0$ for all $a \neq 1$.

Exercise 2.1.4. Show that the min-plus-algebra $\overline{\mathbb{R}}$ and the max-plus-algebra $\overline{\mathbb{R}}$, as defined in Remark 1.3.2, are semifields. What are the neutral elements for addition and multiplication? Show that the logarithm defines an isomorphism of semirings $\log : \mathbb{T} \to \overline{\mathbb{R}}$. Show that multiplication with -1 defines an isomorphism of semirings $(-1) : \overline{\mathbb{R}} \to \overline{\mathbb{R}}$.

Let X be a closed subset of $\mathbb{R}^n$. Show that the set $\mathrm{Fun}(X, \overline{\mathbb{R}})$ of functions from X to $\overline{\mathbb{R}}$ inherits the structure of a semiring from the addition and multiplication in $\overline{\mathbb{R}}$. Let $\mathrm{CPL}(X)$ be the smallest subring of $\mathrm{Fun}(X, \overline{\mathbb{R}})$ that contains all functions of the type $ax + b$ with $a \in \mathbb{Z}$ and $b \in \mathbb{T}$. Show that $\mathrm{CPL}(X)$ consists of all **c**onvex **p**iecewise **l**inear functions $f : X \to \overline{\mathbb{R}}$ with integer slopes for which there is a finite covering of X by closed subsets Z_i such that $f|_{Z_i}$ is linear for each i.

Exercise 2.1.5. Let $f_1 : S \to R_1$ and $f_2 : S \to R_2$ be two morphisms of semirings. Define the tensor product $R_1 \otimes_S R_2$ as the set of finite sums $\sum a_i \otimes b_i$ of tensors $a_i \otimes b_i$ of elements $a_i \in R_1$ and $b_i \in R_2$, subject to the same relations as in the case of the tensor product of rings. Show that this forms a semiring that comes with morphisms $\iota_i : R_i \to R_1 \otimes_S R_2$ $(i = 1,2)$, sending $a \in R_1$ to $a \otimes 1$ and $b \in R_2$ to $1 \otimes b$, respectively.

Formulate and prove the usual properties of the tensor product: (1) The tensor product is the colimit (or pushout) of the morphism f_1 and f_2; (2) every bilinear morphism from $R_1 \times R_2$ defines a unique morphism from $R_1 \otimes R_2$; (3) the functor $- \otimes_S R$ is left adjoint to the functor $\mathrm{Hom}_S(R, -)$.

Exercise 2.1.6. Show that the category of semirings is complete and cocomplete. More precisely show the following.

(1) Show that the natural numbers $\mathbb{N}$ form an initial object and that the trivial ring $\{0 = 1\}$ forms a terminal object in SRings.

(2) Let $\{R_i\}_{i \in I}$ be a family of semirings. Then the Cartesian product $\prod_{I \in I} R_i$ together with componentwise addition and multiplication is a semiring, and the projections $\pi_j : \prod R_i \to R_j$ are semiring homomorphisms. The semiring $\prod_{I \in I} R_i$ together with the projections π_j is a product of the R_i.

(3) Let $f, g : R_1 \to R_2$ be two morphisms of semirings. Show that $\mathrm{eq}(f,g) = \{a \in R_1 | f(a) = g(a)\}$ is a subsemiring of R_1 and that the $\mathrm{eq}(f,g)$ together with the inclusion $\mathrm{eq}(f,g) \to R_1$ is an equalizer of f an g.

(4) Let $f,g: R_1 \to R_2$ be two morphisms of semirings. Show that there exists a coequalizer of f and g. *Hint:* Use Lemma 2.4.8 to show that there exists a congruence generated by the relations $f(a) \sim g(a)$ where $a \in R_1$.

(5) Let $\{R_i\}_{i\in I}$ be a finite family of semirings. Show that it has a coproduct, which we denote by $\bigotimes_{i\in I} R_i$. *Hint:* Use filtered colimits (i.e. "unions") of finite tensor products over $\mathbb{N}$.

Exercise 2.1.7. Show that a morphism $f: R_1 \to R_2$ is a monomorphism if and only if it is injective. Show that f is an isomorphism if and only if f is bijective. Show that every surjective morphism is an epimorphism. Give an example of an epimorphism that is not surjective (*hint:* cf. Exercise 2.7.3).

Exercise 2.1.8. Let $f: R \to S$ be a morphism of semirings. Show that the set theoretic image $\operatorname{im} f = f(R)$ is a subsemiring of S. Show that $\operatorname{im} f$ together with the restriction $f': R \to \operatorname{im} f$ of f and the inclusion $\operatorname{im} f \to S$ is the categorical image of f. Conclude that every morphism factors into an epimorphism followed by a monomorphism.

2.2 First properties

We list some first properties that characterize important subclasses of semirings.

Definition 2.2.1. A semiring R is

- *without zero divisors* if for any $a,b \in R$, the equality $ab = 0$ implies that $a = 0$ or $b = 0$;
- *integral* (or *multiplicatively cancellative*) if $0 \neq 1$ and for any $a,b,c \in R$ the equality $ac = bc$ implies $c = 0$ or $a = b$;
- *strict* if $a + b = 0$ implies $a = b = 0$ for all $a,b \in R$;
- *(additively) cancellative* if for any $a,b,c \in R$ the equality $a + c = b + c$ implies $a = b$;
- *(additively) idempotent* if $1 + 1 = 1$.

Lemma 2.2.2. *Let R be a semiring.*

(1) If $0 = 1$, then R is trivial*, i.e. R consists of the single element $0 = 1$.*

(2) If R is idempotent and cancellative, then R is trivial.

(3) If R is idempotent, then $a + a = a$ for all $a \in R$.

(4) If R is idempotent, then R is strict.

(5) If R is integral, then R is without zero divisors.

Proof. If $1 = 0$, then we have for every $a \in R$ that $a = 1 \cdot a = 0 \cdot a = 0$. Thus (1).

If R is idempotent and cancellative, then $1 + 1 = 1 = 1 + 0$ implies $1 = 0$. Thus (2).

If $1 + 1 = 1$, then we have for every $a \in R$ that $a + a = a(1 + 1) = a \cdot 1 = a$. Thus (3).

If R is idempotent and $a + b = 0$, then we have $a = a + a + b = a + b = 0$ and similarly $b = 0$. Thus (4).

If R is integral and $ab = 0$, then $ab = 0 = 0 \cdot b$ implies $b = 0$ or $a = 0$. Thus (5). □

Note that a nontrivial semiring without zero divisors does not have to be integral, in contrast to the situation for rings. An example verifying this claim is the tropical polynomial ring $\mathbb{T}[T]$, cf. Exercise 2.4.5; see Exercise 2.3.3 for another example.

Exercise 2.2.3. Verify which of the semirings from Example 2.1.3 and Exercise 2.1.4 are without zero divisors, integral, strict, cancellative or idempotent.

Exercise 2.2.4. Show that the morphism $\iota : R \to R \otimes_{\mathbb{N}} \mathbb{Z}$, sending a to $a \otimes 1$, satisfies the following properties: $R_{\mathbb{Z}} = R \otimes_{\mathbb{N}} \mathbb{Z}$ is a ring and every semiring morphism $f : R \to S$ into a ring S factors uniquely through ι. Show that R is cancellative if and only if $\iota : R \to R_{\mathbb{Z}}$ is injective. Show that R contains an *additive inverse of* 1, i.e. an element a such that $1 + a = 0$, if and only if R is a ring. Show that in this case $\iota : R \to R_{\mathbb{Z}}$ is an isomorphism.

2.3 Semigroup algebras and polynomial semirings

Definition 2.3.1. Let R be a semiring and A a multiplicatively written abelian semigroup with neutral element 1_A. The *semigroup algebra of A over R* is the semigroup ring $R[A]$ of finite R-linear combinations $\sum r_a a$ of elements $a \in A$, i.e. the sum contains only finitely many nonzero coefficients $r_a \in R$. The addition of $R[A]$ is defined by the formula

$$\left[\sum r_a a\right] + \left[\sum s_a a\right] \;=\; \sum (r_a + s_a) a$$

and the product is defined by the formula

$$\left[\sum r_a a\right] \cdot \left[\sum s_a a\right] \;=\; \sum_{a=bc} (r_b \cdot s_c) a.$$

The zero of $R[A]$ is the empty sum 0, i.e. the linear combination $\sum r_a a$ with $r_a = 0$ for all a, and the one of $R[A]$ is the linear combination $1 = \sum r_a a$ for which $r_{1_A} = 1$ and $r_a = 0$ for $a \neq 1_A$.

If A is the free abelian semigroup on the set of generators $\{T_i\}_{i \in I}$, then we write $R[A] = R[T_i]_{i \in I}$ or $R[A] = R[T_1, \dots, T_n]$ if $I = \{1, \dots, n\}$. We call $R[T_i]$ the *free algebra over R in $\{T_i\}$* or the *polynomial semiring over R in $\{T_i\}$*.

We allow ourselves to omit zero terms from the sums $\sum r_a a$, i.e. we may write $sb + tc$ for the element $\sum r_a a$ of $R[A]$ with $r_b = b$, $r_c = t$ and $r_a = 0$ for $a \neq b, c$. We simply write a for the element $1a$ of $R[A]$ and r for the element $r1_A$ of $R[A]$.

Exercise 2.3.2. Show that $R[A]$ is a semiring. Show that the map $\iota_R : R \to R[A]$ with $\iota_R(r) = r$ is an injective morphism of semirings. Show that the map $\iota_A : A \to R[A]$ with $\iota_A(a) = a$ is a *multiplicative map*, i.e. $\iota_A(1_A) = 1$ and $\iota_A(ab) = \iota_A(a) \cdot \iota_A(b)$ for all $a, b \in A$. Show that for every semiring morphism $f_R : R \to S$ and every multiplicative map $f_A : A \to S$, there is a unique semiring morphism $f : R[A] \to S$ such that $f_A = f \circ \iota_A$ and $f_R = f \circ \iota_R$. Use this to formulate and prove the universal property for a polynomial semiring over R.

