Tutorial 1: Existence of free energy
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In this tutorial, we study two methods that can be used to prove the existence of the quenched free energy associated with the random pinning model, that will be discribed in Lecture 4.

1 Random pinning of a polymer at an interface

Configurations of the polymer. Let $n \in \mathbb{N}$ and consider a polymer made of n monomers. The allowed configurations of this polymer are modeled by the n-steps trajectories of a 1-dimensional random walk $S = (S_i)_{i \geq 0}$. We focuss on the case where $S_0 = 0$ and $(S_i - S_{i-1})_{i \geq 1}$ is an i.i.d. sequence of random variables satisfying

$$P(S_1 = 1) = P(S_1 = -1) = P(S_1 = 0) = 1/3.$$

We denote by $\mathcal{W}_n$ the set of all n-steps trajectories of S.
Disorder: randomness of the pinning intensity. Let $\omega = (\omega_i)_{i \geq 1}$ be an i.i.d. sequence of bounded random variables such that for $l \in \mathbb{N}$ the interaction intensity between the l^{th} monomer and the interface takes the value ω_l. Note that ω and S are independent, and write $\mathbb{P}$ for the law of ω. Set $R > 0$ such that ω $\mathbb{P}$-a.s. $|\omega_1| \leq R$.

Interaction polymer-interface. The flat interface that interacts with the polymer is located at height 0, so that the polymer hits this interface every time S comes back to 0. Thus, with every $S \in \mathcal{W}_n$ we associate the energy

$$H_n^{\omega,\beta}(S) = \beta \sum_{i=1}^{n} \omega_i \, 1_{\{S_i = 0\}},$$

where $\beta \in (0, \infty)$ stands for the inverse temperature.

Partition function and free energy. For fixed n, the quenched (frozen disorder) partition function and free energy are defined as

$$Z_n(\omega,\beta) = E\big(e^{H_N^{\omega,\beta}(S)}\big) \quad \text{and} \quad f_n(\omega,\beta) = \tfrac{1}{n} \log Z_n(\omega,\beta). \tag{1.1}$$

2 Convergence of the free energy

Our goal is to prove the following theorem.

Theorem 2.1 *For $\beta \in \mathbb{R}$ there exists an $f(\beta) \in (0, \beta R)$ such that*

$$\lim_{n\to\infty} \mathbb{E}[f_n(\omega,\beta)] = f(\beta) \tag{2.1}$$

and

$$\lim_{n\to\infty} f_n(\omega,\beta) = f(\beta) \quad \mathbb{P}-a.e.\ \omega \tag{2.2}$$

For technical reasons, we will first prove Theorem 2.1 with the partition function restricted to those trajectories that hit the interface at their right extremity, i.e.,

$$Z_n^0(\omega,\beta) = E\big(e^{H_n^{\omega,\beta}(S)}\, 1_{\{S_n=0\}}\big) \quad \text{and} \quad f_n^0(\omega,\beta) = \tfrac{1}{n}\log Z_n^0(\omega,\beta). \tag{2.3}$$

As stated above, we will prove Theorem 2.1 via two different methods. In Section 2.1 we will state Kingman's Subadditive Ergodic Theorem and see how it can be applied to obtain Theorem 2.1. In Section 2.2 we will reprove Theorem 2.1 by using a concentration of measure argument. The latter method is more involved but also more flexible than the former. Finally, in Section 2.3 we will see that the restriction introduced in (2.3) has no effect on the value of the limiting free energy.

2.1 Method 1: Kingman's theorem

Theorem 2.2 *(Kingman's Subbaditive Ergodic Theorem). Let (Ω, A, μ) be a probability space, let T be an ergodic measure-preserving transformation acting on Ω, and let $(g_n)_{n\geq 1}$ be a sequence of random variables in $L_1(\mu)$ that satisfy the subadditivity relation*

$$g_{m+n} \leq g_n + g_m(T^n) \qquad n, m \geq 1. \tag{2.4}$$

Then

$$\lim_{n\to\infty} \frac{g_n}{n} = \gamma := \inf_{n\geq 1} E_\mu[\tfrac{g_n}{n}], \quad \mu - a.e.\ \omega. \tag{2.5}$$

