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Blueprints and tropical scheme theory

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Preface

These lecture notes accompany a course that I am giving in the term March–June 2018 at IMPA. I intend to add chapters accordingly to the progress of these lectures and to regularly put new versions of these notes online. To make the changes between the different version more visible, each version will carry a distinct date on the front page. To make it possible to print these notes chapter by chapter, chapters will start on odd pages and contain a partial bibliography. To make changes in older parts of the lectures visible, each chapter carries the date of the last changes on its initial page.

Aim of these notes

In these notes, we will introduce blueprints and blue schemes and explain how this theory can be used to endow the tropicalization of a classical variety with a schematic structure.

Once the basic constructions are explained, we discuss balancing conditions and connections to related theories as skeleta of Berkovich spaces, toroidal embeddings and log-structures. We put a particular weight on explaining open problems in this very young branch of tropical geometry.

Main references

The main aim of this course is to explain (parts of) the material of the papers [GG14] and [GG16] by Jeffrey and Noah Giansiracusa, [MR14] and [MR16] by Maclagan and Rincón, and [Lor15] by the author. There will be plenty of secondary references, which we will cite at the appropriate places.

A useful alternative source are the lecture notes [YALE17] of lectures at YALE, which were held by various experts in the area and organized by Mincheva and Payne.

I am grateful for any kind of feedback that helps me to improve these notes!

Chapter 1

Why tropical scheme theory?

In this first chapter, we explain the purpose of tropical scheme theory, its main achievements as of today and some of the central question of this new branch of tropical geometry. At the end of this chapter, we give a brief outline of the previsioned structure of the rest of these notes.

1.1 Tropical varieties

In brevity, a tropical variety is a balanced polyhedral complex. In this section, we explain this definition, starting with the case of a tropical curve, which is easier to formulate than its higher dimensional analogue.

Definition 1.1.1. A *tropical curve (in $\mathbb{R}^n$)* is an embedded graph Γ in $\mathbb{R}^n$ with possibly unbounded edges together with a weight function

$$m: \ \mathrm{Edge}\,\Gamma \longrightarrow \mathbb{Z}_{>0}$$

such that all edges have rational slopes and such that the following so-called *balancing condition* is satisfied for every vertex p of Γ: for every edge e containing p, let $v_e \in \mathbb{Z}^n$ be the *primitive vector*, which is the smallest nonzero vector pointing from p in the direction of e; then

$$\sum_{p\in e} m(e)\cdot v_e \ = \ 0.$$

Example 1.1.2. In Figure 1.1, we depict a tropical curve in $\mathbb{R}^2$, explaining the balancing condition at the three vertices of the curve.

The generalization of the involved notions to higher dimensions requires some preparation and leads us to the following definitions.

Definition 1.1.3. A *halfspace in* $\mathbb{R}^n$ is a subset of $\mathbb{R}^n$ of the form

$$H \ = \big\{ (x_1,\dots,x_n) \in \mathbb{R}^n \,\big|\, a_1x_1+\cdots+a_nx_n \geqslant b \big\}$$

with $a_1,\dots,a_n,b \in \mathbb{R}$. The halfspace H is *rational* if $a_1,\dots,a_n \in \mathbb{Q}$.

Definition 1.1.4. A *(rational) polyhedron P (in $\mathbb{R}^n$)* is an intersection of finitely many (rational) halfspaces in $\mathbb{R}^n$. A *face* of a polyhedron P is a nonempty intersection of P with a halfspace H such that the boundary of H does not contain interior points of P.

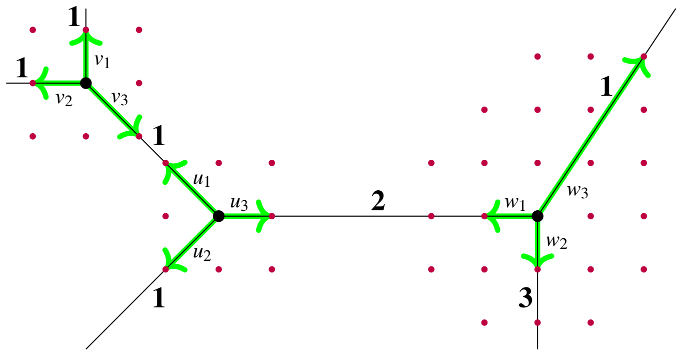

Figure 1.1: A tropical curve in $\mathbb{R}^2$ and the balancing condition

Note that the polyhedron P is a face of itself and that every face of a (rational) polyhedron is again a (rational) polyhedron.

