partial hyperbolicity in 3-manifolds

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setting

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- M^3 closed Riemannian 3-manifold

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- M^3 closed Riemannian 3-manifold
- $f : M \to M$ partially hyperbolic diffeomorphism

partial hyperbolicity

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$f : M^3 \to M^3$ is partially hyperbolic

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$$TM = \underset{\substack{\uparrow \\ \text{contracting}}}{E^s} \oplus \underset{\substack{\uparrow \\ \text{intermediate}}}{E^c} \oplus \underset{\substack{\uparrow \\ \text{expanding}}}{E^u}$$

partial hyperbolicity

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$f : M^3 \to M^3$ is partially hyperbolic

$$TM = \underset{\substack{\uparrow \\ \text{1-dim}}}{E^s} \oplus \underset{\substack{\uparrow \\ \text{1-dim}}}{E^c} \oplus \underset{\substack{\uparrow \\ \text{1-dim}}}{E^u}$$

example

example (conservative)

$$f : \mathbb{T}^2 \times \mathbb{T}^1 \to \mathbb{T}^2 \times \mathbb{T}^1$$

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$$f : \mathbb{T}^2 \times \mathbb{T}^1 \to \mathbb{T}^2 \times \mathbb{T}^1$$

such that

$$f = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \times id$$

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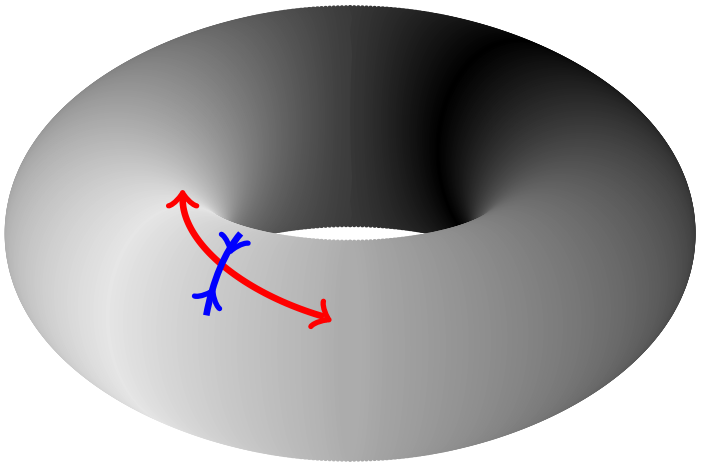

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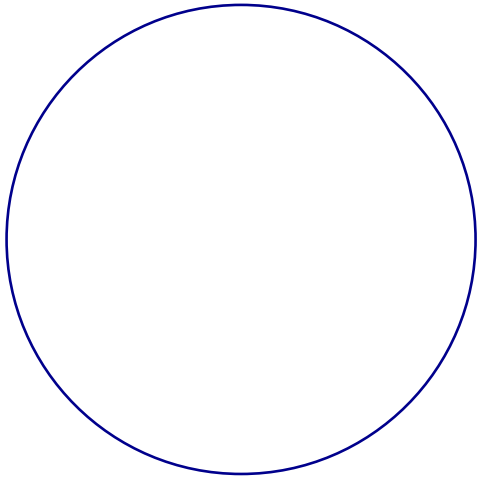

example

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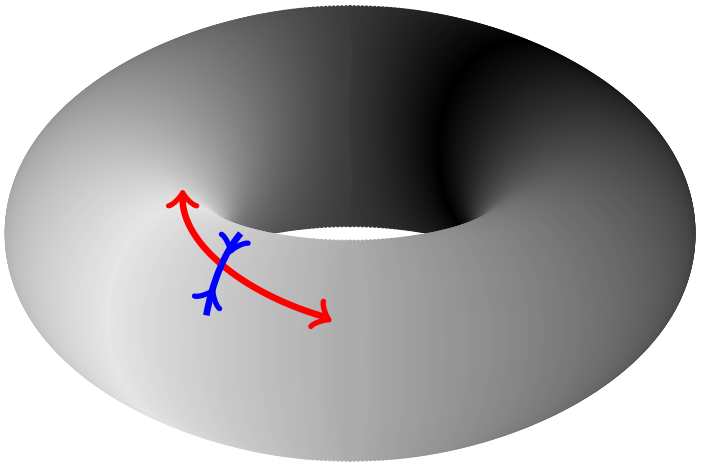
$$f : \mathbb{T}^2 \times \mathbb{T}^1 \to \mathbb{T}^2 \times \mathbb{T}^1$$

such that

$$f = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \times R_\theta$$



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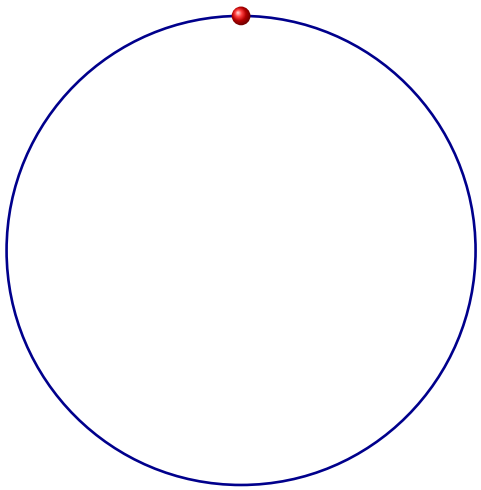


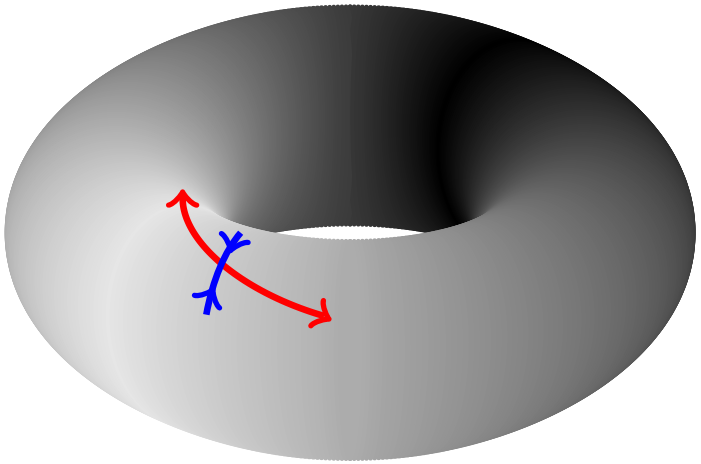
example

example (non-conservative)

$$f : \mathbb{T}^2 \times \mathbb{T}^1 \to \mathbb{T}^2 \times \mathbb{T}^1$$

such that

$$f = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \times NPSP$$



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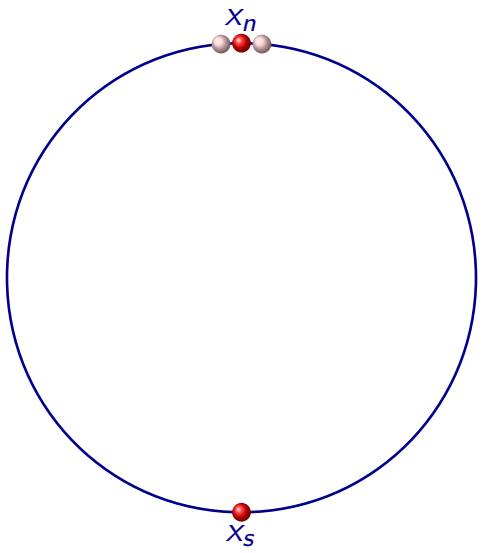


open problems

problems

- ergodicity

open problems

problems

- ergodicity
- dynamical coherence

open problems

problems

- ergodicity
- dynamical coherence
- classification

open problems

problems

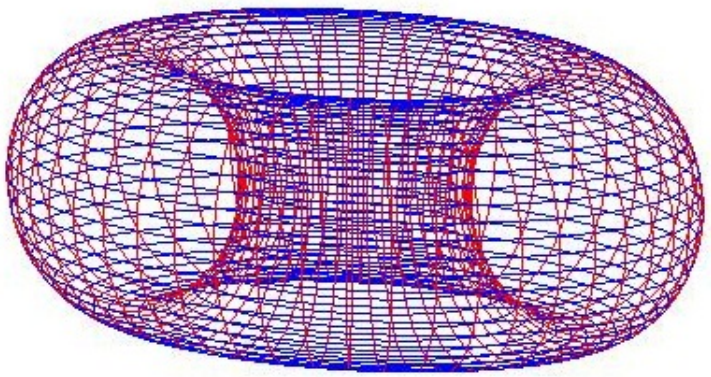
- ergodicity
- dynamical coherence
- classification

