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The dynamics of the Jouanolou foliation on the complex projective 2-space

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Abstract. We prove that the Jouanolou foliation of degree $k \leq 5$ admits no nontrivial minimal sets. The proof uses reliable computations based on interval arithmetic.

1. *Introduction*

A differential equation of the form

$$\begin{aligned}\dot{x} &= \frac{dx}{dt} = P(x,y) \\ \dot{y} &= \frac{dy}{dt} = Q(x,y), \qquad (x,y) \in \mathbf{C}^2,\end{aligned} \tag{1}$$

where P and Q are polynomials with complex coefficients, and $t \in \mathbf{C}$, defines a complex one-dimensional foliation $\mathcal{F}$ on the complex projective two-dimensional space $\mathbf{C}\mathrm{P}(2)$. This foliation is defined as the natural extension to $\mathbf{C}\mathrm{P}(2)$ of the differential equation

$$P(x,y)\,dy - Q(x,y)\,dx = 0. \tag{2}$$

We can write

$$\begin{aligned}P(x,y) &= P_k(x,y) + xR_k(x,y) \\ Q(x,y) &= Q_k(x,y) + yR_k(x,y),\end{aligned}$$

where R_k is a homogeneous polynomial of degree k, and P_k and Q_k are polynomials of degree at most k [**7**]. Then, after the change of variables $x = X/Z$ and $y = Y/Z$, we obtain that equation (2) is equivalent to

$$\begin{vmatrix} dX & dY & dZ \\ X & Y & Z \\ \widetilde{P}_k & \widetilde{Q}_k & R_k \end{vmatrix} = 0, \tag{3}$$

where $\widetilde{P}_k(X,Y,Z) = Z^k P_k(X/Z, Y/Z)$ and $\widetilde{Q}_k(X,Y,Z) = Z^k Q_k(X/Z, Y/Z)$

are the "homogeneizations" of P_k and Q_k, respectively [**3**]. Then we say that $\mathcal{F}$ *has degree* k.

Reciprocally, every complex foliation on $\mathbf{C}\mathrm{P}(2)$ is defined by an equation (3) and has an affine expression given in (2) [**7**]. Clearly, we can cover $\mathbf{C}\mathrm{P}(2)$ by three affine charts, corresponding to $Z = 1$ (given in (2)), $Y = 1$, and $X = 1$.

A *singular point* of the foliation $\mathcal{F}$ is a point where some affine expression of $\mathcal{F}$ vanishes; for instance, in (2), a singular point is a common zero of P and Q. We assume that the singular points of $\mathcal{F}$ are isolated (i.e., P and Q have no common factor). Thus, if $\mathcal{F}$ has degree k, then $\mathcal{F}$ will have $N = k^2 + k + 1$ singular points [**1**].

A *minimal set* of $\mathcal{F}$ is an invariant (i.e., union of leaves and singular points), closed, nonempty subset of $\mathbf{C}\mathrm{P}(2)$ such that it is minimal with these three properties. We say that a minimal set is *nontrivial* if it is not a singular point.

A fundamental problem about the dynamics of these foliations $\mathcal{F}$ on $\mathbf{C}\mathrm{P}(2)$ was first posed in [**3**]:

PROBLEM. *Are there foliations of* $\mathbf{C}\mathrm{P}(2)$ *with nontrivial minimal sets?*

Although this problem is not yet solved, several properties of nontrivial minimal sets $\mathcal{M}$ have been established, provided they exist [**3**]:

(i) $\mathcal{M}$ is unique;

(ii) there is no $\mathcal{F}$-invariant transverse measure supported on $\mathcal{M}$, and so all the leaves in $\mathcal{M}$ have exponential growth;

(iii) $\mathcal{M}$ intersects every one-dimensional algebraic subset of $\mathbf{C}\mathrm{P}(2)$, and thus $\mathcal{F}$ cannot have algebraic leaves. Moreover, there are leaves in $\mathcal{M}$ with nontrivial linear holonomy [**2**].

The following example was studied by Jouanolou [**6**], and has been useful in proving that the set of foliations on $\mathbf{C}\mathrm{P}(2)$, of degree $k \geq 2$, not admitting algebraic leaves is open and dense in the space of all foliations of degree k [**6, 7**]:

$$\mathcal{D}_k : \quad \begin{vmatrix} dX & dY & dZ \\ X & Y & Z \\ Y^k & Z^k & X^k \end{vmatrix} = 0.$$

In particular, $\mathcal{D}_k$ has no algebraic one-dimensional leaves if $k \geq 2$. Thus, it is natural to ask whether this example admits a nontrivial minimal set.