Definition 1.1.5. A *polyhedral complex (in $\mathbb{R}^n$)* is a finite collection Δ of polyhedra in $\mathbb{R}^n$ such that the following two conditions are satisfied:

(1) each face of a polyhedron in Δ is in Δ;

(2) the intersection of two polyhedra in Δ is a face of both polyhedra or empty.

Definition 1.1.6. Let Δ be a polyhedral complex. The *support* of Δ is

$$|\Delta| \;=\; \bigcup_{P\in\Delta} P.$$

The *dimension* of Δ is $\dim\Delta \;=\; \max\{\dim P\,|\,P\in\Delta\}$. The polyhedral complex Δ is *equidimensional* if

$$|\Delta| \;=\; \bigcup_{\dim P=\dim\Delta} P$$

and Δ is *rational* if every polyhedron P in Δ is rational.

Exercise 1.1.7. Let H be a rational subvector space of $\mathbb{R}^n$, i.e. H has a basis in $\mathbb{Q}^n$. Show that the image of $\mathbb{Z}^n\subset\mathbb{R}^n$ under the quotient map $\pi:\mathbb{R}^n\to\mathbb{R}^n/H$ is a lattice, i.e. a discrete subgroup Λ that is isomorphic to $\mathbb{Z}^k$ where $k=n-\dim H$. The isomorphism $\Lambda\simeq\mathbb{Z}^k$ extends to an isomorphism $\mathbb{R}^n/H\simeq\mathbb{R}^k$ of vector spaces, i.e. we can identify π with a surjection $\pi':\mathbb{R}^n\to\mathbb{R}^k$ that maps $\mathbb{Z}^n$ to $\mathbb{Z}^k$. Show that the image $\pi'(P)$ of a rational polyhedron P in $\mathbb{R}^n$ is a rational polyhedron in $\mathbb{R}^k$.

Let P be a rational polyhedron in $\mathbb{R}^n$ and $x_0\in P$. Show that the subvector space H spanned by $\{x-x_0|x\in P\}$ is rational and does not depend on the choice of x_0. Choose an isomorphism $\mathbb{R}^n/H\simeq\mathbb{R}^k$ as above. Conclude that the image $\overline{P}$ of P in $\mathbb{R}^k$ is a 0-dimensional rational polyhedron. More generally, let Q be rational polyhedron that contains P as a face. Show that the image $\overline{Q}$ of Q in $\mathbb{R}^k$ is a rational polyhedron of dimension $\dim Q-\dim P$.

We call the image $\overline{Q}$ under the quotient map $\pi':\mathbb{R}^n\to\mathbb{R}^k$, as considered in Exercise 1.1.7, the *image of Q modulo the affine linear span of P*. If Q is a rational polyhedron of dimension $\dim Q=\dim P+1$ that contains P as a face, then the image $\overline{Q}$ of Q in $\mathbb{R}^k$ is a one dimensional rational polyhedron that contains $\overline{P}$ as a boundary point. Thus we can speak of the *primitive vector* $v_{\overline{Q}}$ *of* $\overline{Q}$ *at* $\overline{P}$, which is the smallest nonzero vector in $\mathbb{R}^k$ with integral coefficients that is pointing from $\overline{P}$ in the direction of $\overline{Q}$.

Definition 1.1.8. A *tropical variety (in* $\mathbb{R}^n$*)* is an equidimensional and rational polyhedral complex Δ together with a weight function

$$m : \left\{ P \in \Delta \middle| \dim P = \dim \Delta \right\} \longrightarrow \mathbb{Z}_{>0}$$

such that for every polyhedron $P \in \Delta$ with $\dim P = \dim \Delta - 1$, the top dimensional polyhedra in Δ containing P satisfy the balancing modulo the affine linear span of P, i.e.

$$\sum_{P \subsetneq Q} m(Q) v_{\overline{Q}} \;=\; 0$$

where $\overline{P}$ and $\overline{Q}$ are the images of P and Q modulo the affine linear span of P and where $v_{\overline{Q}}$ is the primitive vector of $\overline{Q}$ at $\overline{P}$.