Exercise 2.3.3. Let $A = \{1, \epsilon\}$ be the semigroup with $\epsilon^2 = \epsilon$ and $\mathbb{B}$ the Boolean numbers (cf. Example 2.1.3). Show that $\mathbb{B}[A]$ has 4 elements. Determine the addition and multiplication table for $\mathbb{B}[A]$. Show that $\mathbb{B}[A]$ is without zero divisors, but not integral.

2.4 Quotients and congruences

Definition 2.4.1. Let R be a semiring. A *congruence on R* is an equivalence relation $\mathfrak{c}$ on R that is *additive* and *multiplicative*, i.e. (a, b) and (c, d) in $\mathfrak{c}$ imply $(a + c, b + d)$ and (ac, bd) in $\mathfrak{c}$ for all $a, b, c, d \in R$.

Exercise 2.4.2. Let R be a ring. Show that for every ideal I of R, the set $\{(a,b)|a-b\in I\}$ is a congruence on R and that every congruence is of this form.

Exercise 2.4.3. Let $k,n\in\mathbb{N}$. Show that the set

$$\mathfrak{c}_{k,n} = \left\{ (m+rk, m+sk)\in\mathbb{N}\times\mathbb{N} \,\middle|\, m,r,s\in\mathbb{N} \text{ and } m\geqslant n \text{ or } r=s=0 \right\}$$

is a congruence on $\mathbb{N}$ and that every congruence of $\mathbb{N}$ is of this form.

Given a congruence $\mathfrak{c}$ on R, we often write $a\sim_{\mathfrak{c}} b$, or simply $a\sim b$, if there is no danger of confusion, to express that (a,b) is an element of $\mathfrak{c}$. The following proposition shows that congruences define quotients of semirings.

Proposition 2.4.4. *Let R be a semiring and $\mathfrak{c}$ be a congruence. Then the associations $[a]+[b]=[a+b]$ and $[a]\cdot[b]=[ab]$ are well-defined on equivalence classes $[a]$ of $\mathfrak{c}$ and turn the quotient $R/\mathfrak{c}$ into a semiring with zero $[0]$ and one $[1]$.*

The quotient map $\pi: R\to R/\mathfrak{c}$ is a morphism of semirings that satisfies the following universal property: every morphism $f: R\to S$ of semiring such that $f(a)=f(b)$ whenever $a\sim b$ in $\mathfrak{c}$ factors uniquely through π.

Proof. Consider $a\sim a'$ and $b\sim b'$. Then $a+b\sim a'+b'$ and $ab\sim a'b'$. Thus the addition and multiplication of $R/\mathfrak{c}$ does not depend on the choice of representative and is therefore well-defined. The properties of a semiring follow immediately, including the characterization of the zero as $[0]$ and the one as $[1]$. That $\pi: R\to R/\mathfrak{c}$ is a semiring homomorphism is tautological by the definition of $R/\mathfrak{c}$.

Let $f: R\to S$ be a semiring morphism such that $f(a)=f(b)$ whenever $a\sim b$ in $\mathfrak{c}$. For f to factor into $\overline{f}\circ\pi$ for a semiring morphism $\overline{f}: R/\mathfrak{c}\to S$, it is necessary that $\overline{f}([a])=\overline{f}\circ\pi(a)=f(a)$. This shows that $\overline{f}$ is unique if it exists. Since $a\sim b$ implies $f(a)=f(b)$, we conclude that $\overline{f}$ is well-defined as a map. The verification of the axioms of a semiring morphism are left as an exercise. □

Exercise 2.4.5. Let $n\geqslant 1$. Show that $R=\mathbb{T}[T_1,\dots,T_n]$ is without zero divisors, but not integral. Show that the relation $\{(f,g)\in R\times R|f(x)=g(x) \text{ for all } x\in\mathbb{T}^n\}$ is a congruence on R; cf. section 1.3 for definition of $f(x)$. Show that the quotient $R/\mathfrak{c}$ is integral and isomorphic to $\mathrm{CPL}(\mathbb{R}^n)$; cf. Exercise 2.1.4 for the definition of $\mathrm{CPL}(\mathbb{R}^n)$.

Conversely, every quotient is characterized by a congruence. More precisely, for every semiring morphism, there is a congruence that characterizes which elements in the domain become identified in the image.

Definition 2.4.6. Let $f: R\to S$ be a morphism of semirings. The *congruence kernel of f* is the relation $\mathfrak{c}(f)=\{(a,b)\in R\times R|f(a)=f(b)\}$ on R.

Lemma 2.4.7. *The congruence kernel $\mathfrak{c}(f)$ of a morphism $f: R\to S$ of semirings is a congruence on R.*

Proof. That $\mathfrak{c}=\mathfrak{c}(f)$ is an equivalence relation follows from the following calculations: $f(a)=f(a)$ (reflexive); $f(a)=f(b)$ implies $f(b)=f(a)$ (symmetry); $f(a)=f(b)$ and $f(b)=f(c)$ imply $f(a)=f(c)$ (transitive). Additivity and multiplicativity follow from: $f(a)=f(b)$ and $f(c)=f(d)$ imply $f(a+c)=f(a)+f(c)=f(b)+f(d)=f(b+d)$ and $f(ac)=f(a)\cdot f(c)=f(b)\cdot f(d)=f(bd)$. This shows that $\mathfrak{c}$ is a congruence. □

As a consequence of this lemma, we see that for a semiring R, the associations

$$\begin{array}{ccc} \{\text{congruences on } R\} & \longleftrightarrow & \{\text{quotients of } R\} \\ \mathfrak{c} & \longmapsto & R \to R/\mathfrak{c} \\ \mathfrak{c}(\pi) & \longleftarrow & \pi : R \twoheadrightarrow R' \end{array}$$

are mutually inverse bijections. We will see in section 2.5 that we do not have a correspondence between quotients and ideals, as in the case of rings. In so far, one has to work with congruences when one wants to describe quotients of semirings.

Lemma 2.4.8. *Let R be a semiring and $S \subset R \times R$ a subset. Then there is a smallest congruence $\mathfrak{c} = \langle S \rangle$ containing S. The quotient map $\pi : R \to R/\langle S \rangle$ satisfies the following universal property: every morphisms $f : R \to R'$ with the property that $f(a) = f(b)$ whenever $(a,b) \in S$ factors uniquely through π.*

Proof. It is readily verified that the intersection of congruences is again a congruence. As a consequence, the intersection of all congruences containing S is the smallest congruence containing S.

Given any morphism $f : R \to R'$ with the property that $f(a) = f(b)$ whenever $(a,b) \in S$, then the congruence kernel $\mathfrak{c}(f)$ must contain S and thus $\mathfrak{c} = \langle S \rangle$. Using Proposition 2.4.4, we see that f factors uniquely through π. □

This lemma shows that we can construct new semirings from known ones by prescribing a number of relations: let R be a semiring and $\{a_i \sim b_i\}$ a set of relations on R, i.e. $S = \{(a_i, b_i)\}$ is a subset of $R \times R$. Then we define $R/\langle a_i \sim b_i \rangle$ as the quotient semiring $R/\langle S \rangle$.

Exercise 2.4.9. Show that $\mathbb{B}[T]/\langle T^2 \sim T \rangle$ is isomorphic to the semigroup algebra $\mathbb{B}[A]$ where $A = \{1, \epsilon\}$ is the semigroup with $\epsilon = \epsilon^2$; cf. Exercise 2.3.3. Determine all congruences on $\mathbb{B}[A]$.

Exercise 2.4.10. Let R be a semirings and $\mathfrak{c}$ a congruence on R. Show that $\mathfrak{c}$ is a subsemiring of $R \times R$ containing the image of the diagonal map $\Delta : R \to R \times R$.

Let $f : R \to S$ be a homomorphism of semirings. Show that the congruence kernel of f together with the inclusion into $R \times R$ is the equalizer of the morphisms $f \circ \mathrm{pr}_1$ and $f \circ \mathrm{pr}_2$ from $R \times R$ to S where $\mathrm{pr}_i : R \times R \to R$ is the i-th canonical projection ($i = 1, 2$).

2.5 Ideals

While the concept of congruences is the correct generalization of ideals from rings to semirings that characterizes quotients of semirings, there are other more straight-forward generalizations of ideals, which carry over other properties from rings to semirings. In this section, we will examine two such notions: ideals and k-ideals.

Definition 2.5.1. Let R be a semiring. An *ideal of R* is a subset I of R such that 0, ac and $a+b$ are elements of I for all $a, b \in I$ and $c \in R$. A *k-ideal* or a *subtractive ideal of R* is an ideal I of R such that $a + c = b$ with $a, b \in I$ and $c \in R$ implies $c \in I$.

Once we make sense of the concept of a (semi)module over R, we could characterize an ideal of R as a submodule of R. The relevance of (prime) ideals of semirings lies in the fact that they are the good notion of points of the spectrum of R. We will come back to this in the chapter on semiring schemes.

The relevance of k-ideals is easier to explain. Namely, they form the class of subsets that is characterized as the 0-fibres, or kernels, of semiring morphisms. We assume, without evidence, that the “k” in “k-ideal” stands for “kernel”. The name k-ideal seems to be coined by Henriksen in [Hen58].

Definition 2.5.2. Let $f : R \to S$ be a semiring morphism. The *(ideal) kernel of f* is the inverse image $\ker(f) = f^{-1}(0)$ of 0.

Let S be a subset of R. The *congruence generated by S* is the congruence $\mathfrak{c}(S)$ generated by $\{(a,0)|a \in S\}$.