In this paper, we prove that the answer is "no" for $k \leq 5$, and so all leaves of $\mathcal{D}_k$ approach a singular point:

THEOREM 1. *For* $k \leq 5$, *the foliation* $\mathcal{D}_k$ *does not admit a nontrivial minimal set.*

The strategy of the proof is as follows:

(i) Exploit the symmetries associated to $\mathcal{D}_k$ to find small, thin regions of $\mathbf{C}\mathrm{P}(2)$ that every minimal set of $\mathcal{D}_k$ must cross. We prove in §2 that two small sectors of angle π/N of the unit disk on the coordinate planes $\mathbf{C} \times 0$ and $0 \times \mathbf{C}$ suffice.

(ii) Find a sphere centered at the real singular point $(1,1)$ that is transversal to $\mathcal{D}_k$. Then, every orbit that enters this sphere must accumulate on $(1,1)$. The radius of this sphere as a function of k is given in §3.

(iii) Show that all orbits starting in the two sectors of step (i) enter this transversal sphere in finite time. In fact, we show in §4 that it is enough to consider the *real* flow.

For the last two steps, we give reliable computational proofs based on interval arithmetic [**9**].

2. *The symmetries of* $\mathcal{D}_k$

The set $\operatorname{sing}(\mathcal{D}_k)$ of singularities of $\mathcal{D}_k$ can be obtained easily from its affine expression:

$$\begin{aligned} \dot{x} &= y^k - x^{k+1} \\ \dot{y} &= 1 - x^k y. \end{aligned} \tag{4}$$

As mentioned in §1, $\operatorname{sing}(\mathcal{D}_k)$ is the set of common zeros of $y^k - x^{k+1}$ and $1 - x^k y$. Thus, $\operatorname{sing}(\mathcal{D}_k) = \{\, (\zeta^j, \zeta^{-jk}) : j = 0, \dots, N-1 \,\}$, where $N = k^2+k+1$ and ζ is a primitive N-th root of unity. All these singular points lie on the real two-dimensional torus $S^1 \times S^1 = \{\, (x,y) \in \mathbf{C}^2 : |x| = |y| = 1 \,\}$.

The foliation $\mathcal{D}_k$ is invariant by the group of diffeomorphisms of $\mathbf{C}\mathrm{P}(2)$ generated by $S(X,Y,Z) = (Y,Z,X)$ and $T(X,Y,Z) = (\zeta X, \zeta^{-k} Y, Z)$. Thus, the map $\tau(x,y) = (\zeta x, \zeta^{-k} y)$ is leaf-preserving for the differential equation (4). In particular, τ preserves the set of singular points of $\mathcal{D}_k$. Since the Jacobian matrix of (4) at the point $(1,1)$ has eigenvalues with nonreal quotient [**7**], *all* the singular points of $\mathcal{D}_k$ are in the Poincaré domain [**4**].

LEMMA 1. *Consider the real differential equation in* $\mathbf{R}^2$*:*

$$\begin{aligned} \dot{x} &= y^k - x^{k+1} \\ \dot{y} &= 1 - x^k y. \end{aligned}$$

If k is even, then the ω-limit set of every point in $\mathbf{R}^2$ is the singular point $(1,1)$. If k is odd, then the ω-limit set of $(0,0)$ is the singular point $(1,1)$.

Proof. As described above, this equation has only one real singular point, $(1,1)$. Figure 1 shows a large square centered at the origin, containing the singular point (the large dot). The point P is the point on the top side of the square where the field is horizontal. The curve passing through P is part of its orbit. In both cases, the region shown in thick lines contains the singular point $(1,1)$ and the field always points toward the interior of the region, and so this region is an attracting region for the singular point, in the sense that all orbits entering the region accumulate on $(1,1)$. The assertion follows because the square is arbitrary. (Actually, when k is odd, we see that the ω-limit set of every point in the first quadrant is the singular point $(1,1)$.) □