$\Big\} \rightarrow$ $\boxed{\text{Anosov torus}}$

Anosov torus

Anosov torus

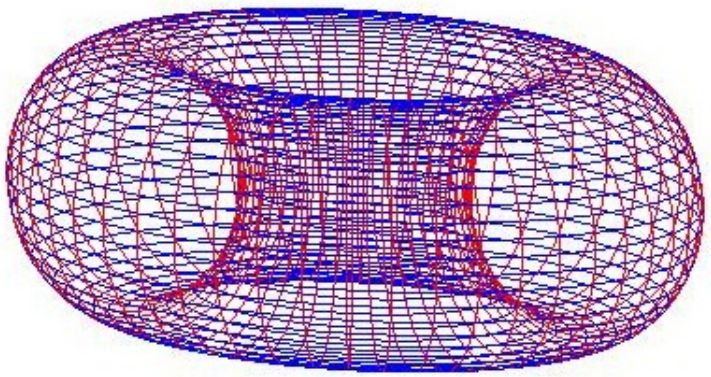
- T embedded 2-torus



Anosov torus

Anosov torus

- T embedded 2-torus
- $\exists g : M \to M$ diffeo s.t.



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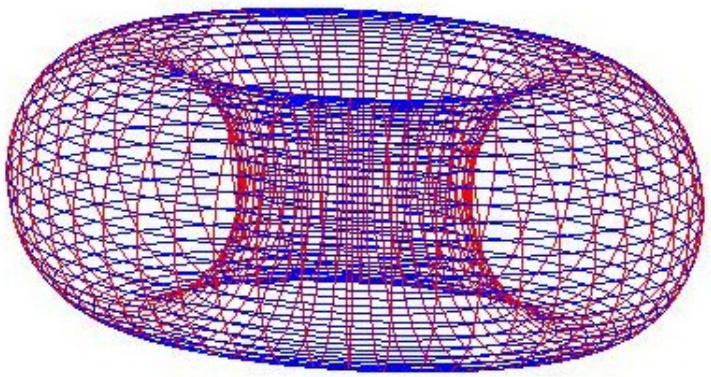
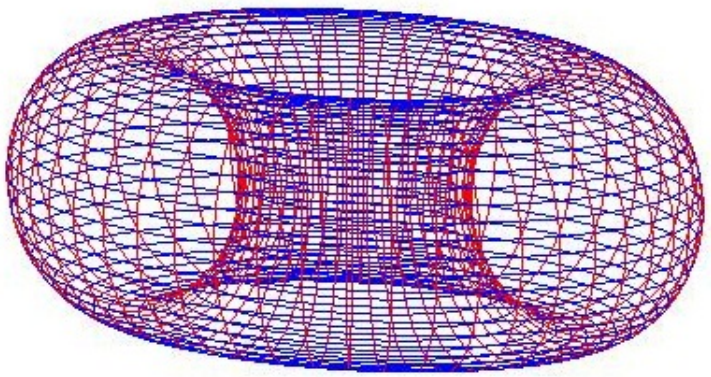
  1. $g(T) = T$



Anosov torus

Anosov torus

- T embedded 2-torus
- $\exists g : M \to M$ diffeo s.t.
  1. $g(T) = T$
  2. $g|_T$ hyperbolic



most ph are ergodic

conjecture (pugh-shub)

partially hyperbolic diffeomorphisms

$\cup$

C^1-open and C^r-dense set of ergodic diffeomorphisms

most ph are ergodic

hertz-hertz-ures08 (this setting)

partially hyperbolic diffeomorphisms

$\cup$

C^1-open and C^∞-dense set of ergodic diffeomorphisms

open problem

open problem

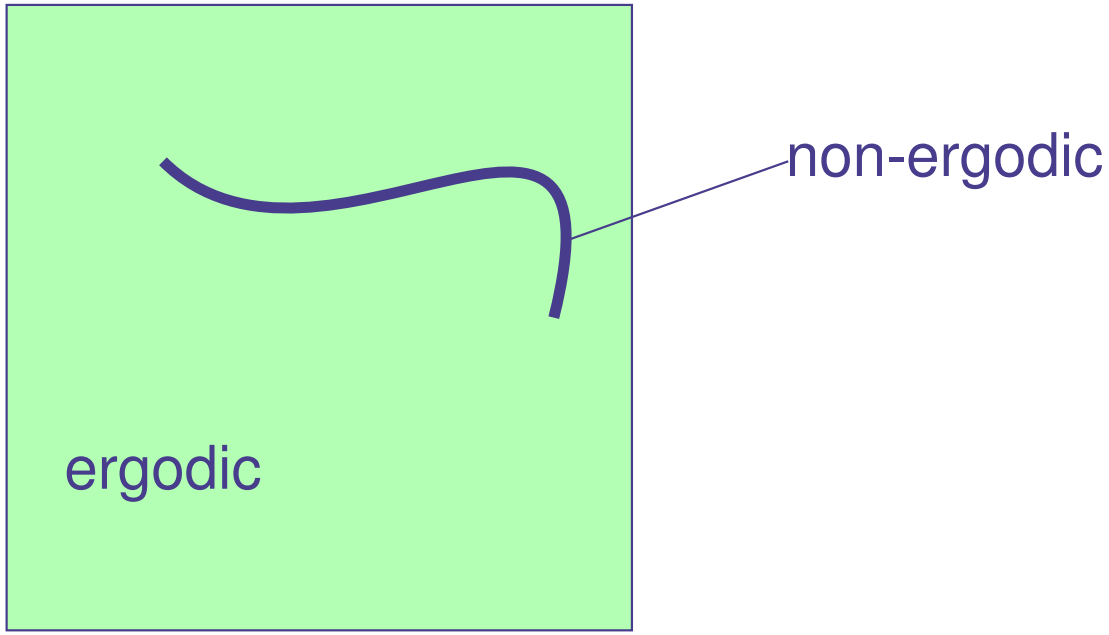
describe non-ergodic partially hyperbolic diffeomorphisms

open problem

open problem

describe non-ergodic partially hyperbolic diffeomorphisms

non-ergodicity



open problem

open problem

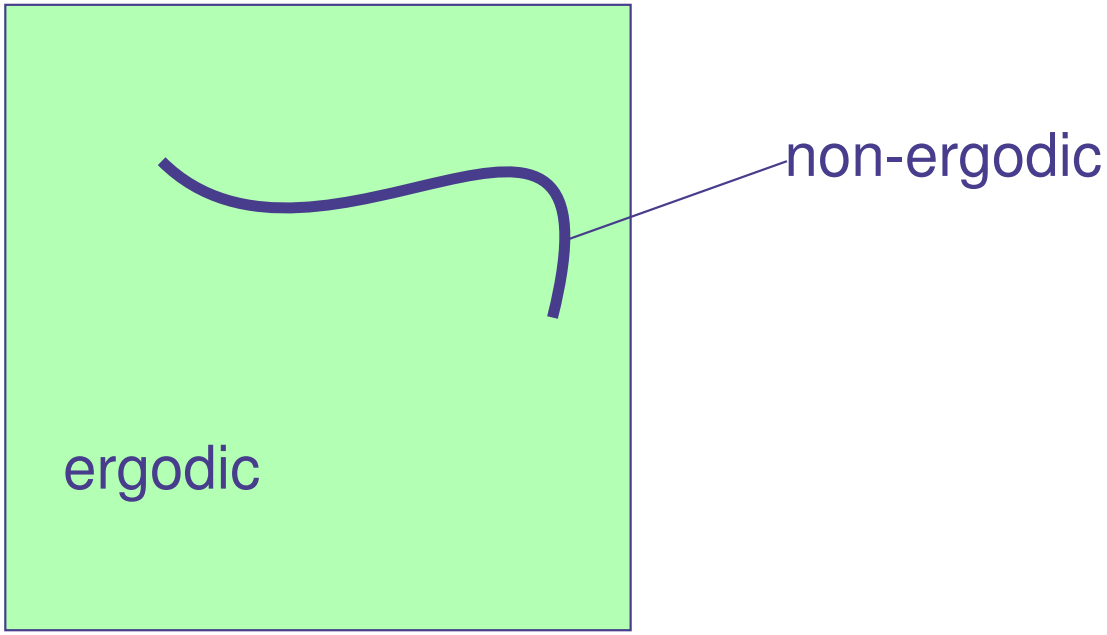
describe 3-manifolds
supporting non-ergodic partially hyperbolic diffeomorphisms

