1 Galton-Watson Trees

Exercise 1.1 (Catalan number). *1. Show that there exists a bijection between the set $\mathcal{B}_n$ of rooted, oriented binary trees with $2n$ edges and the set $\mathbf{A}_n$ of rooted, oriented (general) trees with n edges.*

2. The generating function of $\mathcal{B}_n$ is by definition

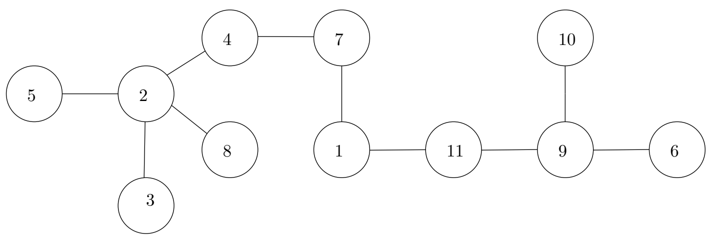
$$B(z) = \sum_{n \geqslant 0} z^{2n} \# \mathcal{B}_n.$$

Show that B satisfy $B(z) = 1 + z^2 B(z)^2$, or equivalently $zB(z) = z(1 + (zB(z))^2)$.

3. Apply Lagrange inversion formula (or solve the equation) to find

$$\# \mathcal{B}_n = \frac{1}{n+1} \binom{2n}{n}.$$

A *Cayley tree* is a labeled tree with n vertices without any orientation nor distinguished point. In other words it is a spanning tree on $\mathbb{K}_n$, the complete graph over n vertices. See figure below.



Exercise 1.2. *Let T be a (rooted, oriented) Galton-Watson tree with Poisson(1) offspring distribution. We denote by $|T|$ the number of edges of T and put*

$$P(z) = \sum_{n \geqslant 0} \mathbb{P}(|T| = n) z^n.$$

1. Show that $P = \mathrm{e}^{-1} \exp(zP)$.

2. Apply Lagrange Inversion Theorem to get

$$\mathbb{P}(|T| = n-1) = \frac{n^{n-1} \mathrm{e}^{-n}}{n!}, \text{ for all } n \geqslant 1.$$

3. Let T_n be a Galton-Watson tree with Poisson(1) offspring distribution conditioned to have n vertices. Then assign the labels $\{1, ..., n\}$ uniformly at random to the vertices of T_n and forget the ordering and the root of T_n. Show that the resulting unordered labeled tree is uniform over the set $\mathcal{C}_n$ of Cayley trees over $\{1, ..., n\}$. Deduce

$$\# \mathcal{C}_n = n^{n-2}.$$

Lagrange Inversion Theorem. Let $\phi(u) = \sum_{k \geqslant 0} \phi_k u^k$ be a power series of $\mathbb{C}[[u]]$ with $\phi_0 \neq 0$. Then, the equation $y = z\phi(y)$ admits a unique solution in $\mathbb{C}[[z]]$ whose coefficients are given by

$$y(z) = \sum_{n=1}^{\infty} y_n z^n, \qquad \text{with } y_n = \frac{1}{n} [u^{n-1}] \phi(u)^n,$$

where $[u^{n-1}]\phi(u)^n$ stands for the coefficient in front of u^{n-1} in $\phi(u)^n$.

Exercise 1.3 (Proof of Lagrange Inversion Theorem). *1. Show that the coefficients of y as a series of z are determined by the coefficients of ϕ and the equation $y = z\phi(y)$ in $\mathbb{C}[[z]]$.*

2. Assume that ϕ is a polynomial function.

(a) Show that y is analytic around 0.

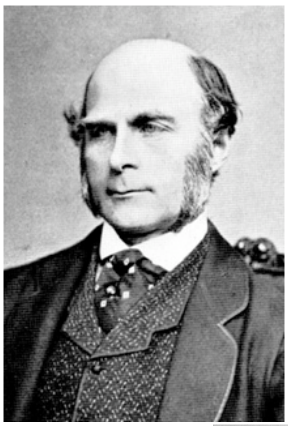
(b) Using Cauchy formula, show that

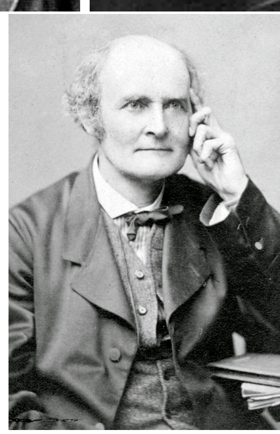
$$n[z^n]y = ny_n = \frac{1}{2i\pi}\int_{\mathcal{C}} \frac{y'(z)dz}{z^n}, \qquad \text{where } \mathcal{C} \text{ is a small anticlockwise contour around } 0.$$

(c) With a change of variable, prove the Lagrange Inversion Theorem in this case.