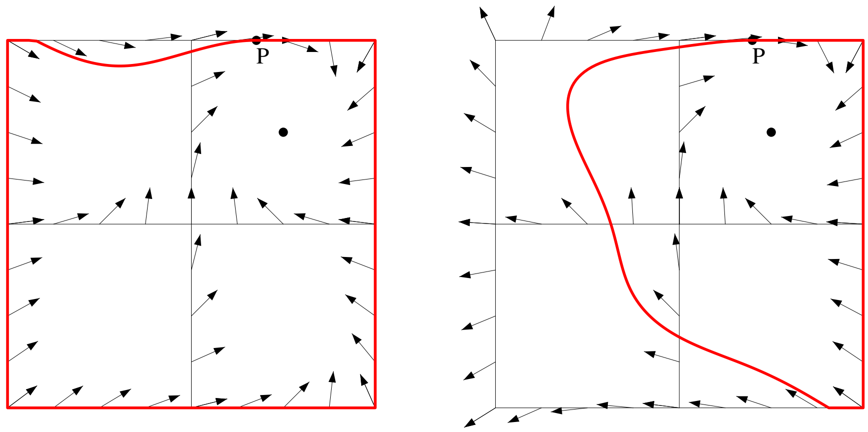

FIGURE 1. Attracting regions for the real Jouanolou foliation, for k even (left) and for k odd (right).

PROPOSITION 1. *The leaf of $\mathcal{D}_k$ passing through $(0,0) \in \mathbf{C}^2$ accumulates on all singular points of $\mathcal{D}_k$.*

Proof. By Lemma 1, the leaf passing through $(0,0) \in \mathbf{C}^2$ accumulates on $(1,1)$. The real phase space described in Figure 1 is reproduced in each of the real planes $\zeta^j\mathbf{R} \times \zeta^{-jk}\mathbf{R} \subseteq \mathbf{C}^2$, for $j = 1, \dots, N-1$, because the map $(x_j, y_j) = \tau^j(x,y) = (\zeta^j x, \zeta^{-jk} y)$ takes the leaves of (4) to the solutions of the differential equation

$$\begin{aligned} \dot{x}_j &= y_j^k - x_j^{k+1} \\ \dot{y}_j &= 1 - x_j^k y_j, \end{aligned}$$

which, of course, is the same as (4).

Thus, the leaf of $\mathcal{D}_k$ passing through the point $(0,0) \in \mathbf{C}^2$ also accumulates on $\tau^j(1,1) = (\zeta^j, \zeta^{-jk})$; and this holds for all j. □

LEMMA 2. *If $\mathcal{M}$ is a nontrivial minimal set of $\mathcal{D}_k$, then $\mathcal{M}$ intersects the polydisc $\{\,(x,y) \in \mathbf{C}^2 : |x| \le 1, |y| \le 1\,\}$. More precisely, $\mathcal{M}$ has a nontrivial intersection either with the disc $D_1 = \{\,(x,0) \in \mathbf{C}^2 : |x| \le 1\,\}$ or with the disc $D_2 = \{\,(0,y) \in \mathbf{C}^2 : |y| \le 1\,\}$*

Proof. By Property (iii) of §1, $\mathcal{M}$ must intersect the x-axis. The symmetry $S^2(X,Y,Z) = (Z,X,Y)$, seen in $\mathbf{C}\mathrm{P}(2)$, sends the y-axis centered at $0 \in \mathbf{C}^2$ to the x-axis centered at infinity. More precisely, suppose that $(u,v) = (Y/X, Z/X)$ is the chart $X = 1$. Then, $v = 1/x$, $u = y/x$, and

$$(u = 0) \cap (y = 0) = \{\,(x,0) : x \neq 0\,\} = \{\,(0,v) : v \neq 0\,\}\,.$$

Thus, $S^2(D_2) = \{\,(0,v) : |v| \le 1\,\}$. By the uniqueness of $\mathcal{M}$ (Property (i) of §1), $S^{-2}(\mathcal{M}) = \mathcal{M}$ and so if $\mathcal{M}$ intersects the x-axis only at points outside D_1, then $\mathcal{M}$ will intersect the disc D_2. □

From Lemma 2, it follows that in order to prove that a nontrivial minimal set for $\mathcal{D}_k$ does not exist it is enough to show that the leaves of $\mathcal{D}_k$ passing through $D_1 \cup D_2$ accumulate on the singularities of $\mathcal{D}_k$. In fact, since $\mathcal{D}_k$ is invariant by the symmetry $\tau(x, y) = (\zeta x, \zeta^{-k} y)$ and also by complex conjugation (because its coefficients are real), it is enough to verify this in sectors of D_1 and D_2 of angle π/N. The simplest such sectors are $S_1 = S_0 \times 0 \subseteq D_1$ and $S_2 = 0 \times S_0 \subseteq D_2$, where

$$S_0 = \{\, z \in \mathbf{C}^2 : |z| \leq 1,\ 0 \leq \arg(z) \leq \pi/N \,\},$$

which we shall consider in §4.

