On the location of the maximum of a continuous stochastic process

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Uniqueness of the location

Let $(X(s),\ s \in K)$ be a continuous stochastic process. Define

$$M = M(X) := \max_{s \in K} X(s),$$

and

$$\arg\max(X) := \{s \in K \ : \ X(s) = M\}$$

($K = [a, b]$ or $\mathbb{R}$). In many situations we do expect that

$$\arg\max(X) \overset{a.s.}{=} \{Z\}.$$

How to prove it?

Theorem

For $\lambda \in \mathbb{R}$ let

$$m(\lambda) := \mathbb{E}(M^\lambda)\,,$$

where

$$M^\lambda = M(X^\lambda) \text{ and } X^\lambda(s) := X(s) + \lambda s\,.$$

- $m(\lambda)$ is differentiable at $\lambda = 0$ iff

$$\arg\max(X) \overset{a.s.}{=} \{Z\}\,.$$

- In this case we have that

$$\mathbb{E}Z = m'(0)\,.$$

Lemma

(This is an analytical statement, we only need to assume that X is a continuous function.) Let

$$Z_1 := \inf \arg\max(X)\,,$$

and

$$Z_2 := \sup \arg\max(X)\,.$$

Then

$$\lim_{\lambda \to 0^-} \frac{M^\lambda - M}{\lambda} = Z_1 \quad \text{and} \quad \lim_{\lambda \to 0^+} \frac{M^\lambda - M}{\lambda} = Z_2\,.$$

Proof of the Theorem

Notice that

$$\arg\max(X) \overset{a.s.}{=} \{Z\}\,,$$

iff

$$Z_1 \overset{a.s.}{=} Z_2\,.$$

Since $Z_1 \leq Z_2$, this holds iff

$$\mathbb{E}Z_1 = \mathbb{E}Z_2\,,$$

which holds iff $m(\lambda)$ is differentiable at $\lambda = 0$ (previous lemma).

Application

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- Uniqueness was proved by Corwin and Hammond (2011), and more recently by Flores, Quastel and Remenik (2012). Both proofs were strongly based on very particular aspects of the Airy process.
- The previous theorem provides us a very simple and direct proof (which holds under weaker hypothesis).

Corollary

Let $(A(s)\,, s \in \mathbb{R})$ be a continuous and stationary process, and take

$$X(s) := A(s) - s^2\,.$$

Then the location of the maximum of X is a.s. unique and

$$\mathbb{E}Z = 0\,.$$

(The Airy process is stationary and has continuous paths.)

Proof of the Corollary

Notice that

$$A(s) - s^2 + \lambda s = A(s) - (s - \frac{\lambda}{2})^2 + \frac{\lambda^2}{4}\,,$$

By taking expectations of the max and using stationarity, we have that

$$m(\lambda) = m(0) + \frac{\lambda^2}{4}\,,$$

and hence

$$m'(\lambda) = \frac{\lambda}{2}\,.$$

More Applications

- $X(s) = B(s) - s^2$, B two-sided Brownian motion on $\mathbb{R}$. Then
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- $X(s) = B(s) + f(s)$, B standard Brownian motion on $[0, t]$. Then
$$\mathbb{E} Z = m'(0) = \mathrm{Cov}(M, B(t))\,.$$