1.2 Tropicalization of classical varieties

Let k be a field.

Definition 1.2.1. A *nonarchimedean absolute value of* k is a function $v : k \to \mathbb{R}_{\geqslant 0}$ such that for all $a, b \in k$,

(1) $v(0) = 0$ and $v(1) = 1$;

(2) $v(ab) = v(a)v(b)$;

(3) $v(a+b) \leqslant \max\{v(a), v(b)\}$.

An nonarchimedean absolute value is *trivial* if $v(a) = 1$ for all $a \in k^\times$. Otherwise it is called *nontrivial*. An nonarchimedean absolute value is *discrete* if $v(k^\times)$ is a discrete subset of $\mathbb{R}_{\geqslant 0}$.

A *nonarchimedean field* is an algebraically closed field k together with a nontrivial nonarchimedean absolute value v.

Exercise 1.2.2. Let v be a nonarchimedean absolute value on a field k. Show the following assertions.

(1) If v is trivial, then v is discrete. If k is algebraically closed and v is discrete, then v is trivial. Give an example of a discrete absolute value that is not trivial. If v is not discrete, then its image in $\mathbb{R}_{\geqslant 0}$ is dense.

(2) We have $v(k^\times) \subset \mathbb{R}_{>0}$ and $v(-1) = 1$. If $v(a) \neq v(b)$, then $v(a+b) = \max\{v(a), v(b)\}$. Conclude that if $\sum_{i=1}^n a_i = 0$ in k, then at least two terms $v(a_k)$ and $v(a_l)$ with $k \neq l$ assume the maximum $\max\{v(a_i)\}$.

For the rest of this chapter, we fix a nonarchimedean field (k, v). Let $X \subset (k^\times)^n$ be an algebraic variety, i.e. the zero set of Laurent polynomials $f_1, \dots, f_r \in k[T_1^{\pm 1}, \dots, T_n^{\pm 1}]$.

Definition 1.2.3. The *tropicalization of* X is defined as the topological closure $X^{\mathrm{trop}} = \overline{\mathrm{trop}(X)}$ of the image of X under the map

$$\mathrm{trop} : (k^\times)^n \xrightarrow{(v,\dots,v)} \mathbb{R}^n_{>0} \xrightarrow{(\log,\dots,\log)} \mathbb{R}^n.$$

Example 1.2.4. In Figure 1.2, we illustrate the tropicalization of a genus 1 curve E, embedded sufficiently general in $(k^\times)^2$. More precisely, we illustrate the tropicalization of the compactification $\overline{E}$ of E, which embeds into the projective plane $\mathbb{P}^2$. This means that all unbounded edges of the tropicalization of E gain a second boundary point, which we illustrate by bullets in Figure 1.2. Note that this picture suggests that tropicalizations preserve certain geometric invariants like the genus.

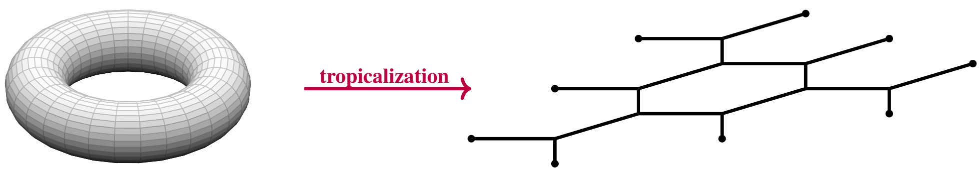

Figure 1.2: Tropicalization of an elliptic curve, including its points at infinity

Theorem 1.2.5 (Structure theorem for tropicalizations)**.** *Let (k,v) be a nonarchimedean field and $X \subset (k^\times)^n$ an equidimensional algebraic variety. Then*

(1) $X^{\text{trop}} = |\Delta|$ for a rational and equidimensional polyhedral complex Δ;

(2) $X \subset (k^\times)^n$ determines a weight function

$$m : \{P \in \Delta \,|\, \dim P = \dim \Delta\} \longrightarrow \mathbb{Z}_{>0}$$

such that (Δ, m) is a tropical variety.