3. Deduce the general case.

Exercise 1.4. ** Who are these charming gentlemen ?*




Note that the fourth one already had a cell-phone.

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2 Metric Geometry

Exercise 2.1. *Give an example of two (non-compact !) metric spaces X and Y such that $d_{GH}(X,Y)=0$ but $X \neq Y$.*

Exercise 2.2 (Topology)**.** *Suppose that $(X_n)_{n\geqslant 0}$ is a sequence of compact metric spaces such that X_n is homeomorphic to X_0 for every $n \geqslant 0$. Show that $X_n \to X$ in the sense of d_{GH} does not imply X homeomorphic to X_0.*

Exercise 2.3 (Gromov's Compactness Theorem)**.** *The goal of this exercise is to characterize pre-compactness for Gromov-Hausdorff distance. A collection $\mathfrak{X}$ of compact metric spaces is* totally bounded *if*

(i) There exists $C>0$ such that for every $X \in \mathfrak{X}$, the diameter of X is bounded above by C,

(ii) For every $\varepsilon > 0$, there exists $N(\varepsilon) \in \mathbb{Z}_+$ such that every $X \in \mathfrak{X}$ admits an ε-net containing no more than $N(\varepsilon)$ points.

1. *Show that every pre-compact collection is totally bounded.*

Now, let $(X_n)_{n\geqslant 1}$ be a totally bounded sequence of compact metric spaces and denote $N(\varepsilon)$ the minimal number of balls of radius ε needed to cover any space X_i. For every n, denote the distance in the space X_n by d_n and let $(x^{(n)}_{k,j})_{k,j\geqslant 1}$ be a sequence of points of X_n such that for every $k \geqslant 1$ the points

$$x^{(n)}_{k,1}, \ldots, x^{(n)}_{k,N(1/k)}, \text{ form a } 1/k\text{-net in } X_n.$$

2. *Show that we can extract a subsequence $(n_i)_{i\geqslant 1}$ such that for every $(k,j),(k',j') \in \mathbb{Z}_+^2$*

$$\lim_{n\to\infty} d_n\big(x^{(n)}_{k,j}, x^{(n)}_{k',j'}\big) \text{ exists.}$$

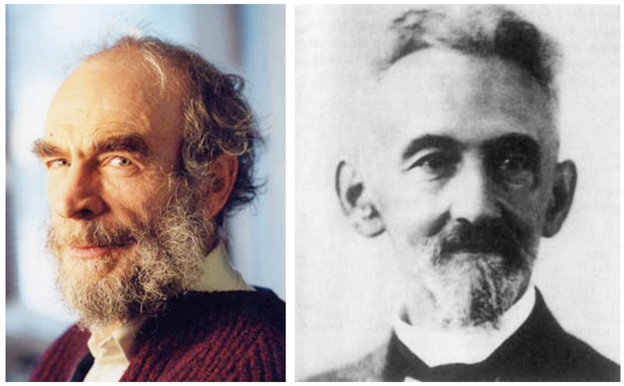
To simplify notation, we suppose that there is no need to take a subsequence. Consider the abstract space $X = \{(k,j)\}_{k,j\geqslant 1}$ endowed with $\Delta((k,j),(k',j')) = \lim d_n\big(x^{(n)}_{k,j}, x^{(n)}_{k',j'}\big)$.