3. *A large transversal sphere around the real singularity*

In §2, we have limited the location of a possible minimal set to a small, thin region of $\mathbf{C}\mathrm{P}(2)$. We shall now find a large attracting sphere around a singularity.

PROPOSITION 2. *The sphere in $\mathbf{C}^2$ of radius R centered at $(1, 1)$ is transversal to $\mathcal{D}_k$, where the value of R for $k \leq 5$ is given in Table 1.*

Proof. We have to show that for all points (x, y) on the boundary of that sphere the normal to the sphere at (x, y) is not parallel to the field, i.e.,

$$\langle (x-1, y-1), (y^k - x^{k+1}, 1 - x^k y)\rangle_{\mathbf{C}} \neq 0.$$

It is enough to show that the real part of this inner product in never zero (the real part is also the real inner product). So, we show that

$$f(x_1, x_2, y_1, y_2) = \mathrm{Re}\langle (x-1, y-1), (y^k - x^{k+1}, 1 - x^k y)\rangle_{\mathbf{C}} \ll 0,$$

where $x = x_1 + ix_2$ and $y = y_1 + iy_2$, by showing that

$$\max f(x_1, x_2, y_1, y_2) \ll 0$$

for all points (x_1, x_2, y_1, y_2) on the unit sphere in $\mathbf{R}^4$.

This is now a global optimization problem in $\mathbf{R}^4$. By using a program for reliable global optimization based on interval arithmetic (such as described in [**5**]), we get a computational proof that $f(x_1, x_2, y_1, y_2) \leq M < 0$, where the upper bound M is given in Table 1 (rounded up to two significant figures). □

TABLE 1. Radius R of transversal spheres and upper bound M as a function of the degree k.

k	2	3	4	5
R	1.02	0.57	0.36	0.26
M	-0.017	-0.0020	-0.0064	-0.0015

4. *All orbits approach the real singularity*
Proposition 2 implies that if an orbit enters the transversal sphere then the orbit remains inside it and exponentially approaches the real singularity [**3**]. Therefore, given the results of §2, to show that all orbits approach the singular set, it is enough to show that all orbits starting in the sectors S_1 and S_2 enter the transversal sphere. We prove this by covering S_1 and S_2 with a set of rectangles and showing that each rectangle is mapped into the transversal sphere by the real flow.

For this part of proof, we used AWA, a program for the reliable solution of initial value problems [**8**]. This program is able to give a reliable bound for the location at a given time of *all* orbits starting in a given box. More precisely, given a box B_0 and a time t_1, the program uses interval arithmetic to compute a box B_1 such that every orbit starting in B_0 is inside B_1 at time t_1 (the program also proves that the solution exists at t_1). For a survey of reliable computer solution of initial value problems, see [**10**].

To cover the sectors with rectangles that can be verified by AWA to go into the transversal sphere, we used a simple adaptive algorithm: Starting with a single rectangle covering the sector, we refined the covering as needed, according to the output of AWA. At each step, we took one rectangle from the list and tested it with AWA. If AWA was able to show that the rectangle is taken into the transversal sphere, then the rectangle was accepted; otherwise, the rectangle was bisected across its longest side and the two pieces were put into the list for future testing. Rectangles that safely did not intersect the sector were ignored. The algorithm is given more formally below. Similar adaptive schemes are used in global optimization [**5**].

Input: Rectangle B covering sector S.
Output: List of rectangles covering S such that each rectangle has been proved by AWA to be taken into the transversal sphere.

```
ℒ ← {B}
while ℒ is not empty do
    select a rectangle X from ℒ
    if X intersects S then
        run AWA over X until X is taken into the sphere or AWA gives up
        if X was taken into the sphere then
            output X
        else
            subdivide X into X1 and X2
            put X1 and X2 into ℒ
```

That this algorithm actually stops, instead of refining the cover forever, is a computational proof that all orbits starting in the given sector S do enter the transversal sphere in finite time. The output for $k \leq 5$ is shown in Figures 2 to 5. For more information on the use of AWA in this algorithm, see the Appendix.