The first part of the structure theorem has been proven by Bieri and Groves in their 1984 paper [BG84], which precedes tropical geometry by around 15 years and uses a slightly different setup than we do in our statement. The second part has been proven by Speyer in his thesis [Spe05]. A formulation of the structure theorem that is very close to ours appears in Maclagan and Sturmfels' book [MS15] as Theorem 3.3.6.

1.3 Two problems with the concept of a tropical variety

There are two oddities with the concept of a tropical variety that create difficulties for the development of algebro-geometric tools for tropical geometry and their application to tropicalizations of classical varieties.

The first problem is that the polyhedral complex Δ with $|\Delta| = X^{\text{trop}}$ is not determined by the classical variety $X \subset (k^\times)^n$. In other words,

the tropicalization of a classical variety is *not* a tropical variety.

The second problem relates to the functions of a tropical variety. The explanation of this issue requires some preliminary definitions.

Definition 1.3.1. The *tropical semifield* is the set $\mathbb{T} = \mathbb{R}_{\geqslant 0}$ together with the addition

$$a + b = \max\{a, b\}$$

and the usual multiplication

$$a \cdot b = ab$$

of nonnegative real numbers a, b.

Together with these operations $\mathbb{T}$ is indeed a semifield, i.e. it satisfies all of the axioms of a field except for the existence of additive inverses. The tropical semifield allows for the following reformulation of Definition 1.2.1: a nonarchimedean absolute value is a multiplicative map $v : k \to \mathbb{T}$ that is *subadditive*, i.e. $v(a+b) \leqslant v(a) + v(b)$ where the latter sum is taken with respect to the addition in $\mathbb{T}$.

Remark 1.3.2. In these lecture notes, we adopt the "max-times"-convention for the tropical numbers, which is less common than the "max-plus" or the "min-plus"-convention. To explain, the map $\log : \mathbb{T} \to \overline{\mathfrak{R}}$ defines an isomorphism of semirings between the tropical semifield $\mathbb{T}$ and the *max-plus-algebra* $\overline{\mathfrak{R}} = (\mathbb{R} \cup \{-\infty\}, \max, +)$. Multiplication of with (-1) defines an isomorphism $\overline{\mathfrak{R}} \to \overline{\mathbb{R}}$ between the max-plus-algebra with the *min-plus-algebra* $\overline{\mathbb{R}} = (\mathbb{R} \cup \{\infty\}, \min, +)$.

A priori, it is a matter of choice, with which semifield one works. But depending on the situation, some choices are more natural than others. When considering tropical varieties as polyhedral complexes, then the piecewise linear structure of the tropical variety is only visible in the logarithmic picture, i.e. one is led to work with the max-plus or the min-plus-algebra.

When working with tropical polynomials and tropical functions, in particular when compared to classical polynomials and functions, then it is more natural and less confusing to work with the max-times-convention.

Definition 1.3.3. The *tropical polynomial algebra in $T_1, \dots, T_n$* is the set

$$\mathbb{T}[T_1, \dots, T_n] \;=\; \Big\{ \sum_{J=(e_1,\dots,e_n)} a_J T_1^{e_1} \cdots T_n^{e_n} \,\Big|\, a_J \in \mathbb{T} \text{ and } a_J = 0 \text{ for almost all } J \Big\},$$

which is a semiring with respect to the usual addition and multiplication rules for polynomials where we apply the tropical addition $a_I + a_J = \max\{a_i, a_J\}$ to add coefficients.

A tropical polynomial $f = \sum a_J T_1^{e_1} \cdots T_n^{e_n}$ defines the function

$$\begin{array}{cccc} f(-): & \mathbb{T}^n & \longrightarrow & \mathbb{T}. \\ & x = (x_1, \dots, x_n) & \longmapsto & f(x) = \max\left\{a_J x_1^{e_1} \cdots x_n^{e_n}\right\} \end{array}$$

We are prepared to explain the second problem with tropical varieties. Namely, different polynomials can define the same function, as demonstrated in the following example.