Proposition 2.5.3. *The kernel* $\ker(f)$ *of a morphism of semiring $f : R \to S$ is a k-ideal and every k-ideal appears as a kernel. More precisely, if I is a k-ideal of R and $\mathfrak{c} = \mathfrak{c}(I)$ is the congruence generated by I, then I is the kernel of $\pi : R \to R/\mathfrak{c}$ and $a \sim_{\mathfrak{c}} b$ if and only if there are elements $c,d \in I$ such that $a+c = b+d$.*

Proof. We begin with the verification that $\ker(f)$ is a k-ideal. Clearly $0 \in \ker(f)$. Let $a,b \in \ker(f)$ and $c \in R$. Then $f(ac) = f(a)f(c) = 0 \cdot f(c) = 0$ and $f(a+b) = f(a) + f(b) = 0$, thus ac and $a+b$ are in $\ker(f)$. If $a+c = b$, then $f(c) = 0 + f(c) = f(a) + f(c) = f(b) = 0$ shows that $c \in \ker(f)$. Thus $\ker(f)$ is a k-ideal.

In order to verify the second claim of the proposition, we begin with showing that the relation

$$\mathfrak{c}' \;=\; \big\{\, (a,b) \in R \times R \,|\, a+c = b+d \text{ for some } c,d \in I \,\big\}$$

is a congruence. Reflexivity and symmetry are immediate from the definition. Transitivity is shown as follows: if $a \sim_{\mathfrak{c}'} b \sim_{\mathfrak{c}'} b'$, then there are elements $c,d,c',d' \in I$ such that $a+c = b+d$ and $b+c' = b'+d'$. Adding c' to the former and d to the latter equation yields $a+c+c' = b+d+c' = b'+d+d'$. Since I is closed under sums, $c+c'$ and $d+d'$ are in I and thus $a \sim_{\mathfrak{c}'} b'$. This shows that $\mathfrak{c}'$ is an equivalence relation.

We continue with the verification of additivity and multiplicativity of $\mathfrak{c}'$. Let $a \sim_{\mathfrak{c}'} b$ and $a' \sim_{\mathfrak{c}'} b'$, i.e. $a+c = b+d$ and $a'+c' = b'+d'$ for some $c,d,c',d' \in I$. Adding these equations yields $a+a'+c+c' = b+b'+d+d'$ where $c+c'$ and $d+d'$ are in I. Thus $a+a' \sim_{\mathfrak{c}'} b+b'$, which establishes additivity. Multiplying these equations yields

$$aa' + ac' + a'c + cc' \;=\; (a+c)(a'+c') \;=\; (b+d)(b'+d') \;=\; bb' + bd' + b'd + dd'.$$

Since $ac' + a'c + cc'$ and $bd' + b'd + dd'$ are in I, we have $aa' \sim_{\mathfrak{c}'} bb'$, which shows multiplicativity of $\mathfrak{c}'$. Thus $\mathfrak{c}'$ is a congruence.

As the next step, we verify that $\mathfrak{c}'$ is equal to the congruence $\mathfrak{c}$ generated by I. Since $a+0 = 0+0$, we see that $\mathfrak{c}'$ contains the generating set $\{(a,0)|a \in I\}$ of $\mathfrak{c}$. Thus $\mathfrak{c}$ is contained in $\mathfrak{c}'$. Conversely, consider a relation $a \sim_{\mathfrak{c}'} b$ in $\mathfrak{c}'$, i.e. $a+c = b+d$ for some $b,d \in I$. Then $b \sim_{\mathfrak{c}} 0 \sim_{\mathfrak{c}} d$ and, by the additivity of $\mathfrak{c}$,

$$a \;=\; a+0 \;\sim_{\mathfrak{c}}\; a+b \;=\; b+d \;\sim_{\mathfrak{c}}\; b+0 \;= b,$$

i.e. $a \sim_{\mathfrak{c}} b$ in $\mathfrak{c}$. This shows that $\mathfrak{c} = \mathfrak{c}'$, as claimed.

Finally, we show that I is the kernel of $\pi : R \to R/\mathfrak{c}$, i.e. $I = \{a \in R | a \sim_{\mathfrak{c}} 0\}$. By the definition of $\mathfrak{c} = \mathfrak{c}(I)$, it is clear that $I \subset \ker(\pi)$. By the characterization of $\mathfrak{c}$ as $\mathfrak{c}'$, we have $a \in I$ if and only if there are elements $c,d \in I$ such that $a+c = 0+d = d$. Since I is a k-ideal, this equation implies that $a \in I$. Thus $I = \ker(\pi)$ as claimed. This finishes the proof of the proposition. □

To conclude, ideals, k-ideals and congruences are different generalizations of ideals to semirings, which do not coincide in general. There are ways to pass from one class to the other, which follows from our previous results.

Namely, with a congruence $\mathfrak{c}$ on a semiring R, we can associate the kernel of the projection $\pi_{\mathfrak{c}} : R \to R/\mathfrak{c}$, which is a k-ideal; with a k-ideal I, we can associate the congruence $\mathfrak{c}(I)$ generated by I. We have that the kernel of $R \to R/\mathfrak{c}(I)$ is I and the congruence $\mathfrak{c}(\ker \pi_{\mathfrak{c}})$ is contained in $\mathfrak{c}$, but in general not equal to $\mathfrak{c}$.

On the other end, every k-ideal is tautologically an ideal. With an ideal I of R, we can associate the smallest k-ideal containing I, which is the kernel of $R \to R/\mathfrak{c}(I)$. We summarize this discussion in the following picture.

"submodules"		**"kernels"**		**"quotients"**
$\{\text{ideals of } R\}$	$\underset{\twoheadrightarrow}{\hookleftarrow}$	$\{k\text{-ideals of } R\}$	$\underset{\hookrightarrow}{\twoheadleftarrow}$	$\{\text{congruences on } R\}$

Exercise 2.5.4. Describe all ideals, k-ideals and congruences for $\mathbb{N}$; cf. Exercise 2.4.3. Describe the maps from the above diagram in this example.

Exercise 2.5.5. Let $A = \{1, \epsilon\}$ be the semigroup with $\epsilon^2 = \epsilon$ and $R = \mathbb{B}[A]$ the semigroup algebra, which has been already the protagonist of Exercises 2.3.3 and 2.4.9. Determine all ideals, k-ideals and congruences of $\mathbb{B}[A]$ and describe the above maps between ideals, k-ideals and congruences explicitly.

2.6 Prime ideals

In the last two sections of this chapter, we turn to topics of relevance for scheme theory, which are prime ideals and localizations, respectively.

Definition 2.6.1. A (k-)ideal I of R is *proper* if it is not equal to R. It is *maximal* if it is proper and if $I \subset J$ implies $I = J$ for any other proper (k)-ideal. It is *prime* if its complement $S = R - I$ is a multiplicative subset of R.

Note that a k-ideal I is a prime k-ideal if and only if it is a prime ideal. In so far, we can use the attribute "prime" unambiguously for ideals and k-ideals. Note, however, that the k-ideal generated by a prime ideal does not need to be prime; we provide proof in Example 2.6.2 below.

The situation for maximal (k-)ideals is more subtle. A k-ideal that is a maximal ideal is tautologically a maximal k-ideal. But the converse fails to be true in general, as demonstrated in Example 2.6.2. This means that we have to make a clear distinction between maximal ideals and maximal k-ideals.

Example 2.6.2. Consider the semiring $R = \mathbb{B}[T]/\langle T^2 \sim T \sim T+1 \rangle$, which is a quotient of the semiring $\mathbb{B}[A]$ from Exercises 2.3.3, 2.4.9 and 2.5.5. It consists of the elements $0, 1, T$ and its unit group is $R^\times = \{1\}$. The proper ideals of R are $(0) = \{0\}$ and $(T) = \{0, T\}$, which are both prime ideals, but only (0) is a k-ideal.

This example demonstrates the following effects:

- (0) is a maximal k-ideal, but it is not a maximal ideal since it is properly contained in the proper ideal (T).
- (T) is a prime ideal, but the k-ideal generated by (T), which is R, is not a prime k-ideal.

- The quotient $R/(0)$ of R by the maximal k-ideal (0), which is equal to R, is not a semifield.
- The quotient $R/(T)$ of R by the k-ideal generated by (T), which is the trivial semiring $R/R = \{0\}$, is not a semifield.

Being warned that (k-)ideals for semirings fail to satisfy certain properties that we are used to from ideal theory of rings, we begin with the proof of properties that extend to the realm of semirings.

Lemma 2.6.3. *Let R be a semiring and I a k-ideal of R. Then I is prime if and only if R/I is nontrivial and without zero divisors.*

Proof. The k-ideal I is prime if and only if for all $a,b \in R$, $ab \in I$ implies that $a \in I$ or $b \in I$. Passing to the quotient R/I, this means that $[ab] = [0]$ implies $[a] = [0]$ or $[b] = [0]$ where we use that the kernel of $R \to R/I$ is I, cf. Proposition 2.5.3. This latter condition is equivalent to R/I being nontrivial and without zero divisors. □

Remark 2.6.4. As shown in Example 2.6.2, the usual characterization of maximal ideals as those ideals whose quotient is a field does not hold for semirings. We can only give the following quite tautological characterization of maximal k-ideals: a k-ideal I is maximal if and only if the zero ideal $\{0\}$ of R/I is a maximal k-ideal.

Lemma 2.6.5. *Every maximal ideal of a semiring is a prime ideal.*

Proof. Let R be a semiring and $\mathfrak{m}$ a maximal ideal. Consider $a,b \in R$ such that $ab \in \mathfrak{m}$, but $a \notin \mathfrak{m}$. We want to show that $b \in \mathfrak{m}$.

First we note that the set $\mathfrak{m} \cup \{a\}$ generates the ideal $(1) = R$ since $\mathfrak{m}$ is maximal and does not contain a. We claim that $cb \in \mathfrak{m}$ for all $c \in B$. This claim is certainly true for elements $c \in \mathfrak{m}$ and for multiples of a. Thus it is also true for all linear combinations of elements of $\mathfrak{m} \cup \{a\}$, which are all elements of R since $R = \langle \mathfrak{m} \cup \{a\} \rangle$. Thus the claim.