3. *Show that Δ is a pseudo-metric on X and that the completion $\overline{X}$ of X with respect to Δ is compact.*

4. *Prove that $X_n \to \overline{X}$ in the Gromov-Hausdorff sense.*

Exercise 2.4. ** Prove that a sequence of length spaces homeomorphic to the two-dimensional sphere $\mathbb{S}_2$ cannot converge to the standard two-dimensional closed ball $\mathbb{B}_2$.*

Exercise 2.5. *Who are these charming gentlemen ?*



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3 Stick-Breaking Construction of the CRT

Exercise 3.1 (Aldous-Broder Algorithm and stick breaking construction of the CRT.)**.** *Given a finite connected graph $G = (V, E)$, how can we sample a uniform spanning tree of G ? In general it is hard to list all the spanning trees of a given graph. However, there exist stochastic algorithms that sample from this set without knowing it. The algorithm we describe here is due to Aldous & Broder.*
Recipe: Let $G = (V, E)$ be a connected graph and $v_0 \in V$ a vertex. We perform a simple random walk on G starting from v_0. For any $v \in G$, we denote by T_v the hitting time of v by the walk and by E_{T_v} the edge traversed by the walk just before it hits v.

1. *Show that the set of edges $\left\{E_{T_v}, v \in V \backslash \{v_0\}\right\}$, defines a spanning tree of G.*

Theorem 3.1. *This (unrooted) spanning tree is uniform over the set of all spanning trees of G.*

Now, we will focus on the case of $G = \mathbb{K}_n$ the complete graph over n vertices [1]*. Denote by $(X_k)_{k \geqslant 0}$ the simple random walk on $\mathbb{K}_n$ (the dependence in n is implicit). That is $(X_k)_{k \geqslant 0}$ is a sequence of independent variables uniformly distributed over the vertices of $\mathbb{K}_n$. We introduce the first time the walk hits its past and the corresponding vertex*

$$T_1^{(n)} = \inf\left\{k \geqslant 1 : X_k \in \{X_0, ..., X_{k-1}\}\right\} \text{ and } P_1^{(n)} = X_{T_1^{(n)}}.$$

We define $T_2^{(n)} = \inf\{k > T_1^{(n)} : X_k \in \{X_0, ..., X_{k-1}\}\}$, $P_2^{(n)} = X_{T_2^{(n)}}$ ... by induction.
We also recall "the birthday paradox": Assume that a year has n days and that people are born equally likely each day of the year. Then among $\sqrt{n}$ people chosen at random two of them are born the same day with a big probability.

2. *What is the rough order of $T_1^{(n)}$ as $n \to \infty$ (don't look below !).*
3. *Show that $n^{-1/2} T_1^{(n)}$ converges in distribution as $n \to \infty$ and identify the limit law.*
4. *More generally, show that for any $k \in \mathbb{Z}_+$,*
$$n^{-1}\left(\frac{(T_1^{(n)})^2}{2}, \frac{(T_2^{(n)})^2}{2}, ..., \frac{(T_k^{(n)})^2}{2}\right),$$
converges in distribution towards the first k points of a standard Poisson point process on $\mathbb{R}_+$ with intensity 1.
5. *Show that for any $k \in \mathbb{Z}_+$, conditionally on $(T_1^{(n)}, ..., T_k^{(n)}, X_0, X_1, ..., X_{T_k - 1})$ the point $P_k^{(n)}$ is uniformly distributed over $\{X_0, ..., X_{T_k^{(n)}-1}\}$.*
6. *Describe the continuous limit of the construction of the spanning tree over $\mathbb{K}_n$.*

Exercise 3.2. ** Prove Theorem 3.1. in the case of the complete graph.*

[1] We have seen (tutorial 1) that the number of spanning trees of $\mathbb{K}_n$ is n^{n-2}

Exercise 3.3. *Who are these charming gentlemen ?*

References

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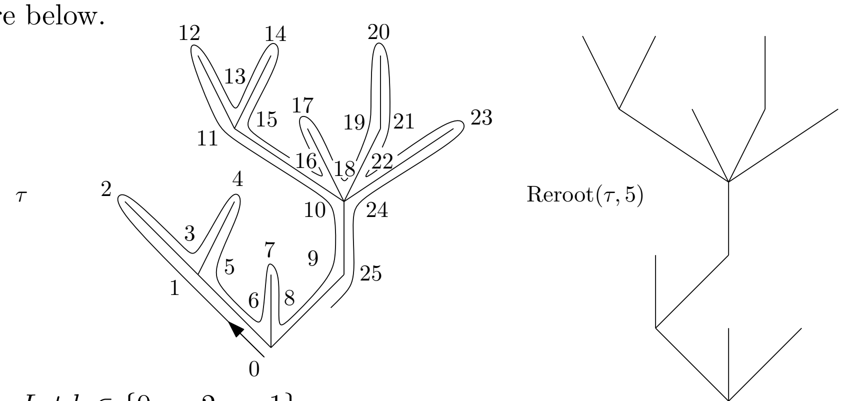
4 Properties of the CRT