Example 1.3.4. Consider $f_1 = T^2 + 1$ and $f_2 = T^2 + T + 1$. Then

$$f_1(x) = x^2 + 1 = \max\{x^2, 1\} = \max\{x^2, x, 1\} = f_2(x)$$

for all $x \in \mathbb{T}$.

In other words,

tropical functions are *not* the same as tropical polynomials.

To understand why tropical scheme theory promises to resolve these digressions, let us have a look at classical algebraic geometry.

For varieties over an algebraically closed field, Hilbert's Nullstellensatz guarantees that functions are the same as polynomials. However, if one tries to generalize the concept of a variety to arbitrary field or even rings, one faces the same problem: different polynomials can define the same function.

Grothendieck surpassed this problem with the invention of schemes. Since the functions of a tropical variety do not form a ring, but merely a semiring, it is clear that Grothendieck's concept of a scheme does not find applications in tropical geometry.

However, $\mathbb{F}_1$-geometry has provided a theory of so-called semiring schemes, cf. the papers [Dur07] of Durov, [TV09] of Toën-Vaquié and [Lor12] of the author. This theory and its refinement in terms of blueprints provides an appropriate framework for tropical scheme theory.

1.4 Semiring schemes

In this section, we give an idea of the definition of a semiring scheme. Similar to a scheme, it is built from the spectra of semirings. In order to understand this relation between tropical varieties and semiring schemes that we have in mind, we explain this concept in analogy to classical varieties and schemes, concentrating on the affine situation. More details about the construction of semiring schemes will be explained in later parts of these notes.

Let k be an algebraically closed field and $X \subset k^n$ a variety, i.e. the zero set of polynomials $f_1, \dots, f_r \in k[T_1, \dots, T_n]$. Let

$$I \;=\; \big\{ f \in k[T_1, \dots, T_n] \,\big|\, f(x_1, \dots, x_n) = 0 \text{ for all } (x_1, \dots, x_n) \in X \big\}.$$

be its ideal of definition and $A \;=\; k[T_1, \dots, T_n]/I$ its ring of regular functions.

The associated scheme is the spectrum of A, which is the set $\operatorname{Spec} A$ of all prime ideals of A together with the topology generated by the principal open subsets

$$U_h \;=\; \{ \mathfrak{p} \subset A \,|\, h \notin \mathfrak{p} \}$$

for $h \in A$ and with the structure sheaf

$$\begin{array}{rccc} \mathcal{O}: & \{\text{open subsets of } \operatorname{Spec} A\} & \longrightarrow & \text{Rings.} \\ & U_h & \longmapsto & A[h^{-1}] \end{array}$$

We can recover the variety X from $\operatorname{Spec} A$ as follows. The ring of regular functions $A = k[T_1, \dots, T_n]/I$ equals the ring of global sections

$$\mathcal{O}(\operatorname{Spec} A) \;=\; A[1^{-1}] \;=\; A.$$

The variety X is equal to the set of k-rational points of $\operatorname{Spec} A$, i.e. we have a canonical bijection

$$X \longrightarrow \operatorname{Hom}_k(A, k) \;=\; \operatorname{Hom}_k(\operatorname{Spec} k, \operatorname{Spec} A)$$

that sends a point $x = (x_1, \dots, x_n)$ of X to the evaluation map

$$\operatorname{ev}_x : \; h \;\mapsto\; h(x).$$

Its inverse sends a homomorphism $f : A \to k$ to the point $\big(f(T_1), \dots, f(T_n)\big)$ of X.

The definition of $\operatorname{Spec} A$ extends to any semiring A as follows. There are natural extensions of the notions of prime ideals and localizations from rings to semirings.

Definition 1.4.1. The *spectrum of A* is the set $\mathrm{Spec} A$ of all prime ideals of A together with the topology generated by the principal open subsets

$$U_h = \{\mathfrak{p} \subset A \,|\, h \notin \mathfrak{p}\}$$

for $h \in A$ and with the structure sheaf

$$\begin{array}{rccc} \mathcal{O}: & \{\text{open subsets of } \mathrm{Spec} A\} & \longrightarrow & \text{Semirings} \\ & U_h & \longmapsto & A[h^{-1}] \end{array}$$

A semiring scheme is a topological space together with a sheaf in the category of semiring that is locally isomorphic to the spectra of semirings. A detailed definition of all this terminology will be given in later chapters.