We conclude that $b = 1 \cdot b \in \mathfrak{m}$ as desired, which completes the proof. □

Lemma 2.6.6. *Every maximal k-ideal of a semiring is a prime k-ideal.*

Proof. We can prove this affirmation along the lines of the proof of Lemma 2.6.5. However, in the present case, we can only use that R is equal to the k-ideal generated by $\mathfrak{m} \cup \{a\}$, which requires an additional step in the proof that $cb \in \mathfrak{m}$ for all $c \in R$. Namely, the proof of Lemma 2.6.5 shows only $cb \in \mathfrak{m}$ for all c in the ideal I generated by $\mathfrak{m} \cup \{a\}$.

Consider an equality $d + c = e$ with d and e in I. Then we have $db + cb = eb$ and $db, eb \in \mathfrak{m}$. Since $\mathfrak{m}$ is a k-ideal, we also have $cb \in \mathfrak{m}$. This shows that $cb \in \mathfrak{m}$ for all $c \in R$, and we can continue as in the proof of Lemma 2.6.5. □

Lemma 2.6.7. *Let $f : R \to R'$ be a morphism of semirings and I an ideal of R'. Then $f^{-1}(I)$ is an ideal of R. If I is prime, then $f^{-1}(I)$ is prime. If I is a k-ideal, then $f^{-1}(I)$ is a k-ideal.*

Proof. We verify that $f^{-1}(I)$ is an ideal. Obviously, it contains 0. If $a,b \in f^{-1}(I)$ and $c \in R$, then $f(a+b) = f(a) + f(b) \in I$ and $f(ca) = f(c)f(a) \in I$. Thus $a+b, ca \in f^{-1}(I)$. This shows that $f^{-1}(I)$ is an ideal.

Assume that I is prime, i.e. $S = R' - I$ is a multiplicative set. Then $f^{-1}(S) = R - f^{-1}(I)$ is a multiplicative set of R and thus $f^{-1}(I)$ is a prime ideal of R.

Assume that I is a k-ideal and consider an equality $a + c = b$ in R with $a,b \in f^{-1}(I)$. Then $f(a) + f(c) = f(b)$ and $f(a), f(b) \in I$, which implies that $f(c) \in I$. Thus $c \in f^{-1}(I)$, which shows that $f^{-1}(I)$ is a k-ideal. □

Remark 2.6.8. There is also a concept of prime congruences. More precisely, there are two possible variants. Let $\mathfrak{c}$ be a congruence on R. Then $\mathfrak{c}$ is a *weak prime congruence on R* if $R/\mathfrak{c}$ is nontrivial and without zero divisors, and $\mathfrak{c}$ is a *strong prime congruence on R* if $R/\mathfrak{c}$ is integral.

However, we do not intend to discuss congruence schemes in these notes and therefore do not pursue the topic of prime congruences. Note that as of today, there is no satisfying theory of congruences schemes for semirings, but that such a theory relies on solving some open problems concerning the structure sheaf of congruence spectra. To explain this issue in more fancy words: one is led to work with a Grothendieck pre-topology on the category of semirings that is not subcanonical. This requires a sophisticated setup that establishes substitutes of certain standard facts for subcanonical topologies.

Exercise 2.6.9. Determine all prime (k-)ideals, all maximal (k-)ideals and all weak and strong prime congruences of $\mathbb{N}$ and $\mathbb{B}[A]$ where $A = \{1, \epsilon\}$ is the semigroup with $\epsilon^2 = \epsilon$. Let $f : \mathbb{N} \to \mathbb{Z}$ be the inclusion of the natural numbers into the integers. Describe the map $\mathfrak{p} \to f^{-1}(\mathfrak{p})$ from the set of prime ideals of $\mathbb{Z}$ to the set of prime ideals of $\mathbb{N}$ explicitly. Is it injective? Is it surjective?

Exercise 2.6.10. Let R be a semiring and I a proper (k-)ideal of R. Show that R has a maximal (k-)ideal that contains I. *Hint:* The usual proof for rings works also for this case. In particular, the claim relies on the axiom of choice aka Zorn's lemma.

2.7 Localizations

Definition 2.7.1. Let R be a semiring and $S \subset R$ be a *multiplicative subset of R*, i.e. a subset that contains 1 and is closed under multiplication. The *localization of R at S* is the quotient $S^{-1}R$ of $S \times R$ by the equivalence relation that identifies (s,r) with (s',r') whenever there is a $t \in S$ such that $tsr' = ts'r$ in R. We write $\frac{r}{s}$ for the equivalence class of (s,r). The addition and multiplication of $S^{-1}R$ are defined by the formulas

$$\frac{r}{s} + \frac{r'}{s'} \;=\; \frac{sr' + s'r}{ss'} \quad \text{and} \quad \frac{r}{s} \cdot \frac{r'}{s'} \;=\; \frac{r'r}{ss'}.$$

The zero of $S^{-1}R$ is $\frac{0}{1}$ and its one is $\frac{1}{1}$.

We write $R[h^{-1}]$ for $S^{-1}R$ if $S = \{h^i\}_{i \in \mathbb{N}}$ for an element $h \in R$ and call $R[h^{-1}]$ the *localization of R at h*. We write $R_\mathfrak{p}$ for $S^{-1}R$ if $S = R - \mathfrak{p}$ for a prime ideal $\mathfrak{p}$ of R and call $R_\mathfrak{p}$ the *localization of R at* $\mathfrak{p}$. Assume that $S = R - \{0\}$ is a multiplicative subset of R. Then we write $\operatorname{Frac}(R)$ for $S^{-1}R$ and call it the *semifield of fractions of R*.

If I is an ideal of R, then we write $S^{-1}I$ for the ideal of $S^{-1}R$ that is generated by $\{\frac{a}{1} | a \in I\}$.

Lemma 2.7.2. *Let R be a semiring, I an ideal of R and S a multiplicative subset of R. Then*

$$S^{-1}I \;=\; \left\{ \tfrac{a}{s} \in S^{-1}R \,\middle|\, a \in I, s \in S \right\}.$$

Proof. It is clear that $S^{-1}I$ contains the set $\{\frac{a}{1} | a \in I\}$ of generators of $S^{-1}I$. If we have proven that the set $I_S = \{\frac{a}{s} | a \in I, s \in S\}$ is an ideal, then it follows that it contains $S^{-1}I$. The reverse inclusion follows from the observation that for $\frac{a}{s} \in I_S$, we have $\frac{a}{s} = \frac{1}{s} \cdot \frac{a}{1} \in S^{-1}I$.

We are left with showing that I_S is an ideal. It obviously contains $\frac{0}{1}$. Given $\frac{a}{s} \in I_S$ and $\frac{b}{t} \in S^{-1}R$, then $\frac{a}{s} \cdot \frac{b}{t} = \frac{ab}{st} \in I_S$ since $ab \in I$. Given $\frac{a}{s}, \frac{b}{t} \in I_S$, then $a, b \in I$ and $ta + sb \in I$. Thus $\frac{a}{s} + \frac{b}{t} = \frac{ta+sb}{st}$ is an element of I_S. This verifies that I_S is an ideal of $S^{-1}I$ and finishes the proof of the lemma. □

Exercise 2.7.3. Let R be a semiring and S a multiplicative subset of R. Show that the map $\iota_S : R \to S^{-1}R$, defined by $\iota_S(a) = \frac{a}{1}$, is a morphism of semirings that maps S to the units of $S^{-1}R$. Show that it satisfies the usual universal property of localizations: every morphism $f : R \to R'$ of semirings that maps S to the units of R' factors uniquely through ι_S. Show that ι_S is an epimorphism.

Exercise 2.7.4. The subset $S = R - \{0\}$ is a multiplicative subset if and only if R is nontrivial and without zero divisors. Assuming that S is a multiplicative subset, show that $\mathrm{Frac}\, R$ is a semifield. Show that the morphism $\iota_S : R \to \mathrm{Frac}(R)$ is injective if and only if R is integral. Describe an example where $R \to \mathrm{Frac}(R)$ is not injective.

Proposition 2.7.5. *Let R be a semiring, S a multiplicative subset of R and $\iota_S : R \to S^{-1}R$ the localization morphism. Then the maps*

$$\begin{array}{ccc} \{\textit{prime ideals } \mathfrak{p} \textit{ of } R \textit{ with } \mathfrak{p}\cap S = \emptyset\} & \longleftrightarrow & \{\textit{prime ideals of } S^{-1}R\} \\ \mathfrak{p} & \stackrel{\Phi}{\longmapsto} & S^{-1}\mathfrak{p} \\ \iota_S^{-1}(\mathfrak{q}) & \stackrel{\Psi}{\longleftarrow\!\shortmid} & \mathfrak{q} \end{array}$$

are mutually inverse bijections. A prime ideal $\mathfrak{p}$ of R with $\mathfrak{p}\cap S = \emptyset$ is a k-ideal if and only if $S^{-1}\mathfrak{p}$ is a k-ideal.

Proof. To begin with, we verify that both Φ and Ψ are well-defined. Let $\mathfrak{p}$ be a prime ideal of R such that $\mathfrak{p}\cap S = \emptyset$. Then $S^{-1}\mathfrak{p} = \{\frac{a}{s} | a \in \mathfrak{p}, s \in S\}$ by Lemma 2.7.2. Consider $\frac{a}{s}, \frac{b}{t} \in S^{-1}R$ such that $\frac{a}{s}\cdot\frac{b}{t} = \frac{ab}{st} \in S^{-1}\mathfrak{p}$, i.e. $ab \in \mathfrak{p}$. Then $a \in \mathfrak{p}$ or $b \in \mathfrak{p}$ and thus $\frac{a}{s} \in S^{-1}\mathfrak{p}$ or $\frac{b}{t} \in S^{-1}\mathfrak{p}$. This shows that $S^{-1}\mathfrak{p}$ is a prime ideal of $S^{-1}R$ and that Φ is well-defined.