5. *Conclusion*

Experiments with ordinary numerical methods for the solution of initial value problems indicate that the region on the y-axis that is taken into the transversal sphere shrinks fast as k grows (as Table 1 suggests, so does the radius R of this sphere). For $k > 5$, this basin of attraction seems to still contain S_2, but just barely. Note the very fine coverings in Figures 4 and 5 already required for $k = 4$ and $k = 5$. We expect that much finer coverings would be needed for $k > 5$, specially because AWA uses a fixed-order Taylor series. Already for $k = 5$ we had to switch to a Taylor series of order 60 to handle the last eight rectangles covering S_2. A reliable integration method that uses adaptive order and step size might be able to handle more cases than AWA.

Theorem 1 implies that there is a neighbourhood of $\mathcal{D}_k$ such that no foliation in this neighbourhood admits a nontrivial minimal set. In principle, the computations described in this paper could be adapted to find such a neighbourhood explicitly, but the resulting real equations would be very complicated.

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REFERENCES

[1] P. F. Baum and R. Bott. On the zeros of meromorphic vector-fields. In *Essays on Topology and Related Topics (Mémoires dédiés à Georges de Rham)*. Springer, New York (1970) pp. 29–47. MR 41:6248.

[2] C. Bonatti, R. Langevin, and R. Moussu. Feuilletages de $\mathbf{C}\mathrm{P}(n)$: de l'holonomie hyperbolique pour les minimaux exceptionnels. *Publications Mathématiques de l'IHES* **75** (1992) 123–134. MR 94b:32055.

[3] C. Camacho, A. Lins Neto, and P. Sad. Minimal sets of foliations on complex projective spaces. *Publications Mathématiques de l'IHES* **68** (1988) 187–203. MR 90e:58129.

[4] J. Guckenheimer. Hartman's theorem for complex flows in the Poincaré domain. *Compositio Math.* **24** (1972) 75–82. MR 46:920.

[5] E. Hansen. *Global optimization using interval analysis*. Marcel Dekker, Inc., New York (1992). MR 93i:90002.

[6] J. P. Jouanolou. *Équations de Pfaff algébriques*. Lecture Notes in Mathematics 708. Springer, Berlin (1979). MR 81k:14008.

[7] A. Lins Neto. Algebraic solutions of polynomial differential equations and foliations in dimension two. In *Holomorphic dynamics (Mexico, 1986)*, Lecture Notes in Mathematics 1345. Springer, Berlin-New York (1988) pp. 192–232. MR 90c:58142.

[8] R. J. Lohner. AWA: software for the computation of guaranteed bounds for solutions of ordinary initial value problems. Institut für Angewandte Mathematik, Universität Karlsruhe (1994). ftp://iamk4515.mathematik.uni-karlsruhe.de/pub/awa.

[9] R. E. Moore. *Interval analysis*. Prentice-Hall, Inc., Englewood Cliffs, N.J. (1966). MR 37:7069.

[10] N. S. Nedialkov, K. R. Jackson, and G. F. Corliss. Validated solutions of initial value problems for ordinary differential equations. *Appl. Math. & Comp.* **105** (1999) 21–68.