1.5 Scheme theoretic tropicalization

In this section, we give an outline of the Giansiracusa tropicalization, which associates with a classical variety a semiring scheme whose $\mathbb{T}$-rational points correspond to the set theoretic tropicalization as considered in section 1.2. For the sake of simplicity, we explain this for subvarieties of affine space opposed to suvarieties of a torus, which is the context of section 1.2.

We require some notation. For a multi-index $J = (e_1, \dots, e_n)$, we write $T^J = T_1^{e_1} \cdots T_n^{e_n}$ and $x^J = x_1^{e_1} \cdots x_n^{e_n}$. Let $f = \sum a_J T^J \in k[T_1, \dots, T_n]$. We define

$$f^{\mathrm{trop}} = \sum v(a_J) T^J \qquad \in \mathbb{T}[T_1, \dots, T_n].$$

Let $X \subset k^n$ a variety with ideal of definition I.

Definition 1.5.1. The *Giansiracusa tropicalization* of X is the semiring scheme

$$\mathrm{Trop}_v(X) = \mathrm{Spec}\Big(\mathbb{T}[T_1, \dots, T_n] \,/\, \mathrm{bend}_v(I)\Big)$$

where the *bend relations* $\mathrm{bend}_v(I)$ are defined as

$$\mathrm{bend}_v(I) = \Big(f^{\mathrm{trop}} \sim f^{\mathrm{trop}} + v(b_J) T^J \,\Big|\, f + b_J T^J \in I \Big).$$

The main result of Jeffrey and Noah Giansiracusa in [GG16] is the following connection to the set theoretic tropicalization X^{trop} of X, which stays in analogy to the corresponding result for schemes and varieties over an algebraically closed field.

Theorem 1.5.2 (Jeffrey and Noah Giansiracusa '13)**.** *We can recover the tropical variety X^{trop} as a set via a natural bijection*

$$X^{\mathrm{trop}} \xrightarrow{\sim} \mathrm{Hom}_{\mathbb{T}}(\mathrm{Spec}\,\mathbb{T}, \mathrm{Trop}_v(X)).$$

Moreover, in case of a projective variety X, the Giansiracusa brothers associate with $\mathrm{Trop}_v(X)$ a Hilbert polynomial and show that it coincides with the Hilbert polynomial of X. This might be seen as the first striking result of tropical scheme theory.

Diane Maclagan and Felipe Rincón have shown in [MR14] that the embedding of $\mathrm{Trop}_v(X)$ into the n-dimensional tropical torus remembers the weights of the tropical variety X^{trop}, provided one has chosen the structure of a polyhedral complex. To wit, the embedding of a variety X into $(k^\times)^n$ yields an embedding of $\mathrm{Trop}_v(X)$ into the n-dimensional tropical torus $\mathrm{Spec}\,\mathbb{T}[T_1^{\pm 1}, \dots, T_n^{\pm 1}]$.

Theorem 1.5.3 (Maclagan-Rincón '14). *Assume that $X \subset (k^\times)^n$ is equidimensional. Then the weight function m of any realization of X^{trop} as a tropical variety (Δ, m) is determined by the embedding of $\text{Trop}_v(X)$ into $\text{Spec}\,\mathbb{T}[T_1^{\pm 1}, \dots, T_n^{\pm 1}]$.*

In the author's paper [Lor15], the above results are refined and generalized by using blueprints and blue schemes. We mention two applications of this refined approach: the Giansiracusa tropicalization can be applied to more general situations than tropicalizations of subvarieties of toric varieties; for instance, it is possible to endow skeleta of Berkovich spaces with a schematic structure under certain additional hypotheses. Another feature is that the weight function of the tropical variety is already encoded into the structure sheaf of the "blue tropical scheme", which opens the possibility for a theory of abstract tropical schemes, opposed to embedded tropical schemes.

1.6 A central problem in tropical scheme theory

The aforementioned results give hope that the replacement of tropical varieties by tropical schemes will allow for new tools in tropical geometry, such as sheaf cohomology or a cohomological interpretation of intersection theory. However, it is not at all clear what a good notion of a "tropical scheme" might be.