Let $\mathfrak{q}$ be a prime ideal of $S^{-1}R$. By Lemma 2.6.7, $\iota_S^{-1}(\mathfrak{q})$ is a prime ideal of R. Note that $\mathfrak{q}$ is proper and does not contain any element of the form $\frac{s}{t}$ with $s,t \in S$ since $\frac{t}{s}\cdot\frac{s}{t} = 1$. Thus $\iota_S^{-1}(\mathfrak{q})$ intersects S trivially. This shows that Ψ is well-defined.

We continue with the proof that $\Psi\circ\Phi$ is the identity, i.e. $\iota_S^{-1}(S^{-1}\mathfrak{p}) = \mathfrak{p}$ for every prime ideal $\mathfrak{p}$ of R that does not intersect S. The inclusion $\mathfrak{p} \subset \iota_S^{-1}(S^{-1}\mathfrak{p})$ is trivial. The reverse inclusion can be shown as follows. The set $\iota_S^{-1}(S^{-1}\mathfrak{p})$ consists of all elements $a \in R$ such that $\frac{a}{1} = \frac{b}{s}$ for some $b \in \mathfrak{p}$ and $s \in S$. This equation says that there is a $t \in S$ such that $tsa = tb$. Since $b \in \mathfrak{p}$, we have $tsa = tb \in \mathfrak{p}$. Since $ts \notin \mathfrak{p}$, we have $a \in \mathfrak{p}$, as desired.

We continue with the proof that $\Phi\circ\Psi$ is the identity, i.e. $S^{-1}\iota_S^{-1}(\mathfrak{q}) = \mathfrak{q}$ for every prime ideal $\mathfrak{q}$ of $S^{-1}R$. The inclusion $S^{-1}\iota_S^{-1}(\mathfrak{q}) \subset \mathfrak{q}$ is trivial. The reverse inclusion can be shown as follows. Let $\frac{a}{s} \in \mathfrak{q}$. Then $\frac{a}{1} = \frac{s}{1}\cdot\frac{a}{s} \in \mathfrak{q}$ and $a \in \iota_S^{-1}\mathfrak{q}$. Thus $\frac{a}{s} \in S^{-1}\iota_S^{-1}(\mathfrak{q})$, as desired. This concludes the proof of the first claim of the proposition.

We continue with the proof that a prime ideal $\mathfrak{p}$ of R with $\mathfrak{p}\cap S = \emptyset$ is a k-ideal if and only if $S^{-1}\mathfrak{p}$ is a k-ideal. First assume that $S^{-1}\mathfrak{p}$ is a k-ideal and consider an equality $a + c = b$ with $a, b \in \mathfrak{p}$. Then we have $\frac{a}{1} + \frac{c}{1} = \frac{b}{1}$ with $\frac{a}{1}, \frac{b}{1} \in S^{-1}\mathfrak{p}$. Since $S^{-1}\mathfrak{p}$ is a k-ideal, we have $\frac{c}{1} \in S^{-1}\mathfrak{p}$ and thus $c \in \mathfrak{p}$. This shows that $\mathfrak{p}$ is a k-ideal.

Conversely, assume that $\mathfrak{p}$ is a k-ideal and consider an equality $\frac{a}{s} + \frac{c}{u} = \frac{b}{t}$ with $\frac{a}{s}, \frac{b}{t} \in S^{-1}\mathfrak{p}$. This means that $wtua + wstc = wsub$ for some $w \in S$. Since $wtua$ and $wsub$ are elements of the k-ideal $\mathfrak{p}$, also $wstc \in \mathfrak{p}$. Since $\mathfrak{p}$ is prime and $wst \notin \mathfrak{p}$, we have $c \in \mathfrak{p}$ and thus $\frac{c}{u} \in S^{-1}\mathfrak{p}$, as desired. This finishes the proof of the proposition. □

Let R be a semiring, $\mathfrak{p}$ a prime ideal of R and $S = R - \mathfrak{p}$. Then $S^{-1}\mathfrak{p}$ is the complement of the units of $S^{-1}R$ and therefore its unique maximal ideal.

Definition 2.7.6. Let R be a semiring and $\mathfrak{p}$ a prime ideal of R. The *residue field at* $\mathfrak{p}$ is the semiring $k(\mathfrak{p}) = R_\mathfrak{p}/\mathfrak{c}(S^{-1}\mathfrak{p})$ where S is the complement of $\mathfrak{p}$ in R and $\mathfrak{c}(S^{-1}\mathfrak{p})$ is the congruence on $R_\mathfrak{p}$ that is generated by $S^{-1}\mathfrak{p}$.

Let $\mathfrak{p}$ be a prime ideal of a semiring R. Then the residue field at $\mathfrak{p}$ comes with a canonical morphism $R \to k(\mathfrak{p})$, which is the composition of the localization map $R \to R_\mathfrak{p}$ with the quotient map $R_\mathfrak{p} \to k(\mathfrak{p})$. Note that the residue field $k(\mathfrak{p})$ can be the trivial semiring in case that $\mathfrak{p}$ is not a k-ideal. More precisely, we have the following.

Corollary 2.7.7. *Let R be a semiring, $\mathfrak{p}$ a prime ideal of R and $S = R - \mathfrak{p}$. Then the residue field $k(\mathfrak{p})$ is a semifield if $\mathfrak{p}$ is a k-ideal and trivial if not.*

Proof. First assume that $\mathfrak{p}$ is a prime k-ideal. Then $\mathfrak{p}$ is the maximal prime ideal that does not intersect S and thus $\mathfrak{m} = S^{-1}\mathfrak{p}$ is the unique maximal of $S^{-1}R$. By Proposition 2.7.5, $\mathfrak{m}$ is a k-ideal. Thus the kernel of $S^{-1}R \to k(\mathfrak{p})$ is $\mathfrak{m}$, which shows that $k(\mathfrak{p})$ is not trivial. Since $(S^{-1}R)^\times = S^{-1}R - \mathfrak{m}$, we see that $(S^{-1}R)^\times \to k(\mathfrak{p}) - \{0\}$ is surjective, which shows that all nonzero elements of $k(\mathfrak{p})$ are invertible, i.e. $k(\mathfrak{p})$ is a semifield.

Next assume that $\mathfrak{p}$ is not a k-ideal. By Proposition 2.7.5, $\mathfrak{m} = S^{-1}\mathfrak{p}$ is not a k-ideal, which means that the kernel of $S^{-1}R \to k(\mathfrak{p})$ is strictly larger than $\mathfrak{m}$ and therefore contains a unit of $S^{-1}R$. This shows that $k(x)$ must be trivial. □

Corollary 2.7.8. *Let R be a nontrivial semiring. Then there exists a morphism $R \to k$ to a semifield k.*

Proof. By Exercise 2.6.10, R has a maximal k-ideal $\mathfrak{p}$. By Lemma 2.6.6, $\mathfrak{p}$ is prime. By Corollary 2.7.7, $k(\mathfrak{p})$ is a semifield, and the canonical morphism $R \to k(\mathfrak{p})$ verifies the claim of the corollary. □

Exercise 2.7.9. Let R be a semiring and $\mathfrak{p}$ a prime k-ideal of R. Show that $R/\mathfrak{p}$ is nontrivial and without zero divisors and that $k(\mathfrak{p})$ is isomorphic to $\mathrm{Frac}(R/\mathfrak{p})$. What happens if $\mathfrak{p}$ is a prime ideal that is not a k-ideal?

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Chapter 3

Monoids with zero

In this chapter, we introduce and investigate monoids with zero. As we will see that monoids with zero behave like semirings in many aspects. In particular, most results of Chapter 2 have an analogue for monoids with zero. We review these facts in the following and emphasize the analogy with semirings by a similar formal structure of this chapter with Chapter 2. We will see, though, that several facts and constructions are much simpler for monoids than for semirings.

3.1 The category of monoids with zero

Definition 3.1.1. A *monoid with zero* is a set A together with an associative and commutative multiplication $\cdot : A \times A \to A$ and two constants 0 and 1 such that $0 \cdot a = 0$ and $1 \cdot a = a$ for all $a \in A$. We often write ab for $a \cdot b$.

A *morphism between monoids with zeros* A_1 *and* A_2 is a map $f : A_1 \to A_2$ such that $f(0) = 0$, $f(1) = 1$ and $f(ab) = f(a)f(b)$. This defines the category Mon of monoids with zero.

Let A be a monoid with zero. A *submonoid of* A is a multiplicatively closed subset that contains 0 and 1. The *unit group of* A is the subset $A^\times$ of invertible elements of A.

Note that the multiplication of A restricts to $A^\times$ and turns it into an abelian group. Note further that the constants 0 and 1 of a monoid with zero A are uniquely determined by the properties $0 \cdot a = 0$ and $1 \cdot a = a$. Sometimes, we take the liberty to omit an explicit description of these constants and we call a monoid with zero simply a monoid if it clearly contains a zero. Note, however, that the property $f(0) = 0$ of a morphism of monoids with zero is not automatically implied by the other axioms; in other words, not every monoid morphism between monoids with zeros is a morphism of monoids with zeros.

Example 3.1.2. Every semiring R is a monoid with zero if we omit the addition from the structure. We write $R^\bullet$ for the multiplicative monoid of R.

Given a (multiplicatively written) abelian semigroup A with unit 1, we obtain a monoid with zero $A_0 = A \cup \{0\}$ by adding an element 0 satisfying $0 \cdot a = 0$ for all $a \in A_0$.

The *trivial monoid with zero* $\{0 = 1\}$ is a terminal object in Mon. The so-called *field with one element* $\mathbb{F}_1 = \{0, 1\}$ is initial in Mon.