open problem

open problem

describe 3-manifolds
supporting non-ergodic partially hyperbolic diffeomorphisms

3-manifolds

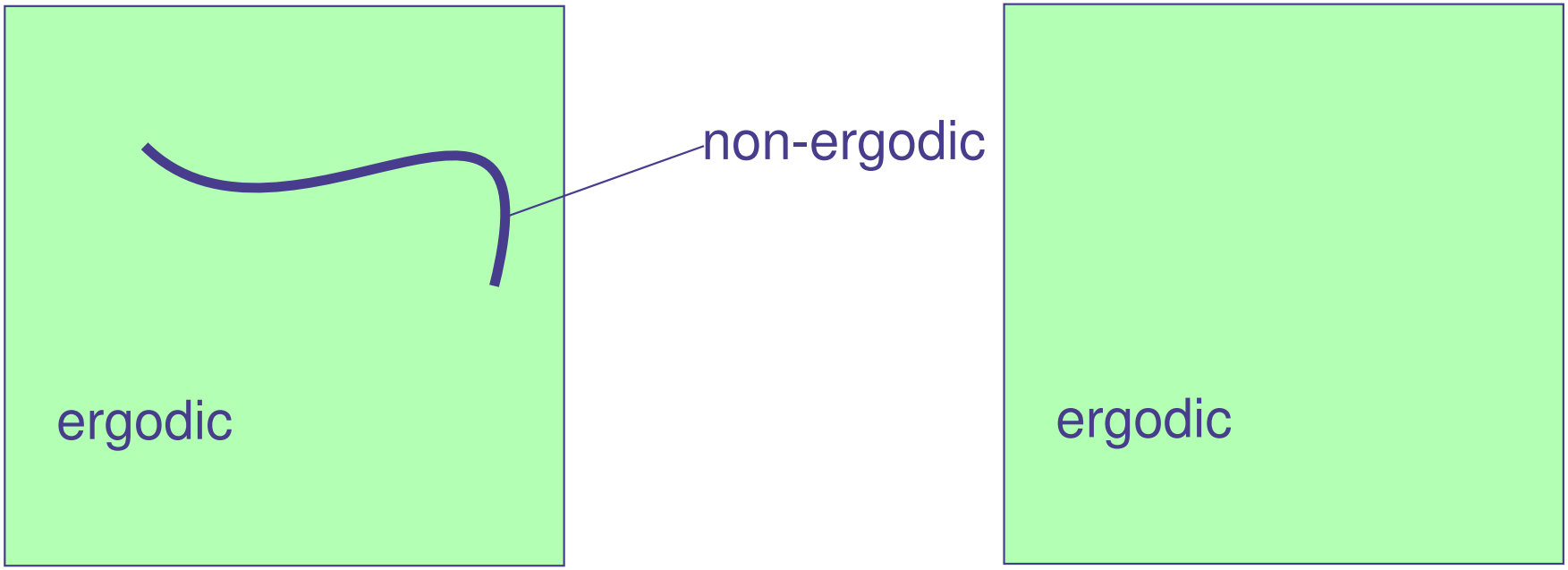


open problem

open problem

describe 3-manifolds
supporting non-ergodic partially hyperbolic diffeomorphisms

3-manifolds



conjecture

subliminal conjecture

most 3-manifolds

conjecture

subliminal conjecture

most 3-manifolds do not support
non-ergodic partially hyperbolic diffeomorphisms

evidence

hertz-hertz-ures08

N 3-nilmanifold,

evidence

hertz-hertz-ures08

N 3-nilmanifold, then either

evidence

hertz-hertz-ures08

N 3-nilmanifold, then either

- $N = \mathbb{T}^3$,

evidence

hertz-hertz-ures08

N 3-nilmanifold, then either

- $N = \mathbb{T}^3$, or
- $\{$partially hyperbolic $\} \subset \{$ ergodic $\}$

evidence

hertz-hertz-ures08

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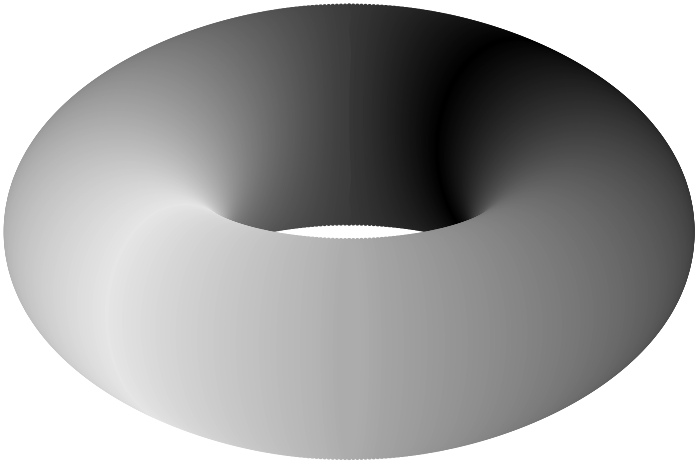
nilmanifolds



evidence

hertz-hertz-ures08

N 3-nilmanifold, then either

- $N = \mathbb{T}^3$, or
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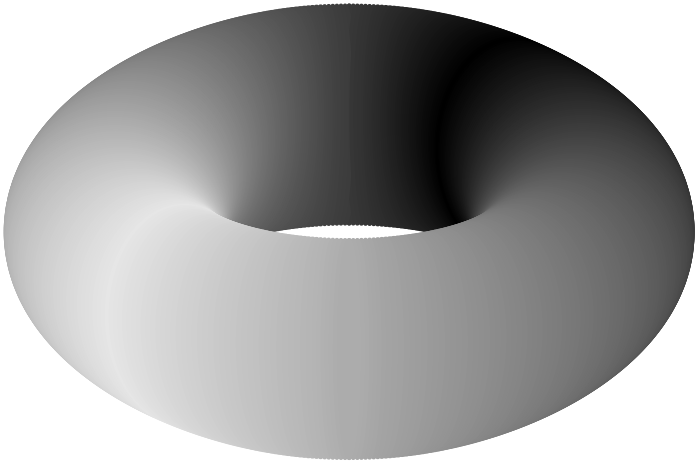
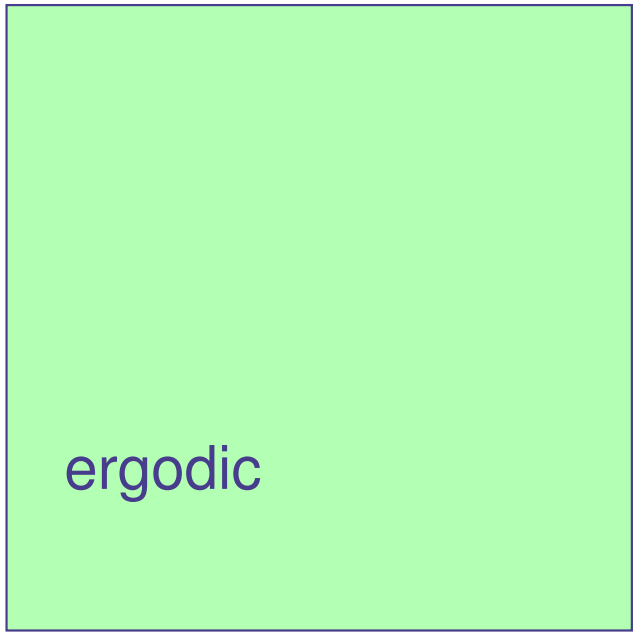
nilmanifolds





non-ergodic conjecture

non-ergodic conjecture (hertz-hertz-ures)

the only 3-manifolds supporting
non-ergodic PH diffeomorphisms are:

non-ergodic conjecture

non-ergodic conjecture (hertz-hertz-ures)