The theory of semiring schemes comes with the notion of a $\mathbb{T}$-scheme, which is a morphism $X \to \text{Spec}\,\mathbb{T}$ of semiring schemes. However, there are too many $\mathbb{T}$-schemes to make this a useful class. For example, every hyperplane in $\mathbb{R}^n$ can be realized as a $\mathbb{T}$-scheme, and such subsets of $\mathbb{R}^n$ cannot satisfy the balancing condition with respect to any polyhedral subdivision and any choice of weight function. Even worse, every intersection of hyperplanes can be realized as $\mathbb{T}$-schemes, and such intersections include all bounded convex subsets of $\mathbb{R}^n$, e.g. the unit ball.

This makes clear that we have to restrict our attention to a subclass of $\mathbb{T}$-schemes in order to obtain a useful class that could replace the class of tropical varieties. Maclagan and Rincon make a suggestion for such a class, which is based on the observation that the ideal of definition of the tropicalization of a classical variety is a valuated matroid. In [MR14] and [MR16], they investigate the class of $\mathbb{T}$-schemes whose ideal of definition is a valuated matroid and show certain desirable properties like chain conditions for "tropical ideals" and the preservation of Hilbert functions.

Unfortunately, this theory encounters some serious difficulties since the class of tropical ideals is, a priori, too restrictive. For instance, the ideals of definition of some prominent spaces in tropical geometry, like linear tropical spaces and Grassmannians, are not tropical ideals. Moreover both the intersection and the sum of two tropical ideals fail to be a tropical ideal in general, which provides obstacles for primary decompositions and intersection theory of schemes, respectively.

It might be the case that there is natural way to associate a "generically generated" tropical ideal with ideals occuring in the situations explained above, but this seems to be a difficult problem. It might be the case that the class of tropical ideals, as considered in [MR14], is too restrictive for a useful theory of "tropical schemes".

In so far, we formulate the central problem of tropical scheme theory in the following way. We would like to find a class $\mathcal{C}$ of $\mathbb{T}$-schemes that satisfies the following criteria:

- $\mathcal{C}$ contains the tropicalizations of all classical varieties and for every tropical variety, $\mathcal{C}$ contains a $\mathbb{T}$-scheme representing it;

- $\mathcal{C}$ contains "universally constructable $\mathbb{T}$-schemes" such as tropical linear spaces and tropical Grassmannians;
- the $\mathbb{T}$-rational points of every $\mathbb{T}$-scheme in $\mathcal{C}$ yields a tropical variety; in particular, this involves a theory of balancing conditions for $\mathbb{T}$-schemes;
- defining ideals of schemes in $\mathcal{C}$ are closed under intersections and sums;
- $\mathcal{C}$ allows for a dimension theory by considering chains of irreducible reduced $\mathbb{T}$-schemes in $\mathcal{C}$; in particular, this involves the notion of an irreducible $\mathbb{T}$-scheme.

A more comprehensive list of open problems in tropical scheme theory was compiled at a workshop in April 2017 at the American Institute of Mathematics, see [AIM17] for a link to the problem list.

1.7 Outline of the previsioned contents of these notes

The central goal of these notes is to explain the material of the previous sections in detail. This includes reviewing some parts of "classical" tropical geometry and introducing semiring schemes, monoid schemes and blue schemes. We intend to discuss the Giansiracusa tropicalization and subsequent results from the papers [GG14] and [GG16] by Jeffrey and Noah Giansiracusa, [MR14] and [MR16] by Maclagan and Rincón, and [Lor15] by the author.

If we achieve this central goal in time, then we intend to treat more advanced topics like scheme theoretic skeleta of Berkovich spaces, schemes over the tropical hyperfield or families of matroids.

The chapters of these notes will be grouped into parts. The first part reviews the algebraic foundations, which are (ordered) semirings, monoids, blueprints, localizations, ideals and congruences. The second part is dedicated to generalized scheme theory and contains the constructions of semiring schemes, monoid schemes and blue schemes. The third part enters the central the theme of these notes, which is scheme theoretic tropicalization.

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