Exercise 3.1.3. Show that Mon is complete and cocomplete. The proof can be done in analogy to the case of semirings, cf. Exercise 2.1.6. In particular, the product of monoids A_i is represented by the Cartesian product $\prod A_i$ and their coproduct is a union over finite tensor products over $\mathbb{F}_1$; the equalizer of two morphisms $f, g : A \to B$ is represented by $\mathrm{eq}(f, g) = \{a \in A | f(a) = g(a)\}$ and their coequalizer is the quotient of B by the congruence generated by the relations $f(a) \sim g(a)$ for $a \in A$.

Definition 3.1.4. A monoid with zero A is *without zero divisors* if for any $a, b \in R$, the equality $ab = 0$ implies that $a = 0$ or $b = 0$. It is *integral* (or *multiplicatively cancellative*) if $0 \neq 1$ and for any $a, b, c \in R$ the equality $ac = bc$ implies $c = 0$ or $a = b$.

Lemma 3.1.5. *An integral monoid with zero is without zero divisors.*

Proof. If R is integral and $ab = 0$, then $ab = 0 = 0 \cdot b$ implies $b = 0$ or $a = 0$. □

Note that as in the case of semirings, a nontrivial monoid with zero and without zero divisor is in general not integral. An example of such a monoid is a semiring with the corresponding properties, e.g. the multiplicative monoid $\mathbb{T}[T]^\bullet$ of the tropical polynomial algebra $\mathbb{T}[T]$.

3.2 Tensor products and free monoids with zero

Definition 3.2.1. Let $f_A : C \to A$ and $f_B : C \to B$ be two morphisms of monoids with zero. The *tensor product of A and B over C* is the set

$$A \otimes_C B \; = \; A \times B / \sim$$

where the equivalence relation $\sim$ is generated by relations of the form $(f_A(c)a, b) \sim (a, f_B(c)b)$ where $a \in A$, $b \in B$ and $c \in C$. We denote the equivalence class of (a, b) by $a \otimes b$. The multiplication of $A \otimes_C B$ is defined by the formula

$$(a \otimes b) \cdot (a' \otimes b') \; = \; aa' \otimes bb'.$$

Its zero is $0 \otimes 0$ and its one is $1 \otimes 1$. The tensor product $A \otimes_C B$ comes with the canonical maps $\iota_A : A \to A \otimes_C B$, sending a to $a \otimes 1$, and $\iota_B : B \to A \otimes_C B$, sending b to $1 \otimes b$.

Exercise 3.2.2. Verify that $A \otimes_C B$ is indeed a monoid with zero and that the canonical maps ι_A and ι_B are morphisms.

Formulate and prove the usual properties of the tensor product: (1) The tensor product is the colimit (or pushout) of the diagram $A \xleftarrow{f_A} C \xrightarrow{f_B} B$; (2) every C-bilinear morphism from $A \times B$ defines a unique morphism from $A \otimes_C B$; (3) the functor $- \otimes_C B$ is left adjoint to the functor $\mathrm{Hom}_C(B, -)$.

Exercise 3.2.3. Let B be monoids with zero and A be a (multiplicatively written) abelian semigroup with neutral element 1. Let $A_0 = A \cup \{0\}$ be the associated monoid with zero; cf. Example 3.1.2. Let $\mathbb{F}_1 \to A_0$ and $\mathbb{F}_1 \to B$ the unique morphisms from the initial object $\mathbb{F}_1$ into A_0 and B, respectively.

Show that the underlying set of $B \otimes_{\mathbb{F}_1} A_0$ is the *smash product* $B \wedge A_0$, which is the quotient of $B \times A_0$ by the equivalence relation generated by $(0, a) \sim (b, 0)$ for all $a \in A_0$ and $b \in B$.

Let $B[A] = B \otimes_{\mathbb{F}_1} A_0$, let $\iota_B : B \to B[A]$ be the canonical map and let $\bar{\iota}_A : A \to B[A]$ be the composition of the inclusion $A \to A_0$ followed by the canonical map $A_0 \to B[A]$. Conclude from Exercise 3.2.2 that $B[A] = B \otimes_{\mathbb{F}_1} A_0$ satisfies the following universal property: for every morphism $f_B : B \to C$ of monoids with zeros and every multiplicative map $f_A : A \to C$ with $f_A(1) = 1$, there is a unique morphism $F : B[A] \to C$ of monoids with zero such that $f_B = F \circ \iota_B$ and $f_A = F \circ \bar{\iota}_A$. Conclude that $B[A]$ is the analogue of a semigroup algebra for monoids with zeros; cf. section 2.3.

Definition 3.2.4. Given a monoid with zero A and a set $\{T_i\}_{i\in I}$, the *free monoid with zero over A in $\{T_i\}$* is the monoid with zero $A[T_i]_{i\in I} = A\otimes_{\mathbb{F}_1} S_0$ where $S = \{\prod T_i^{e_i}\}_{(e_i)\in\bigoplus\mathbb{N}}$ is the multiplicative semigroup of all monomials $\prod T_i^{e_i}$ in the T_i.

If $I=\{1,\dots,n\}$, then we write $A[T_1,\dots,T_n]$ for $A[T_i]_{i\in I}$. We write $a\prod T_i^{e_i}$ for $a\otimes\prod T_i^{e_i}$ and a for the element $a\prod T_i^0$, which we call it a *constant monomial of* $A[T_i]_{i\in I}$. We write $aT_{i_1}^{e_{i_1}}\cdots T_{i_n}^{e_{i_n}}$ for $a\prod T_i^{f_i}$ with $f_{i_k}=e_{i_k}$ for $k=1,\dots,n$ and $f_j=0$ otherwise.

Exercise 3.2.5. Let $f:R_1\to R_2$ be a morphism of semirings. Show that f is also a morphism of the underlying monoids, which we denote by $f^\bullet : R_1^\bullet\to R_2^\bullet$. Show that this defines a functor $(-)^\bullet : \mathrm{SRings}\to\mathrm{Mon}$.

This functor has left adjoint, which can be described as follows. Given a monoid A with zero 0_A, we define A^+ as the semiring $\mathbb{N}[A]/\mathfrak{c}(0_A)$, i.e. the semigroup algebra of A over $\mathbb{N}$ whose zero we identify with 0_A. Show that a morphism $f:A_1\to A_2$ of monoids with zero defines a semiring morphism $f^+ : A_1^+\to A_2^+$ by linear extension. Show that this defines a functor $(-)^+ : \mathrm{Mon}\to\mathrm{SRings}$, which is left adjoint to $(-)^\bullet : \mathrm{SRings}\to\mathrm{Mon}$, i.e. are bijections

$$\mathrm{Hom}_{\mathrm{Mon}}(A,R^\bullet) \xrightarrow{\sim} \mathrm{Hom}_{\mathrm{SRings}}(A^+,R)$$

for all monoids with zeros A and every semirings R, which are functorial in A and R.

3.3 Congruences of monoids

Definition 3.3.1. Let A be a monoid with zero. A *congruence on A* is an equivalence relation $\mathfrak{c}$ on A that is *multiplicative*, i.e. (a,b) and (c,d) in $\mathfrak{c}$ imply $(ac,bd)\in\mathfrak{c}$ for all $a,b,c,d\in A$.

Example 3.3.2. Let R be a semiring and $\mathfrak{c}$ a congruence on R. Then $\mathfrak{c}$ is also a congruence on the monoid $R^\bullet$.

Exercise 3.3.3. Let $k,n\in\mathbb{N}$. Show that the set

$$\mathfrak{c}_{k,n} \;=\; \big\{(T^{m+rk},T^{m+sk})\in\mathbb{F}_1[T]\times\mathbb{F}_1[T]\,\big|\,m,r,s\in\mathbb{N},\text{ and } m\geqslant n \text{ or } r=s=0\big\} \;\cup\; \big\{(0,0)\big\}$$

is a congruence on the free monoid with zero $\mathbb{F}_1[T]$ in T over $\mathbb{F}_1$ and that every congruence of $\mathbb{F}_1[T]$ is of this form.

Let $\mathfrak{c}$ be a congruence on A. Similar to the case of congruences for semirings, we write $a\sim_{\mathfrak{c}} b$, or simply $a\sim b$, to express that (a,b) is an element of $\mathfrak{c}$. The following proposition shows that congruences define quotients of monoids with zeros.

Proposition 3.3.4. *Let A be a monoid with zero and $\mathfrak{c}$ be a congruence on A. Then the association $[a]\cdot[b]=[ab]$ is well-defined on equivalence classes of $\mathfrak{c}$ and turn the quotient $A/\mathfrak{c}$ into a monoid with zero* $[0]$ *and neutral element* $[1]$.

The quotient map $\pi:A\to A/\mathfrak{c}$ is a morphism of monoids with zero that satisfies the following universal property: every morphism $f:A\to B$ such that $f(a)=f(b)$ whenever $a\sim_{\mathfrak{c}} b$ factors uniquely through π.

Proof. Given $a\sim a'$ and $b\sim b'$, we have $ab\sim a'b'$. Thus the multiplication of $A/\mathfrak{c}$ does not depend on the choice of representative and is therefore well-defined. It follows immediately that A is a monoid with zero $[0]$ and neutral element $[1]$ and that π a morphism of monoids with zero.

Let $f : A \to B$ be a morphism such that $f(a) = f(b)$ whenever $a \sim b$ in $\mathfrak{c}$. For f to factor into $\overline{f} \circ \pi$ for a morphism $\overline{f} : A/\mathfrak{c} \to B$, it is necessary that $\overline{f}([a]) = \overline{f} \circ \pi(a) = f(a)$. This shows that $\overline{f}$ is unique if it exists. Since $a \sim b$ implies $f(a) = f(b)$, we conclude that $\overline{f}$ is well-defined as a map. The verification of the axioms of a morphism are left as an exercise. □

Example 3.3.5. Let A be a monoid with zero and without zero divisors. Then $\mathfrak{c} = \{(a,b) \in A \times A | a \neq 0 \neq b\} \cup \{(0,0)\}$ is a congruence. The quotient $A/\mathfrak{c}$ is isomorphic to $\mathbb{F}_1$.