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1. the 3-torus,

non-ergodic conjecture

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1. the 3-torus,
2. the mapping torus of $-id : \mathbb{T}^2 \to \mathbb{T}^2$

non-ergodic conjecture

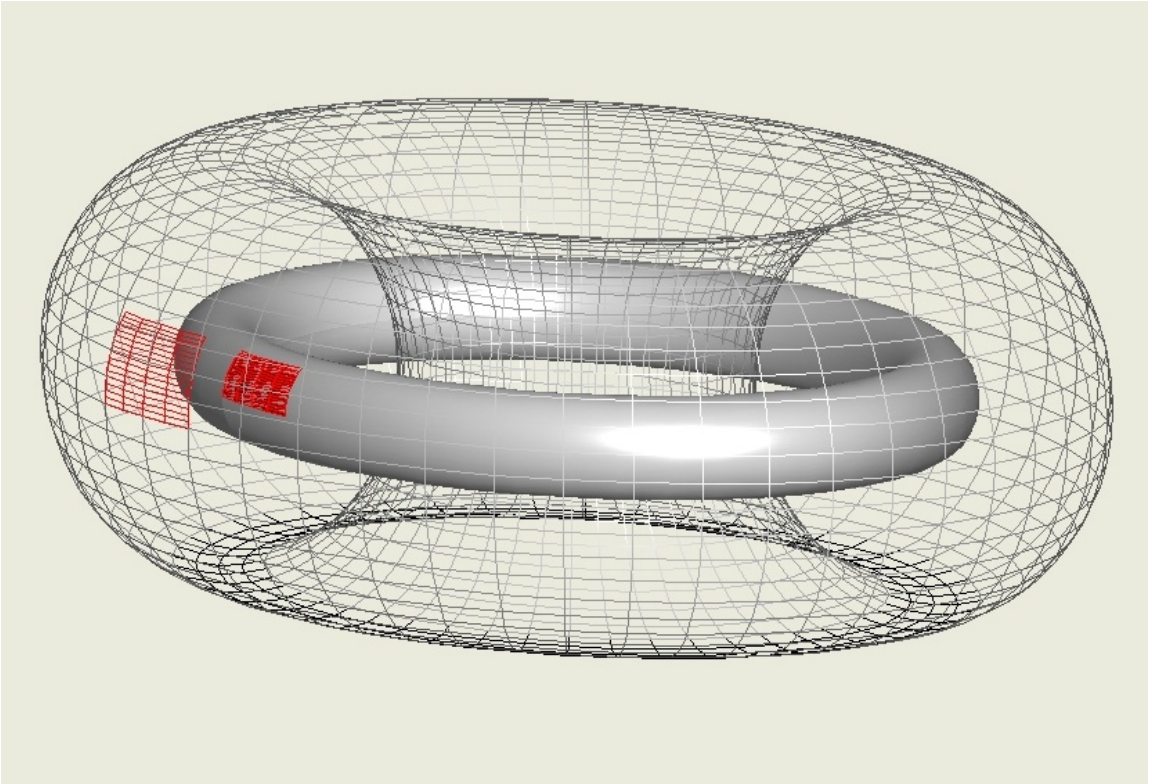
non-ergodic conjecture (hertz-hertz-ures)

the only 3-manifolds supporting
non-ergodic PH diffeomorphisms are:

1. the 3-torus,
2. the mapping torus of $-id : \mathbb{T}^2 \to \mathbb{T}^2$
3. the mapping tori of hyperbolic automorphisms of $\mathbb{T}^2$