1) Let T be the left-shift on $\mathbb{R}^{\mathbb{N}}$. Prove that for $n, m \geq 1$ and $\omega \in \mathbb{R}^{\mathbb{N}}$,

$$\log Z_{n+m}^0(\omega,\beta) \geq \log Z_n^0(\omega,\beta) + \log Z_m^0(T^n(\omega),\beta). \tag{2.6}$$

2) Apply Theorem 2.2 with $(\Omega, A, \mu) = (\mathbb{R}^{\mathbb{N}}, \mathrm{Bor}(\mathbb{R}^{\mathbb{N}}), \mathbb{P})$ and prove Theorem 2.1 with the path restriction.

2.2 Method 2: concentration of measure

This method consists of first proving (2.1), i.e., the convergence of the averaged quenched free energy, and then using a concentration of measure inequality to show that, with large probability, the quenched free energy is almost equal to its expectation and therefore satisfies the same convergence.

1) Use (2.6) and prove that $(\mathbb{E}[\log Z_n^0(\omega,\beta)])_{n\geq 1}$ is a super-additive sequence, i.e., for $n, m \geq 1$,

$$\mathbb{E}[\log Z_{n+m}^0(\omega,\beta)] \geq \mathbb{E}[\log Z_n^0(\omega,\beta)] + \mathbb{E}[\log Z_m^0(\omega,\beta)]. \tag{2.7}$$

2) Deduce that

$$\lim_{n\to\infty} \mathbb{E}[f_n^0(\omega,\beta)] = \sup_{k\geq 1} \mathbb{E}[f_k^0(\omega,\beta)] =: f^0(\beta) \in (0, \beta R). \tag{2.8}$$

At this stage, we need the following inequality.

Theorem 2.3 *There exist $C_1, C_2 > 0$ such that for $m \geq 1$, $K > 0$, $\varepsilon > 0$ and $G : \mathbb{R}^m \mapsto \mathbb{R}$ a K-Lipschitz convex function,*

$$\mathbb{P}\Big(|G(\omega_1, \dots, w_m) - \mathbb{E}(G(\omega_1, \dots, w_m))| > \varepsilon\Big) \leq C_1 e^{-\frac{C_2\,\varepsilon^2}{K}}. \tag{2.9}$$

3) Prove that the function $\omega \in \mathbb{R}^n \mapsto f_n^0(\omega, \beta) \in \mathbb{R}$ is convex and $\beta/\sqrt{n}$-Lipschitz.

(*Hint: pick $\omega, \omega' \in \mathbb{R}^n$ and compute the derivative of $t \mapsto f_n^0(\omega + t(\omega' - \omega), \beta)$.*)

4) Prove that for $\varepsilon > 0$,

$$\sum_{n \geq 1} \mathbb{P}(|f_n^0(\omega, \beta) - \mathbb{E}[f_n^0(\omega, \beta)]| > \varepsilon) < \infty. \tag{2.10}$$

5) Show that, for $\mathbb{P}$-a.e. ω, $f_n^0(\omega, \beta)$ tends to $f^0(\beta)$ as $n \to \infty$, which together with (2.8) proves Theorem 2.1 with the path restriction.

2.3 Removal of the path restriction

The proof of Theorem 2.1 will be completed once we show that restricting the partition function to $\{S_n = 0\}$ does not alter the results. To that aim, we denote by τ the first time at which the random walk S hits the interface. We use that there exists a $C_3 > 0$ such that

$$P(\tau = n) = \frac{C_3}{n^{3/2}}(1 + o(1)) \quad \text{and} \quad P(\tau > n) = \frac{2C_3}{\sqrt{n}}(1 + o(1)). \tag{2.11}$$

6) Consider the last hit of the interface and show that

$$Z_n(\omega, \beta) = \sum_{j=0}^{n} Z_j^0(\omega, \beta)\ P(\tau > n - j). \tag{2.12}$$

7) Prove Theorem 2.1 by combining (2.11), (2.12) and 5) above.