Exercise 3.3.6. Describe the quotients $\mathbb{F}_1[T]/\mathfrak{c}_{k,n}$ for $k,n \in \mathbb{N}$ where $\mathfrak{c}_{k,n}$ are the congruences from Exercise 3.3.3.

Definition 3.3.7. Let $f : A \to B$ be a morphism of monoids with zero. The *congruence kernel of* f is the relation $\mathfrak{c}(f) = \{(a,b) \in A \times A | f(a) = f(b)\}$ on A.

Lemma 3.3.8. *The congruence kernel $\mathfrak{c}(f)$ of a morphism $f : A \to B$ of monoids with zeros is a congruence on A.*

Proof. That $\mathfrak{c} = \mathfrak{c}(f)$ is an equivalence relation follows from the following calculations: $f(a) = f(a)$ (reflexive); $f(a) = f(b)$ implies $f(b) = f(a)$ (symmetry); $f(a) = f(b)$ and $f(b) = f(c)$ imply $f(a) = f(c)$ (transitive). Multiplicativity follows from: $f(a) = f(b)$ and $f(c) = f(d)$ imply $f(ac) = f(a) \cdot f(c) = f(b) \cdot f(d) = f(bd)$. This shows that $\mathfrak{c}$ is a congruence. □

As a consequence of this lemma, we see that for a monoid with zero A, the associations

$$\begin{array}{ccc} \{\text{congruences on } A\} & \longleftrightarrow & \{\text{quotients of } A\} \\ \mathfrak{c} & \longmapsto & A \to A/\mathfrak{c} \\ \mathfrak{c}(\pi) & \longleftarrow\!\shortmid & \pi : A \twoheadrightarrow B \end{array}$$

are mutually inverse bijections. We will see in section 3.4 that we have a similar discrepancy between quotients and ideals as in the case of semirings. In so far, one has to work with congruences when one wants to describe quotients of monoids with zeros.

Lemma 3.3.9. *Let A be a monoid with zero and $S \subset A \times A$ a subset. Then there is a smallest congruence $\mathfrak{c} = \langle S \rangle$ containing S. The quotient map $\pi : A \to A/\langle S \rangle$ satisfies the following universal property: every morphisms $f : A \to B$ with the property that $f(a) = f(b)$ whenever $(a,b) \in S$ factors uniquely through π.*

Proof. It is readily verified that the intersection of congruences is again a congruence. As a consequence, the intersection of all congruences containing S is the smallest congruence containing S.

Given any morphism $f : A \to B$ with the property that $f(a) = f(b)$ whenever $(a,b) \in S$, then the congruence kernel $\mathfrak{c}(f)$ must contain S and thus $\mathfrak{c} = \langle S \rangle$. Using Proposition 2.4.4, we see that f factors uniquely through π. □

This lemma shows that we can construct new monoids with zeros from known ones by prescribing a number of relations: let A be a monoid with zero and $\{a_i \sim b_i\}$ a set of relations on A, i.e. $S = \{(a_i, b_i)\}$ is a subset of $A \times A$. Then we define $A/\langle a_i \sim b_i \rangle$ as the quotient monoid $A/\langle S \rangle$.

Example 3.3.10. In $A = \mathbb{F}_1[T]/\langle T^2 \sim T \rangle$, we have $[T^{2+i}] = [T^{1+i}]$ for all $i \geqslant 0$, thus A consists of the residue classes $[0]$, $[1]$ and $[T]$, and $[T]^2 = [T]$ is an idempotent element of A.

Exercise 3.3.11. Let A be a monoid with zero and $\mathfrak{c}$ a congruence on A. Let $\mathfrak{c}^+$ be the congruence on the semiring A^+ that is generated by $\mathfrak{c} \subset A \times A \subset A^+ \times A^+$. Show that $A^+/\mathfrak{c}^+$ is isomorphic to $(A/\mathfrak{c})^+$.

3.4 Ideals

Definition 3.4.1. Let A be a monoid with zero. An *ideal of A* is a subset I of A such that 0 and ab are elements of I for all $a \in I$ and $b \in A$. Let $f : A \to B$ be a morphism of monoids with zero. The *(ideal) kernel of f* is the inverse image $\ker(f) = f^{-1}(0)$ of 0.

Let S be a subset of A. The *ideal generated by S* is the set $\langle S \rangle = \{as \in A | a \in A, s \in S \cup \{0\}\}$. The *congruence generated by S* is the congruence $\mathfrak{c}(S)$ generated by $\{(a,0)|a \in S\}$.

Note that $\langle S \rangle$ is the smallest ideal of A containing S. In particular, we have $\langle \emptyset \rangle = \{0\}$.

Exercise 3.4.2. Describe all ideals of $\mathbb{F}_1[T]$ and of $\mathbb{N}^\bullet$.

Proposition 3.4.3. *The kernel* $\ker(f)$ *of a morphism of $f : A \to B$ is an ideal and every ideal appears as a kernel. More precisely, if I is an ideal of A and $\mathfrak{c} = \mathfrak{c}(I)$ is the congruence generated by I, then I is the kernel of $\pi : A \to A/\mathfrak{c}$ and $\pi(a) = \pi(b)$ if and only if $a, b \in I$ or $a = b$.*

Proof. We begin with the verification that $\ker(f)$ is an ideal. Clearly $0 \in \ker(f)$. Let $a \in \ker(f)$ and $b \in R$. Then $f(ab) = f(a)f(b) = 0 \cdot f(b) = 0$ and $ab \in \ker(f)$. Thus $\ker(f)$ is an ideal.

It is easily verified that $\mathfrak{c} = \mathfrak{c}(I)$ has the explicit description

$$\mathfrak{c} \;=\; \big\{ (a,b) \in A \times A \,|\, a,b \in I \text{ or } a = b \big\}.$$

It follows that $\pi(a) = \pi(b)$ if and only if $a, b \in I$ or $a = b$, and that $\ker f = \{a \in A | \pi(a) = 0\} = I$, as claimed. □

We summarize: with a congruence $\mathfrak{c}$ on A, we can associate the kernel of the projection $\pi_\mathfrak{c} : A \to A/\mathfrak{c}$, which is an ideal; with an ideal I, we can associate the congruence $\mathfrak{c}(I)$ generated by I. We have that the kernel of $A \to A/\mathfrak{c}(I)$ is I and the congruence $\mathfrak{c}(\ker \pi_\mathfrak{c})$ is contained in $\mathfrak{c}$, but in general not equal to $\mathfrak{c}$. This leads to the following picture.

$$\begin{array}{ccc} \textbf{“kernels”} & & \textbf{“quotients”} \\ \big\{ \text{ideals of } A \big\} & \twoheadleftarrow\!\hookrightarrow & \big\{ \text{congruences on } A \big\} \end{array}$$

Exercise 3.4.4. Compare the ideals of $\mathbb{F}_1[T]$ with the congruences on $\mathbb{F}_1[T]$; cf. Exercises 3.3.3 and 3.4.2. Do the same exercise for $\mathbb{F}_1[T_1, T_2]$.

3.5 Prime ideals

Definition 3.5.1. Let A be a monoid with zero. An ideal I of A is *proper* if it is not equal to A. It is *maximal* if it is proper and if $I \subset J$ implies $I = J$ for any other proper ideal of A. It is *prime* if its complement $S = A - I$ is a multiplicative subset of R.

Let A be a monoid with zero. Then $\mathfrak{m} = A - A^\times$ is an ideal of A, which is necessarily the unique maximal ideal of A. This shows that every monoid with zero A is *local*, i.e. A contains a unique maximal ideal $\mathfrak{m}$ and it satisfies $A = A^\times \cup \mathfrak{m}$.

Exercise 3.5.2. Show that for every subset $J \subset \{1,\dots,n\}$, the ideals

$$\langle T_i | i \in J\rangle \;=\; \{0\} \cup \Big\{ \prod_{i=1}^{n} T_i^{e_i} \in \mathbb{F}_1[T_1,\dots,T_n] \,\Big|\, e_i > 0 \text{ for some } i \in J \Big\}$$

of $\mathbb{F}_1[T_1,\dots,T_n]$ are prime ideals and that every prime ideal of $\mathbb{F}_1[T_1,\dots,T_n]$ is of this form.

Lemma 3.5.3. *Let A be a monoid with zero and I an ideal of A. Then I is prime if and only if A/I is nontrivial and without zero divisors, and I is maximal if and only if $A/I = (A/I)^\times \cup \{[0]\}$.*

Proof. The ideal I is prime if and only if for all $a,b \in A$, $ab \in I$ implies that $a \in I$ or $b \in I$. Passing to the quotient A/I, this means that $[ab] = [0]$ implies $[a] = [0]$ or $[b] = [0]$ where we use that the kernel of $A \to A/I$ is I, cf. Proposition 3.4.3. This latter condition is equivalent to A/I being nontrivial and without zero divisors.

As observed above, I is maximal if and only if $I = A - A^\times$. In this case, A/I is isomorphic to $(A^\times)_0 = A^\times \cup \{0\}$ and thus satisfies $A/I = (A/I)^\times \cup \{[0]\}$. Conversely, if $[a]\cdot[b] = 1$ in A/I, then $ab = 1$ in A since $[a] \neq [0] \neq [b]$ and thus $[a] = \{a\}$ and $[b] = \{b\}$ by Proposition 3.4.3. Thus if $A/I = (A/I)^\times \cup \{[0]\}$, then $I = \ker(A \to A/I) = A - A^\times$. □

Lemma 3.5.4. *Every maximal ideal is a prime ideal.*

Proof. This follows immediately from the characterization of the unique maximal ideal as the complement of the unit group and the fact that the product of non-units is a non-unit. □

Lemma 3.5.5. *Let $f : A \to B$ be a morphism of monoids with zero and I an ideal of B. Then $f^{-1}(I)$ is an ideal of A. If I is prime, then $f^{-1}(I)$ is prime.*

Proof. We verify that $f^{-1}(I)$ is an ideal. Obviously, it contains 0. If $a \in f^{-1}(I)$ and $b \in A$, then $f(ab) = f(a)f(b) \in I$ and $ab \in f^{-1}(I)$. This shows that $f^{-1}(I)$ is an ideal.