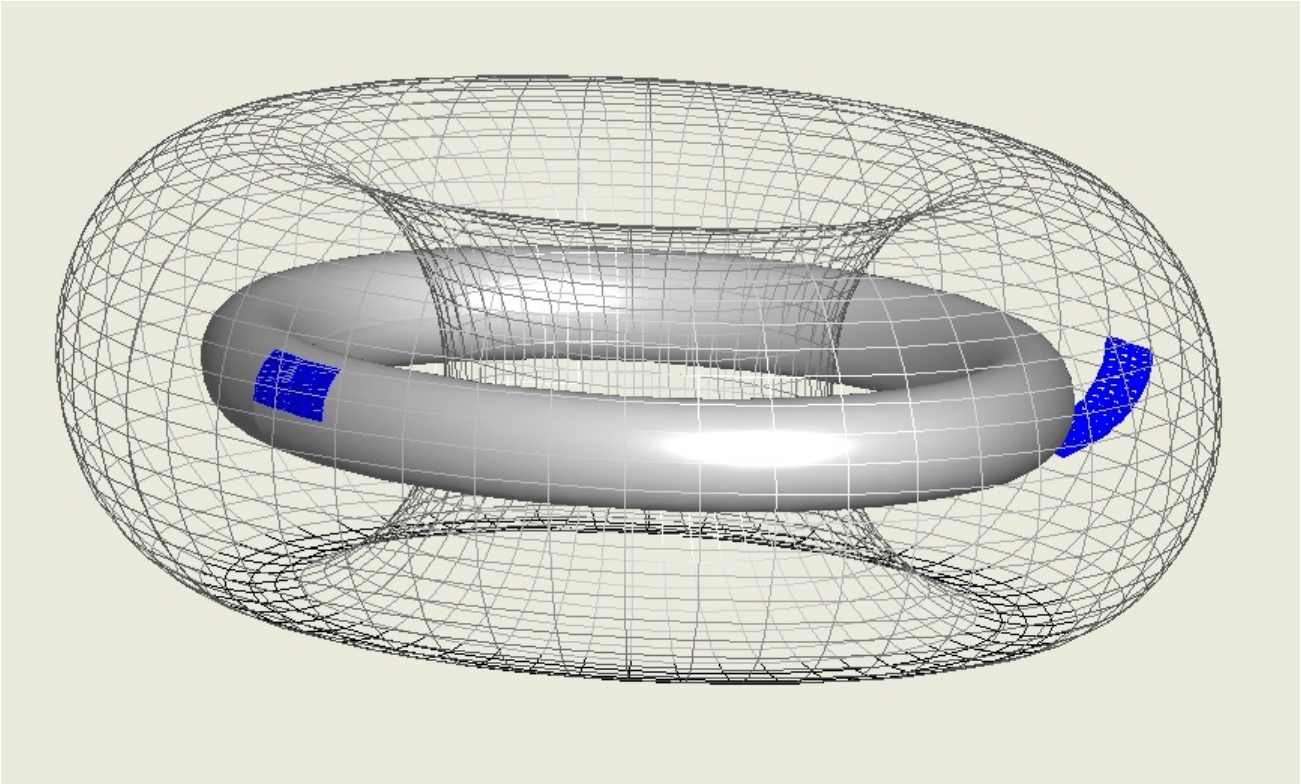
non-ergodic conjecture

1 the 3-torus



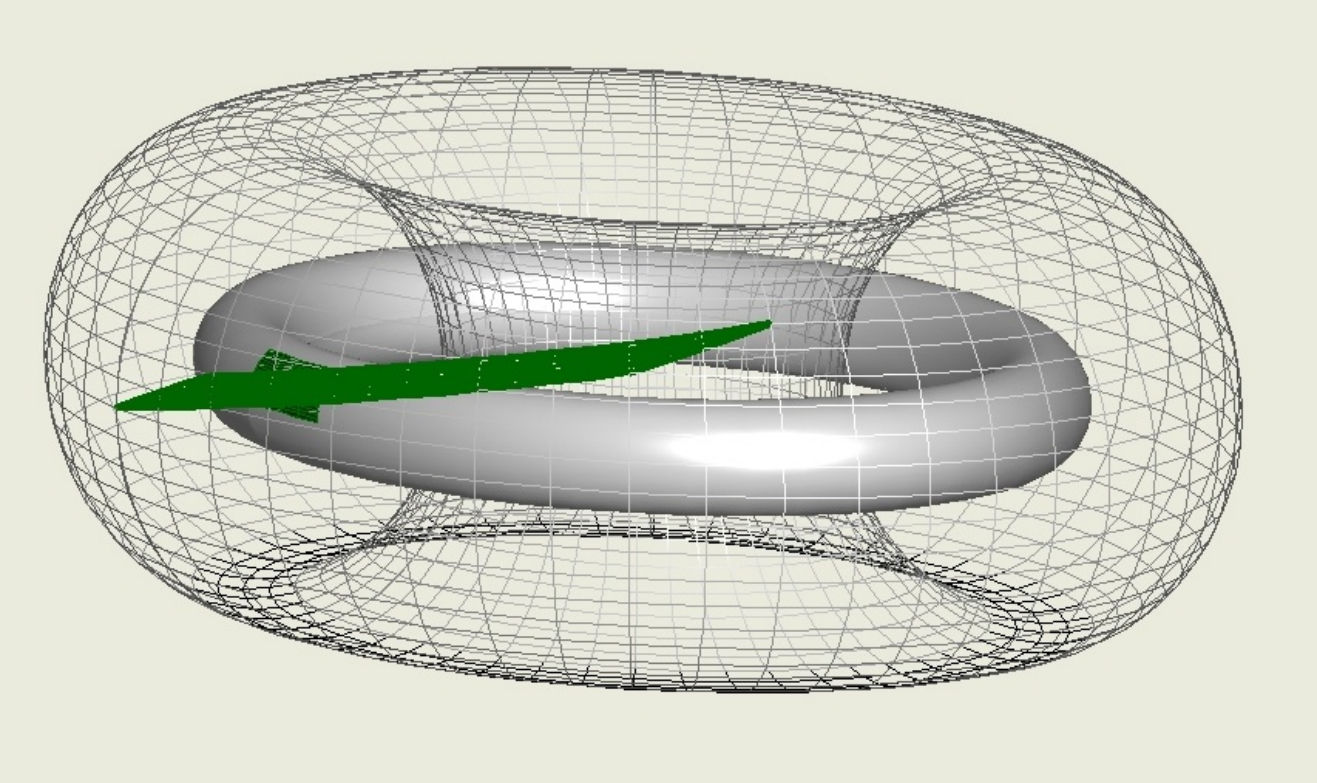
non-ergodic conjecture

2 the mapping torus of $-id$



non-ergodic conjecture

3 the mapping torus of a hyperbolic automorphism



stronger non-ergodic conjecture

stronger non-ergodic conjecture

$f : M \to M$ non-ergodic partially hyperbolic diffeomorphism, then

stronger non-ergodic conjecture

stronger non-ergodic conjecture

$f : M \to M$ non-ergodic partially hyperbolic diffeomorphism, then

- $\exists$ torus tangent to $E^s \oplus E^u$

integrability

integrability

$f : M^3 \to M^3$ is partially hyperbolic

$$TM = E^s \oplus E^c \oplus E^u$$

integrability

integrability

$f : M^3 \to M^3$ is partially hyperbolic

$$\begin{array}{ccccccc} TM & = & E^s & \oplus & E^c & \oplus & E^u \\ & & \uparrow & & & & \uparrow \\ & & \mathcal{F}^s & & & & \mathcal{F}^u \end{array}$$

integrability

integrability

$f : M^3 \to M^3$ is partially hyperbolic

$$\begin{array}{ccccccc} TM & = & E^s & \oplus & E^c & \oplus & E^u \\ & & \uparrow & & \uparrow & & \uparrow \\ & & \mathcal{F}^s & & ? & & \mathcal{F}^u \end{array}$$

dynamical coherence

dynamical coherence

dynamical coherence

dynamical coherence

1. $\exists$ invariant $\mathcal{F}^{cs}$ tangent to $E^s \oplus E^c$

dynamical coherence

dynamical coherence

1. $\exists$ invariant $\mathcal{F}^{cs}$ tangent to $E^s \oplus E^c$
2. $\exists$ invariant $\mathcal{F}^{cu}$ tangent to $E^c \oplus E^u$

dynamical coherence

dynamical coherence

1. $\exists$ invariant $\mathcal{F}^{cs}$ tangent to $E^s \oplus E^c$
2. $\exists$ invariant $\mathcal{F}^{cu}$ tangent to $E^c \oplus E^u$

remark

$\Rightarrow \exists$ invariant $\mathcal{F}^c$ tangent to E^c

open question

longstanding open question

$f : M^3 \to M^3$ partially hyperbolic $\overset{?}{\Rightarrow}$ f dynamically coherent

open question

longstanding open question

$f : M^3 \to M^3$ partially hyperbolic $\overset{?}{\Rightarrow}$ f dynamically coherent

hertz-hertz-ures10

NO

counterexample

hertz-hertz-ures10

$\exists f : \mathbb{T}^3 \to \mathbb{T}^3$ partially hyperbolic

counterexample

hertz-hertz-ures10

$\exists f : \mathbb{T}^3 \to \mathbb{T}^3$ partially hyperbolic

- non-dynamically coherent

counterexample

hertz-hertz-ures10

$\exists f : \mathbb{T}^3 \to \mathbb{T}^3$ partially hyperbolic

- non-dynamically coherent
- non-conservative

counterexample

hertz-hertz-ures10

$\exists f : \mathbb{T}^3 \to \mathbb{T}^3$ partially hyperbolic

- non-dynamically coherent
- non-conservative
- “robust”

theorem

theorem (HHU10)

If $f : M^3 \to M^3$ is partially hyperbolic, then

theorem

theorem (HHU10)

If $f : M^3 \to M^3$ is partially hyperbolic, then

- any invariant $\mathcal{F}^{cu}$ tangent to $E^c \oplus E^u$

theorem

theorem (HHU10)

If $f : M^3 \to M^3$ is partially hyperbolic, then

- any invariant $\mathcal{F}^{cu}$ tangent to $E^c \oplus E^u$
- cannot have compact leaves

structure of the example

structure of the example

let us build an example $f : \mathbb{T}^3 \to \mathbb{T}^3$ such that:

structure of the example

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let us build an example $f : \mathbb{T}^3 \to \mathbb{T}^3$ such that:

- there is an invariant torus T^{cu} tangent to $E^c \oplus E^u$

structure of the example

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let us build an example $f : \mathbb{T}^3 \to \mathbb{T}^3$ such that:

- there is an invariant torus T^{cu} tangent to $E^c \oplus E^u$
- $E^c \oplus E^u$ is uniquely integrable in $\mathbb{T}^3 \setminus T^{cu}$

structure of the example

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let us build an example $f : \mathbb{T}^3 \to \mathbb{T}^3$ such that:

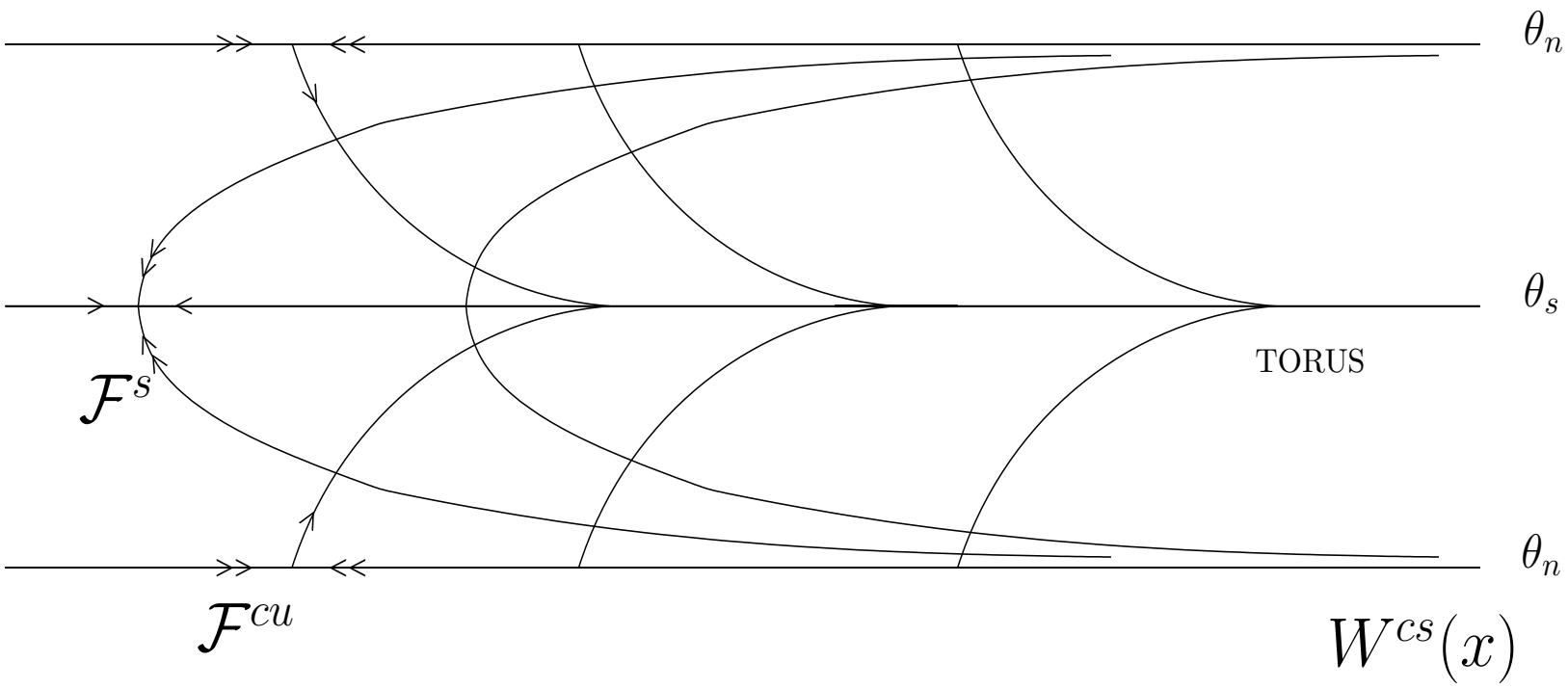
- there is an invariant torus T^{cu} tangent to $E^c \oplus E^u$
- $E^c \oplus E^u$ is uniquely integrable in $\mathbb{T}^3 \setminus T^{cu}$
- $E^c \oplus E^u$ is NOT integrable at T^{cu}

non-integrability at T^{cu}

view of a center-stable leaf

non-integrability at T^{cu}

view of a center-stable leaf



how it is built

f : Anosov $\times$ NP-SP

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$$f(x, \theta) = (Ax, \psi(\theta))$$

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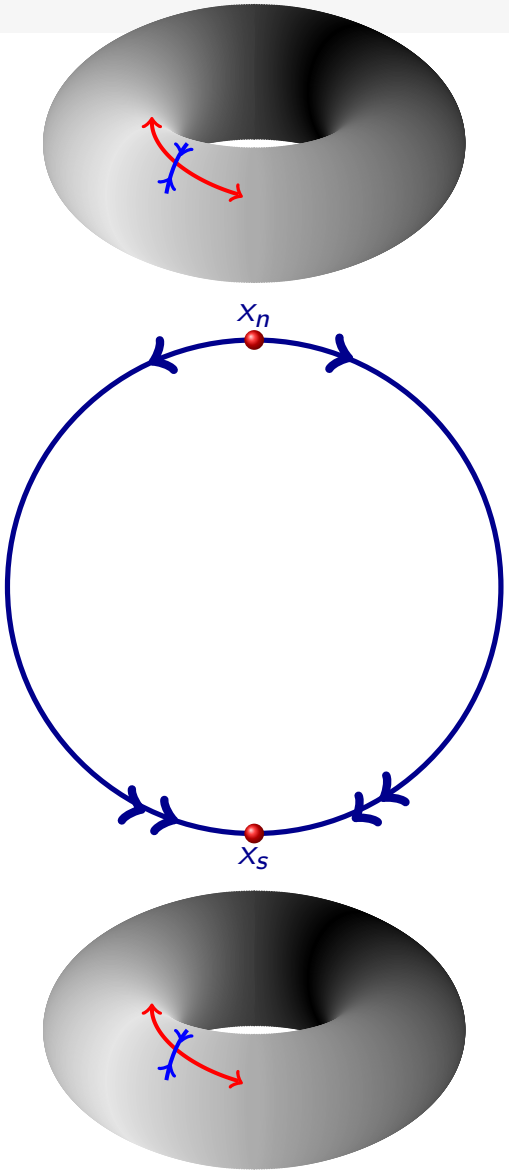

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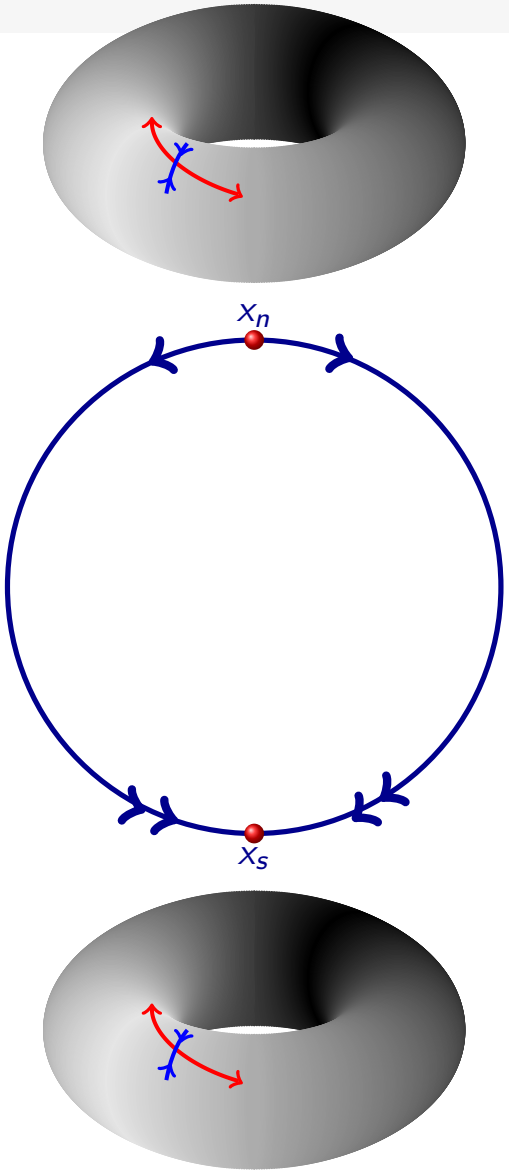
$g \sim$ Anosov $\times$ NP-SP



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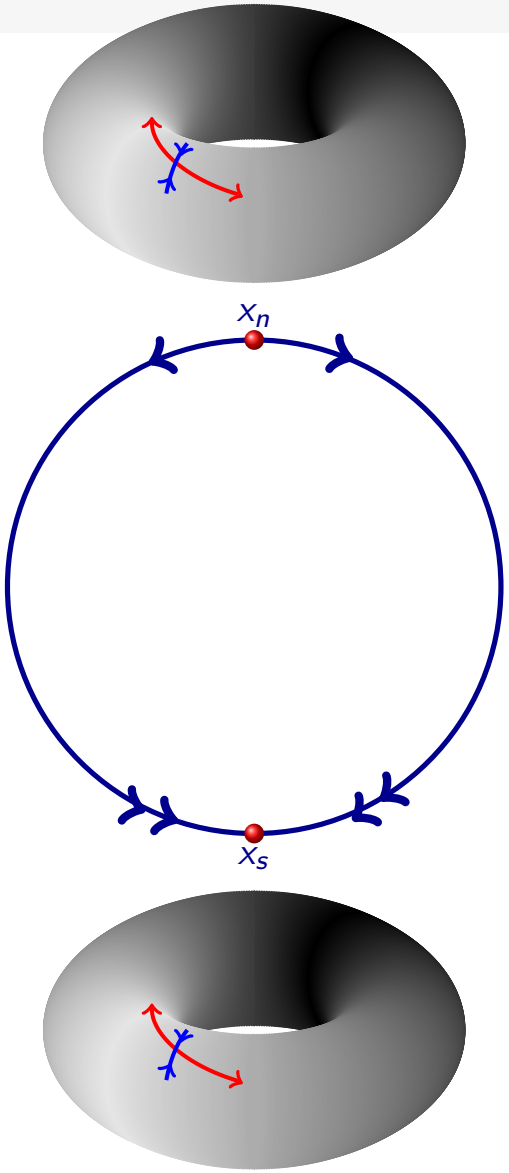
f : Anosov $\times$ NP-SP