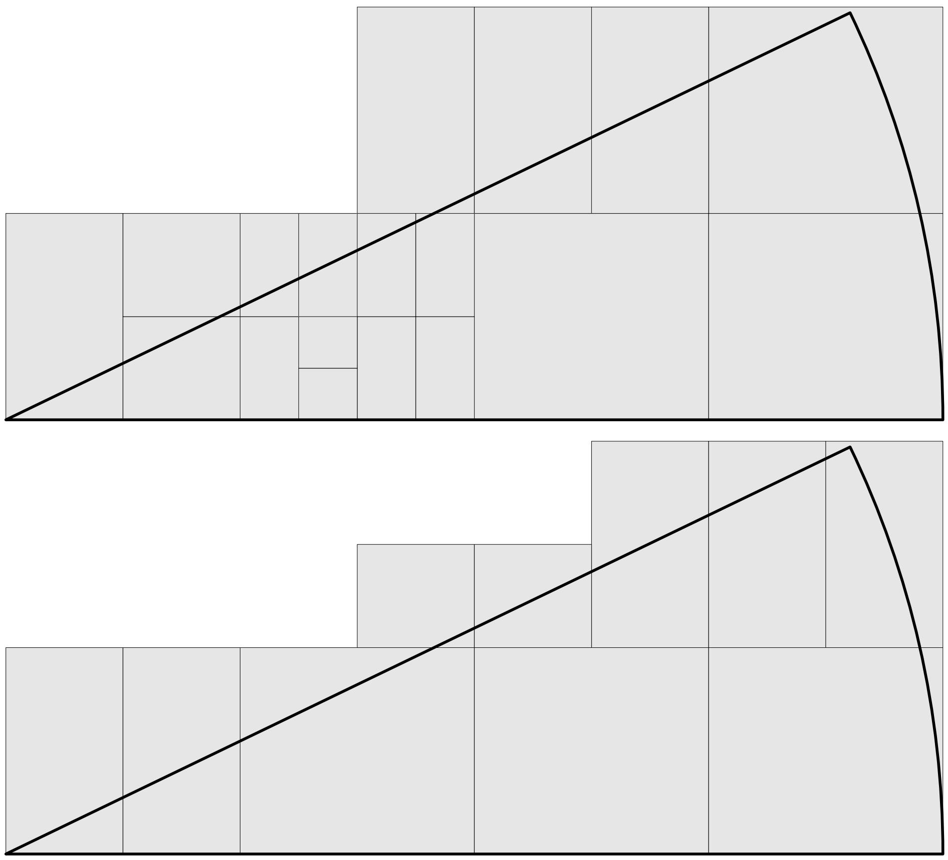

FIGURE 2. Rectangles covering S_1 (top) and S_2 (bottom) for $k = 2$.

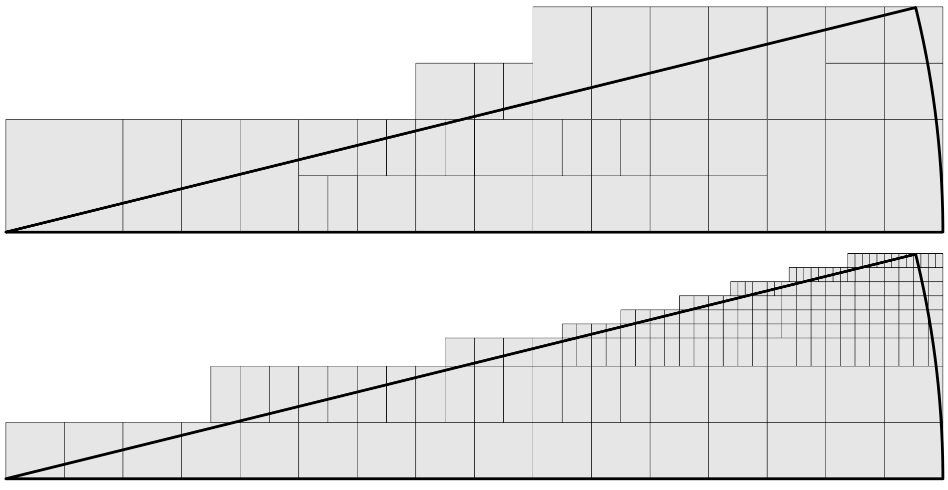

FIGURE 3. Rectangles covering S_1 (top) and S_2 (bottom) for $k = 3$.

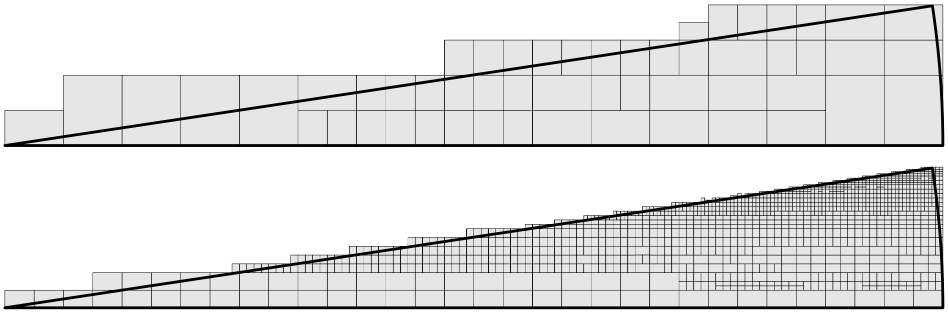

FIGURE 4. Rectangles covering S_1 (top) and S_2 (bottom) for $k = 4$.