Assume that I is prime, i.e. $S = B - I$ is a multiplicative set. Then $f^{-1}(S) = A - f^{-1}(I)$ is a multiplicative set of A and thus $f^{-1}(I)$ is a prime ideal of A. □

Remark 3.5.6. Similar to the case of semirings, there exist two concepts of prime congruences for monoids with zero. Namely, a congruence $\mathfrak{c}$ on a monoid with zero A is a *weak prime congruence on A* if $A/\mathfrak{c}$ is nontrivial and without zero divisors, and $\mathfrak{c}$ is a *strong prime congruence on A* if $A/\mathfrak{c}$ is integral.

3.6 Localizations

Definition 3.6.1. Let A be a monoid with zero and $S \subset A$ be a *multiplicative subset of A*, i.e. a subset that contains 1 and is closed under multiplication. The *localization of A at S* is the quotient $S^{-1}A$ of $S \times A$ by the equivalence relation that identifies (s,a) with (s',a') whenever there is a $t \in S$ such that $tsa' = ts'a$ in A. We write $\frac{a}{s}$ for the equivalence class of (s,a). The multiplication of $S^{-1}A$ is defined by the formula $\frac{a}{s}\cdot\frac{a'}{s'} = \frac{aa'}{ss'}$. The zero of $S^{-1}A$ is $\frac{0}{1}$ and its one is $\frac{1}{1}$.

We write $A[h^{-1}]$ for $S^{-1}A$ if $S = \{h^i\}_{i\in\mathbb{N}}$ for an element $h \in A$ and call $A[h^{-1}]$ the *localization of A at h*. We write $A_\mathfrak{p}$ for $S^{-1}A$ if $S = A - \mathfrak{p}$ for a prime ideal $\mathfrak{p}$ of A and call $A_\mathfrak{p}$ the *localization of A at $\mathfrak{p}$*.

If I is an ideal of A, then we write $S^{-1}I$ for the ideal of $S^{-1}A$ that is generated by $\{\frac{a}{1} | a \in I\}$.

Lemma 3.6.2. *Let A be a monoid with zero, I an ideal of A and S a multiplicative subset of A. Then*

$$S^{-1}I \;=\; \big\{ \tfrac{a}{s} \in S^{-1}A \,\big|\, a \in I, s \in S \big\}.$$

Proof. It is clear that $S^{-1}I$ contains the set $\{\frac{a}{1}|a \in I\}$ of generators of $S^{-1}I$. If we have proven that the set $I_S = \{\frac{a}{s}|a \in I, s \in S\}$ is an ideal, then it follows that it contains $S^{-1}I$. The reverse inclusion follows from the observation that for $\frac{a}{s} \in I_S$, we have $\frac{a}{s} = \frac{1}{s} \cdot \frac{a}{1} \in S^{-1}I$.

We are left with showing that I_S is an ideal. It obviously contains $\frac{0}{1}$. Given $\frac{a}{s} \in I_S$ and $\frac{b}{t} \in S^{-1}A$, then $\frac{a}{s} \cdot \frac{b}{t} = \frac{ab}{st} \in I_S$ since $ab \in I$. This verifies that I_S is an ideal of $S^{-1}I$ and finishes the proof of the lemma. □

Exercise 3.6.3. Let A be a monoid with zero and S a multiplicative subset of A. Show that the map $\iota_S : A \to S^{-1}A$, defined by $\iota_S(a) = \frac{a}{1}$, is a morphism of monoids with zero that maps S to the units of $S^{-1}A$. Show that it satisfies the usual universal property of localizations: every morphism $f : A \to B$ of monoids with zero that maps S to the units of B factors uniquely through ι_S. Show that ι_S is an epimorphism.

Proposition 3.6.4. *Let A be a semiring, S a multiplicative subset of A and $\iota_S : A \to S^{-1}A$ the localization morphism. Then the maps*

$$\begin{array}{ccc} \big\{\textit{prime ideals } \mathfrak{p} \textit{ of } A \textit{ with } \mathfrak{p} \cap S = \emptyset\big\} & \longleftrightarrow & \big\{\textit{prime ideals of } S^{-1}A\big\} \\ \mathfrak{p} & \overset{\Phi}{\longmapsto} & S^{-1}\mathfrak{p} \\ \iota_S^{-1}(\mathfrak{q}) & \overset{\Psi}{\longleftarrow\!\shortmid} & \mathfrak{q} \end{array}$$

are mutually inverse bijections.

Proof. To begin with, we verify that both Φ and Ψ are well-defined. Let $\mathfrak{p}$ be a prime ideal of A such that $\mathfrak{p} \cap S = \emptyset$. Then $S^{-1}\mathfrak{p} = \{\frac{a}{s}|a \in \mathfrak{p}, s \in S\}$ by Lemma 3.6.2. Consider $\frac{a}{s}, \frac{b}{t} \in S^{-1}A$ such that $\frac{a}{s} \cdot \frac{b}{t} = \frac{ab}{st} \in S^{-1}\mathfrak{p}$, i.e. $ab \in \mathfrak{p}$. Then $a \in \mathfrak{p}$ or $b \in \mathfrak{p}$ and thus $\frac{a}{s} \in S^{-1}\mathfrak{p}$ or $\frac{b}{t} \in S^{-1}\mathfrak{p}$. This shows that $S^{-1}\mathfrak{p}$ is a prime ideal of $S^{-1}A$ and that Φ is well-defined.

Let $\mathfrak{q}$ be a prime ideal of $S^{-1}A$. By Lemma 3.5.5, $\iota_S^{-1}(\mathfrak{q})$ is a prime ideal of A. Note that $\mathfrak{q}$ is proper and does not contain any element of the form $\frac{s}{t}$ with $s, t \in S$ since $\frac{t}{s} \cdot \frac{s}{t} = 1$. Thus $\iota_S^{-1}(\mathfrak{q})$ intersects S trivially. This shows that Ψ is well-defined.

We continue with the proof that $\Psi \circ \Phi$ is the identity, i.e. $\iota_S^{-1}(S^{-1}\mathfrak{p}) = \mathfrak{p}$ for every prime ideal $\mathfrak{p}$ of A that does not intersect S. The inclusion $\mathfrak{p} \subset \iota_S^{-1}(S^{-1}\mathfrak{p})$ is trivial. The reverse inclusion can be shown as follows. The set $\iota_S^{-1}(S^{-1}\mathfrak{p})$ consists of all elements $a \in A$ such that $\frac{a}{1} = \frac{b}{s}$ for some $b \in \mathfrak{p}$ and $s \in S$. This equation says that there is a $t \in S$ such that $tsa = tb$. Since $b \in \mathfrak{p}$, we have $tsa = tb \in \mathfrak{p}$. Since $ts \notin \mathfrak{p}$, we have $a \in \mathfrak{p}$, as desired.

We continue with the proof that $\Phi \circ \Psi$ is the identity, i.e. $S^{-1}\iota_S^{-1}(\mathfrak{q}) = \mathfrak{q}$ for every prime ideal $\mathfrak{q}$ of $S^{-1}A$. The inclusion $S^{-1}\iota_S^{-1}(\mathfrak{q}) \subset \mathfrak{q}$ is trivial. The reverse inclusion can be shown as follows. Let $\frac{a}{s} \in \mathfrak{q}$. Then $\frac{a}{1} = \frac{s}{1} \cdot \frac{a}{s} \in \mathfrak{q}$ and $a \in \iota_S^{-1}\mathfrak{q}$. Thus $\frac{a}{s} \in S^{-1}\iota_S^{-1}(\mathfrak{q})$, as desired. This concludes the proof of the proposition. □

Let A be a monoid with zero, $\mathfrak{p}$ a prime ideal of A and $S = A - \mathfrak{p}$. Then $S^{-1}\mathfrak{p}$ is the complement of the units of $S^{-1}A$ and therefore the unique maximal ideal of $S^{-1}A$.

Definition 3.6.5. Let A be a monoid with zero and $\mathfrak{p}$ a prime ideal of A. The *residue field at* $\mathfrak{p}$ is the monoid with zero $k(\mathfrak{p}) = A_{\mathfrak{p}}/\mathfrak{c}(S^{-1}\mathfrak{p})$ where S is the complement of $\mathfrak{p}$ in A and $\mathfrak{c}(S^{-1}\mathfrak{p})$ is the congruence on $A_{\mathfrak{p}}$ that is generated by $S^{-1}\mathfrak{p}$.

Let $\mathfrak{p}$ be a prime ideal of A. Then the residue field at $\mathfrak{p}$ comes with a canonical morphism $A \to k(\mathfrak{p})$, which is the composition of the localization map $A \to A_{\mathfrak{p}}$ with the quotient map $A_{\mathfrak{p}} \to k(\mathfrak{p})$.

Corollary 3.6.6. *Let A be a monoid with zero, $\mathfrak{p}$ a prime ideal of A and $S = A - \mathfrak{p}$. Then $k(\mathfrak{p})$ is nontrivial and $k(\mathfrak{p})^{\times} = k(\mathfrak{p}) - \{0\}$.*

Proof. Note that $\mathfrak{p}$ is the maximal prime ideal that does not intersect S. By Proposition 3.6.4, $\mathfrak{m} = S^{-1}\mathfrak{p}$ is the unique maximal of $S^{-1}A$. Thus the kernel of $S^{-1}A \to k(\mathfrak{p})$ is $\mathfrak{m}$, which shows that $k(\mathfrak{p})$ is nontrivial. Since $(S^{-1}A)^{\times} = S^{-1}A - \mathfrak{m}$, we see that $(S^{-1}A)^{\times} \to k(\mathfrak{p}) - \{0\}$ is surjective, which shows that all nonzero elements of $k(\mathfrak{p})$ are invertible. □

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