$$f(x,\theta) = (Ax, \psi(\theta))$$

$g \sim$ Anosov $\times$ NP-SP

$$g(x,\theta) = (Ax + v(\theta)e_s, \psi(\theta))$$



how it is built

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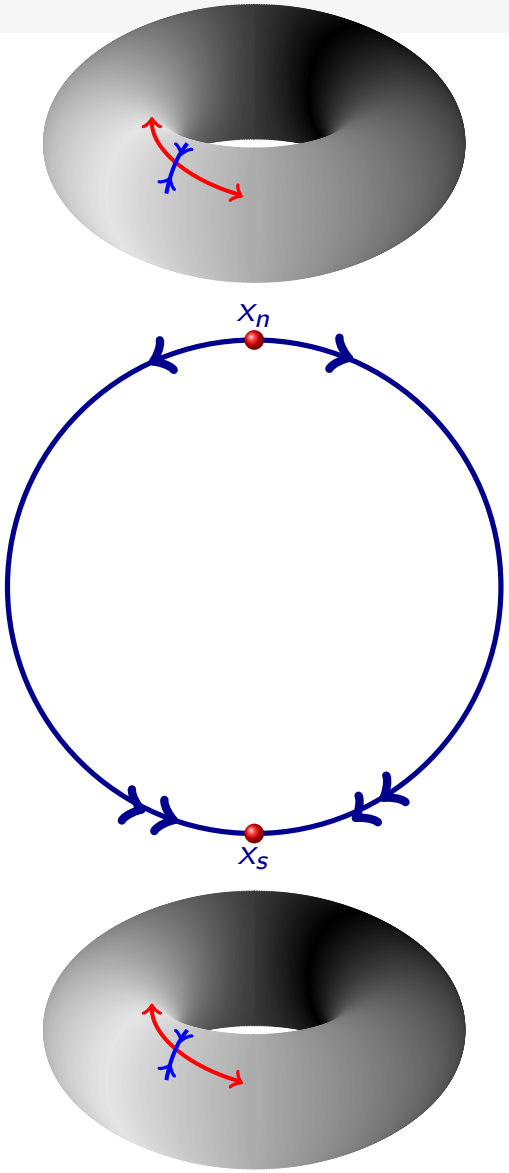
$$g \sim \text{Anosov} \times \text{NP-SP}$$

$$g(x, \theta) = (Ax + v(\theta)e_s, \psi(\theta))$$

$$v(\theta_s) = 0$$



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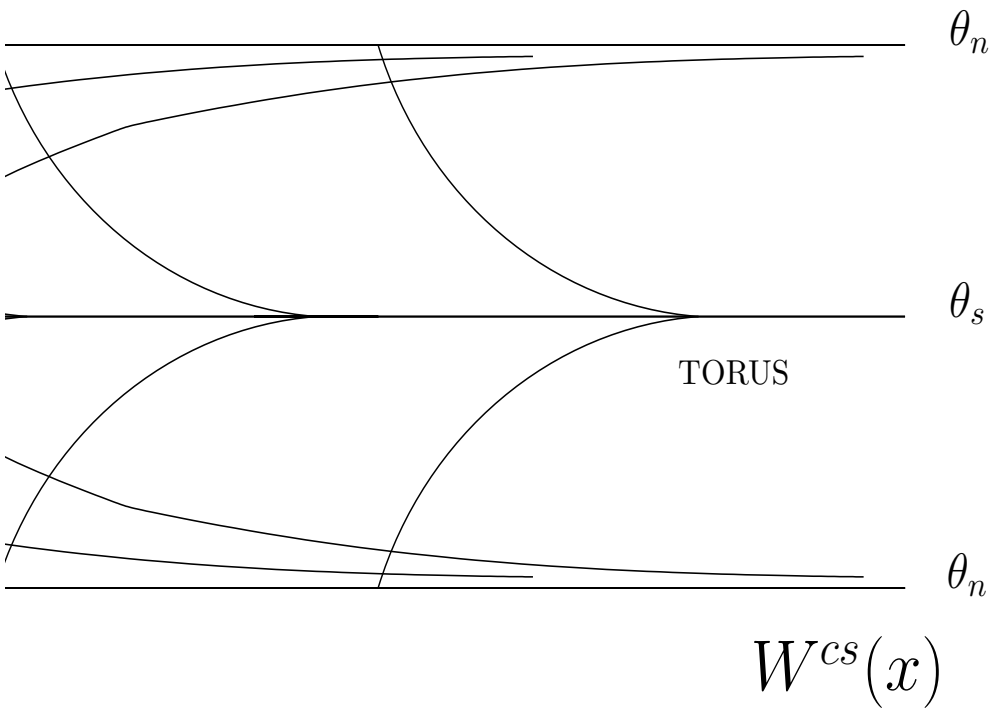

derivative of the perturbation

$$f(x,\theta) = (Ax, \psi(\theta))$$

derivative of the perturbation

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$$Df(x,\theta) = \left(\begin{array}{cc|c} \lambda & 0 & 0 \\ 0 & 1/\lambda & 0 \\ \hline 0 & 0 & \psi'(\theta) \end{array}\right)$$

derivative of the perturbation

$$f(x,\theta) = (Ax, \psi(\theta)) \qquad g(x,\theta) = (Ax + v(\theta)e_s, \psi(\theta))$$

$$Df(x,\theta) = \left(\begin{array}{cc|c} \lambda & 0 & 0 \\ 0 & 1/\lambda & 0 \\ \hline 0 & 0 & \psi'(\theta) \end{array}\right)$$

derivative of the perturbation

$$f(x,\theta) = (Ax, \psi(\theta))$$

$$g(x,\theta) = (Ax + v(\theta)e_s, \psi(\theta))$$

$$Df(x,\theta) = \left(\begin{array}{cc|c} \lambda & 0 & 0 \\ 0 & 1/\lambda & 0 \\ \hline 0 & 0 & \psi'(\theta) \end{array}\right)$$

$$Dg(x,\theta) = \left(\begin{array}{cc|c} \lambda & 0 & v'(\theta) \\ 0 & 1/\lambda & 0 \\ \hline 0 & 0 & \psi'(\theta) \end{array}\right)$$

goals

- goal 1: $g(x,\theta) = (Ax + v(\theta)e_s, \psi(\theta)) \to v$

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goals

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- g is semiconjugated to $A : \mathbb{T}^2 \to \mathbb{T}^2$
- semiconjugacy: $h : \mathbb{T}^3 \to \mathbb{T}^2$

goals

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- goal 2: h preserves $\mathcal{F}_f^{cs}$

goals

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- semiconjugacy: $h : \mathbb{T}^3 \to \mathbb{T}^2$
- goal 2: h preserves $\mathcal{F}_f^{cs}$
- $h(x,\theta) = x - u(\theta)e_s$

semiconjugacy

$$\begin{array}{ccccc} & \mathbb{T}^3 & \xrightarrow{g} & \mathbb{T}^3 & \\ h & \downarrow & & \downarrow & h \\ & \mathbb{T}^2 & \xrightarrow{A} & \mathbb{T}^2 & \end{array} \qquad h(x,\theta) = x - u(\theta)e_s$$

semiconjugacy

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$$h(Ax + v(\theta)e_s, \psi(\theta)) \quad = \quad Ah(x,\theta)$$

semiconjugacy

$$\begin{array}{ccccc} & \mathbb{T}^3 & \xrightarrow{g} & \mathbb{T}^3 & \\ h & \downarrow & & \downarrow & h \\ & \mathbb{T}^2 & \xrightarrow{A} & \mathbb{T}^2 & \end{array} \qquad h(x,\theta) = x - u(\theta)e_s$$

$$\begin{aligned} h(Ax + v(\theta)e_s, \psi(\theta)) &= Ah(x,\theta) \\ Ax + v(\theta)e_s - u(\psi(\theta))e_s &= Ax - \lambda u(\theta)e_s \end{aligned}$$