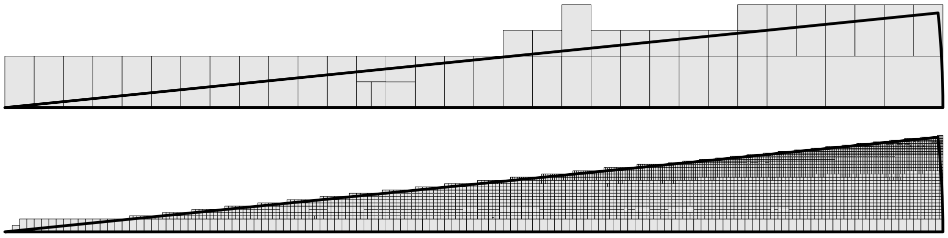

FIGURE 5. Rectangles covering S_1 (top) and S_2 (bottom) for $k = 5$.

Appendix. Details on the use of AWA

The starting box covering the sector was $[0, 1] \times [0, H]$, where $H = \sin(\pi/N)$. The value of H for $k \leq 5$ is given in Table 2 (rounded up to two significant figures).

TABLE 2. Height H of the initial rectangle covering the sectors S_i.

k	2	3	4	5
H	0.44	0.24	0.15	0.11

A typical AWA input file is shown in Figure 6. Note that we wrote equation (4) as an equation in $\mathbf{R}^4$, because AWA only handles real equations. (We used Maple to generate optimized expressions for this real equation.)

In this example, we have $k = 2$ and we integrate the box

$$B_0 = [0.5, 0.75] \times [0, 0.22] \times 0 \times 0 \subseteq S_1$$

from $t = 0$ to $t = 2$, using a Taylor series of order 20 and error tolerances equal to 10^{-16}. This box is the one on the bottom right corner in Figure 2 (top).

The output of AWA for this example included the lines below (slightly edited to fit):

```
t =  2.166768908500671E-001
 4.47775803372305E-001  7.20940501308022E-001  2.74E-001  6.2E-001
-3.88103283336499E-002  2.12697868926960E-001  2.52E-001   ---
 2.03198636666919E-001  2.14228902644709E-001  1.11E-002  5.5E-002
-7.68778525090991E-003  2.68269667132451E-003  1.04E-002   ---
```

```
      X1=U1 X2=U2 Y1=U3 Y2=U4
      t1 = sqr(Y1)
      t2 = sqr(Y2)
      t3 = sqr(X1)
      t5 = sqr(X2)
      t4 = X1*X2
      F1 = t1-t2-t3*X1+3*X1*t5
      F2 = 2*Y1*Y2-3*t3*X2+t5*X2
      F3 = 1-t3*Y1+2*t4*Y2+t5*Y1
      F4 = -t3*Y2-2*t4*Y1+t5*Y2
;;
1 0 2 0 20
4 0
[ 0.5 , 0.75 ]
[ 0 , 0.22 ]
0
0
n
1e-16 1e-16
n
n
n
```

FIGURE 6. A typical AWA input file.

which mean that AWA has proved that at $t = 0.2166768908500671$ all orbits starting at B_0 are inside the box

$$\begin{aligned}B_1 = [&\ \ 0.447775803372305, 0.720940501308022] \times\\ [&-0.0388103283336499, 0.21269786892696] \times\\ [&\ \ 0.203198636666919, 0.214228902644709] \times\\ [&-0.00768778525090991, 0.00268269667132451].\end{aligned}$$

The greatest distance of a vertex of B_1 to $(1, 1)$ is $0.9925439342611965 < 1.02$, and so all points of B_1 are inside the sphere of radius 1.02 centered at $(1, 1)$ (because the sphere is convex and B_1 is the convex hull of its vertices).

Note that we did not have to wait until $t = 2$ was reached, because we monitored the distance of B_1 to the transversal sphere and stopped AWA as soon as the initial box B_0 had been taken into the sphere. In fact, for this example, AWA is not able to reach $t = 2$. Nevertheless, AWA provided enough information for accepting B_0.

The complete set of scripts and output files is available from the authors.