Let $\mathbf{A}_n$ be the set of all rooted oriented trees with n edges. If $\tau \in \mathbf{A}_n$ and $k \in \{0, ..., 2n-1\}$ we define $\mathrm{Reroot}(\tau, k)$ the tree τ re-rooted at the k-th corner in the counter clockwise contour of τ. See figure below.



Exercise 4.1. *Let $k \in \{0, ..., 2n-1\}$.*

1. *Show that* $\mathrm{Reroot}(\cdot, k) : \mathbf{A}_n \to \mathbf{A}_n$ *is a bijection. Deduce that if τ_n is uniform over $\mathbf{A}_n$ then* $\mathrm{Reroot}(\tau_n, k)$ *is also uniform over $\mathbf{A}_n$.*

Let $\tau \in \mathbf{A}_n$ and denote its contour function by $(C_n(t), t \in [0, 2n])$. We also write for $x \in [0,1]$, $\mathbf{C}_n(x) = C_n(2nx)$. For every $r \geqslant$ let $\overline{r} = r - \lfloor r \rfloor$ be the fractional part of r. If $g : [0,1] \longrightarrow [0, +\infty[$ such that $g(0) = g(1) = 0$ and $s, t \in [0,1]$ we recall the notation

$$m_g(s,t) = \inf \big\{ g(u), u \in [s \wedge t, s \vee t] \big\}.$$

2. *Show that the contour function $(\mathfrak{C}_n(x), x \in [0, 2n])$ of the rooted oriented tree* $\mathrm{Reroot}(\tau, k)$ *is given for every $t \in [0,1]$ by*

$$\mathfrak{C}_n(t) = \mathbf{C}_n\left(\frac{k}{2n}\right) + \mathbf{C}_n\left(\overline{\frac{t+k}{2n}}\right) - 2m_{\mathbf{C}_n}\left(\frac{k}{2n}, \overline{\frac{k+t}{2n}}\right).$$

3. *Deduce from the previous considerations that if $(\mathbf{e}(t))_{t \in [0,1]}$ is a normalized Brownian excursion and $x \in [0,1]$ then the process $(\mathfrak{e}(t))_{t \in [0,1]}$ defined by*

$$\mathfrak{e}(t) = \mathbf{e}(x) + \mathbf{e}(\overline{x+t}) - 2m_{\mathbf{e}}(x, \overline{x+t}),$$

has the same distribution as $(\mathbf{e}(t))_{t \in [0,1]}$. What does it imply for the Brownian Continuum Random Tree ?

Let $\mathcal{T}$ be a real tree. For $x \in \mathcal{T}$, the *multiplicity* of x is the number of connected components of $\mathcal{T} \backslash \{x\}$. A *leaf* is a point of multiplicity one. In the following $(\mathbf{e}(t))_{t \in [0,1]}$ denotes a normalized Brownian excursion and $\mathcal{T}_{\mathbf{e}}$ its associated $\mathbb{R}$-tree, in particular we denote $p_{\mathbf{e}} : [0,1] \longrightarrow \mathcal{T}_{\mathbf{e}}$ the canonical projection.

Exercise 4.2.

1. *Show that almost surely $p_{\mathbf{e}}(0)$ is a leaf of $\mathcal{T}_{\mathbf{e}}$. Deduce that for every $x \in [0,1]$, $p_{\mathbf{e}}(x)$ is almost surely a leaf of $\mathcal{T}_{\mathbf{e}}$.*

2. *Show that a.s. the local minima of a Brownian motion are pairwise distinct.*

3. *Deduce that almost surely $\mathcal{T}_{\mathbf{e}}$ has only countable points of multiplicity 3 but no point of multiplicity strictly larger than 3.*

