LARGE RANDOM PLANAR MAPS AND THEIR SCALING LIMITS

Lectures by Jean-François Le Gall and Grégory Miermont (Univ. Paris-Sud)

Description of the course.

There has been much recent interest in understanding the properties of large random planar maps. Recall that planar maps are graphs embedded in the two-dimensional sphere, considered up to continuous deformation, and that the faces of the map are the regions delimited by the edges. Planar maps have been studied extensively in combinatorics, and they have also significant geometrical applications. Random planar maps have been used in theoretical physics, where they serve as models of random geometry in the framework of two-dimensional quantum gravity.

The main goal of these lectures is to discuss scaling limits of large random planar maps viewed as random metric spaces. More precisely, we consider a random planar map $M(n)$ which is uniformly distributed over the set of all planar maps with n faces in a certain class (for instance the class of triangulations, or the class of quadrangulations). The vertex set of $M(n)$ is then equipped with the graph distance rescaled by the factor $n^{-1/4}$ (in such a way that the diameter of the vertex set remains stochastically bounded when n is large). The main goal is to prove that the resulting random metric spaces converge in distribution as n tends to infinity, in the sense of the Gromov-Hausdorff distance between compact metric spaces. This problem was stated by Oded Schramm in his 2006 ICM paper, in the special case of triangulations. Recent results show that, in the case of bipartite planar maps, a limit exists at least along suitable subsequences, and this limit can be written as a quotient space of Aldous' Continuum Random Tree (the CRT) for an equivalence relation which has a simple definition in terms of Brownian labels assigned to the vertices of the CRT. This limiting random metric space, which is called the Brownian map, can be viewed as a "Brownian surface" in the same sense as Brownian motion is the limit of rescaled discrete paths. The Brownian map is almost surely homeomorphic to the two-dimensional sphere, although it has Hausdorff dimension 4. Furthermore, detailed information has been obtained about geodesics in the Brownian map

A key tool for deriving the preceding results is the existence of "nice" bijections between planar maps and various classes of labeled trees. For this reason, the lectures will start with a detailed discussion of discrete trees and their scaling limits. This will include a presentation of Aldous' CRT and of the Brownian snake, which provides an efficient way of looking at Brownian labels on the CRT.

Syllabus (8 lectures)

(1) Discrete plane trees, and convergence of their rescaled contour functions.
(2) The CRT as a random $\mathbf{R}$-tree. Gromov-Hausdorff convergence of discrete trees towards the CRT.
(3) Labelled trees and convergence to the Brownian snake.
(4) Planar maps and the Schaeffer correspondence between quadrangulations and labeled trees.
(5) Tightness of the rescaled planar maps and description of sequential limits.
(6) Gromov-Hausdorff convergence of rescaled maps towards the Brownian map.

(7) The homeomorphism theorem.
(8) Bijections with several marked points and the uniqueness of geodesics.

References.

J. Bouttier, P. Di Francesco, E. Guitter. Planar maps as labeled mobiles. *Electronic J. Combinatorics*, 11 #R69 (2004)

J. Bouttier, E. Guitter. The three-point function of planar quadrangulations. *J. Stat. Mech.* P07020 (2008)

D. Burago, Y. Burago, S. Ivanov. *A Course in Metric Geometry.* Graduate Studies in Mathematics, vol. 33. Amer. Math. Soc., Boston, 2001. (*for basic facts about the Gromov-Hausdorff convergence*)

P. Chassaing, G. Schaeffer. Random planar lattices and integrated superBrownian excursion. *Probab. Th. Rel. Fields*, 128, 161-212 (2004)

B. Duplantier, S. Sheffield. Liouville quantum gravity and KPZ. To appear.

J.F. Le Gall. Random trees and applications. *Probability Surveys* 2, 245-311 (2005) (*for basic facts about discrete random trees, the CRT and the Brownian snake*)

J.F. Le Gall. The topological structure of scaling limits of large planar maps. *Invent. Math.* 169, 621-670 (2007).

J.F. Le Gall, F. Paulin. Scaling limits of bipartite planar maps are homeomorphic to the 2-sphere. *Geometr. Funct. Analysis* 18, 893-918 (2008)

J.F. Le Gall. Geodesics in large random planar maps and in the Brownian map. *Acta Math.*, to appear.

J.F. Marckert, G. Miermont. Invariance principles for random bipartite planar maps. *Ann. Probab.* 35, 1642-1705 (2007)

J.F. Marckert, A. Mokkadem. Limit of normalized quadrangulations. The Brownian map. *Ann. Probab.*, 34 (2006), 2144-2202.

G. Miermont. On the sphericity of scaling limits of random planar quadrangulations. Electron. Comm. Probab. 13, 248-257 (2008)

G. Miermont. Tessellations of random maps of arbitrary genus. *Ann. Sci. École Norm. Sup.* 42, 725-781 (2009)