Exercise 4.3. *Who are these charming gentlemen ?*

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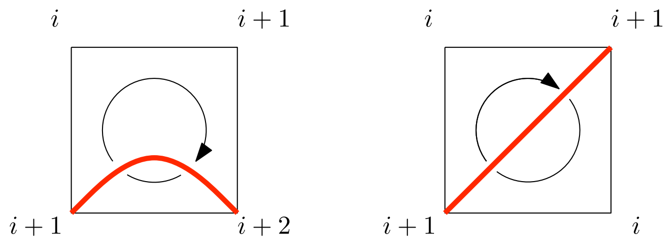
5 Schaeffer's Bijection

Exercise 5.1. *The goal of this exercise is to introduce the reverse construction of the Schaeffer bijection. In the following Q is a finite quadrangulation with n faces with a distinguished oriented edge $\overrightarrow{e}$ with endpoints e_- and e_+. We denote the graph distance in Q by $\mathrm{d_{gr}}$.*

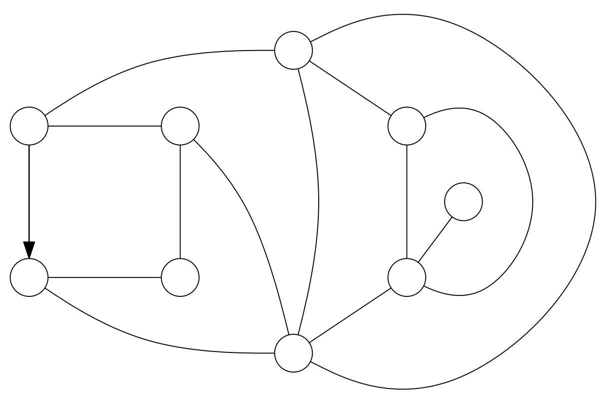
1. *Show that Q is bipartite: if u and v are two neighboring vertices then*

$$\left| \mathrm{d_{gr}}(e_-, u) - \mathrm{d_{gr}}(e_-, v) \right| = 1.$$

Hence the faces of the quadrangulation Q can be decomposed into two subsets: The faces f such that the distance to e_- of the vertices of f are $(i, i+1, i+2, i+1)$ or $(i, i+1, i, i+1)$. We add a "red" edge in each face following the rule given by the figure below.



2. *Apply this construction to this quadrangulation.*[1]



We denote $\mathfrak{T}$ the graph formed by the red edges and the vertices of Q they span. We aim at proving that $\mathfrak{T}$ is a tree that spans $V \backslash \{e_-\}$.

3. *Show that e_- is not a vertex of $\mathfrak{T}$.*

4. *Using Euler's formula, show that it suffices to prove that $\mathfrak{T}$ has no cycle.*

5. *Suppose that we can form a simple cycle $\mathcal{C}$ with the red edges. Pick a vertex u of this cycle whose distance to e_- is minimal among the vertices of the cycle. Show that we can find two vertices v and v' which are separated by $\mathcal{C}$ and such that*

$$\mathrm{d_{gr}}(e_-, v) = \mathrm{d_{gr}}(e_-, v') = \mathrm{d_{gr}}(e_-, u) - 1.$$

6. *Find a contradiction.*

[1] For the connoisseurs this is Le Gall's map.

Exercise 5.2 (Re-rooting). *Let $\mathcal{Q}_n^*$ be the set of all rooted quadrangulations with n faces and an extra oriented edge $\overrightarrow{e}$. If $(q, \overrightarrow{e}) \in \mathcal{Q}_n^*$ we put $\mathfrak{R}(q, \overrightarrow{e})$ for the map q but re-rooted at $\overrightarrow{e}$.*

1. *Show that the image of the uniform distribution on $\mathcal{Q}_n^*$ by $\mathfrak{R}$ is the uniform distribution over the set of rooted quadrangulations with n faces.*

Let Q_n be a uniform rooted quadrangulation with n faces. Conditionally on Q_n, let $(X_n)_{n \geqslant 0}$ be a simple random walk on Q_n starting from the end point of the root-edge of Q_n and denote $(\overrightarrow{E}_1, \overrightarrow{E}_2, ...)$ the oriented edges traversed by the walk.

2. *Prove that for every $k \geqslant 1$, $\mathfrak{R}(Q_n, \overrightarrow{E}_k) = Q_n$, in distribution.*

Exercise 5.3. *Who are these charming gentlemen ?*

References

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