semiconjugacy

$$\begin{array}{ccccc} & \mathbb{T}^3 & \xrightarrow{g} & \mathbb{T}^3 & \\ h & \downarrow & & \downarrow & h \\ & \mathbb{T}^2 & \xrightarrow{A} & \mathbb{T}^2 & \end{array} \qquad h(x,\theta) = x - u(\theta)e_s$$

$$\begin{aligned} h(Ax + v(\theta)e_s, \psi(\theta)) &= Ah(x,\theta) \\ Ax + v(\theta)e_s - u(\psi(\theta))e_s &= Ax - \lambda u(\theta)e_s \\ v(\theta)e_s - u(\psi(\theta))e_s &= -\lambda u(\theta)e_s \end{aligned}$$

semiconjugacy

$$\begin{array}{ccccc} & \mathbb{T}^3 & \xrightarrow{g} & \mathbb{T}^3 & \\ h & \downarrow & & \downarrow & h \\ & \mathbb{T}^2 & \xrightarrow{A} & \mathbb{T}^2 & \end{array} \qquad h(x,\theta) = x - u(\theta)e_s$$

$$\begin{aligned} h(Ax + v(\theta)e_s, \psi(\theta)) &= Ah(x,\theta) \\ Ax + v(\theta)e_s - u(\psi(\theta))e_s &= Ax - \lambda u(\theta)e_s \\ v(\theta)e_s - u(\psi(\theta))e_s &= -\lambda u(\theta)e_s \end{aligned}$$

twisted cohomological equation:

$$\boxed{u \circ \psi - \lambda u = v}$$

cohomological equation

twisted cohomological equation

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cohomological equation

twisted cohomological equation

$$\boxed{u \circ \psi - \lambda u = v}$$

has solution

$$u(\theta) = \frac{1}{\lambda} \sum_{k=1}^{\infty} \lambda^k v(\psi^{-k}(\theta))$$

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$$u(\theta) = \frac{1}{\lambda} \sum_{k=1}^{\infty} \lambda^k v(\psi^{-k}(\theta))$$

- $u(0) = 0$

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has solution

$$u(\theta) = \frac{1}{\lambda} \sum_{k=1}^{\infty} \lambda^k v(\psi^{-k}(\theta))$$

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- $\sum$ converges uniformly in any compact $\not\ni \theta_s$
- $\Rightarrow u'$ continuous for $\theta \neq \theta_s$

calculation of v

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- $u'(\theta) \geq \frac{\varepsilon}{\lambda L^3} \frac{\lambda^K}{\sigma^K}$

open problem

open problem

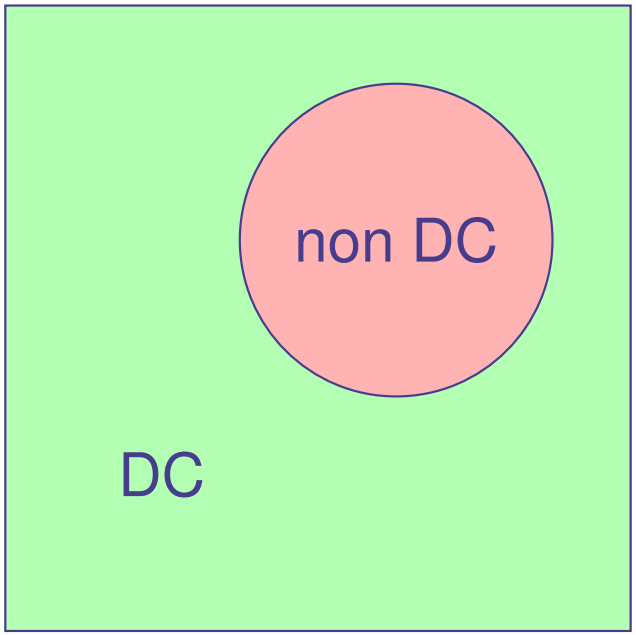
describe 3-manifolds supporting non-dynamically coherent examples

open problem

open problem

describe 3-manifolds supporting non-dynamically coherent examples

3-manifolds



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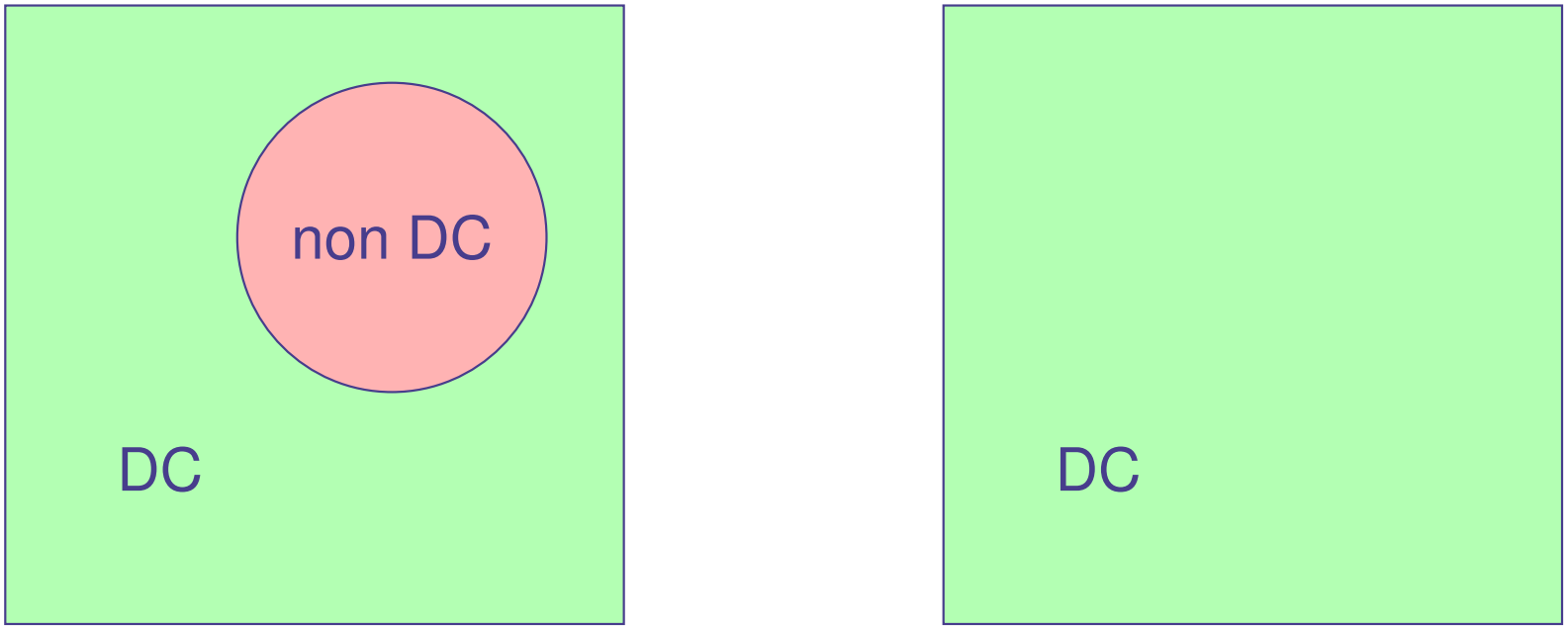
describe 3-manifolds supporting non-dynamically coherent examples

3-manifolds



non-dynamically coherent conjecture

non-dynamically coherent conjecture (hertz-hertz-ures)

$f : M^3 \to M^3$ non-dynamically coherent,

non-dynamically coherent conjecture

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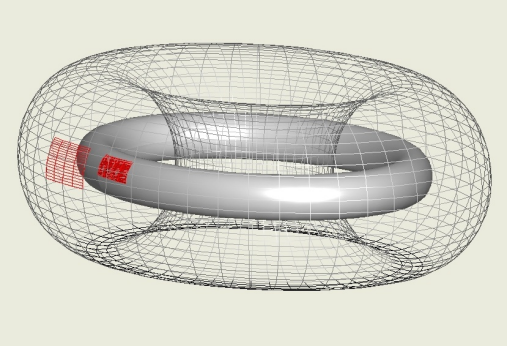

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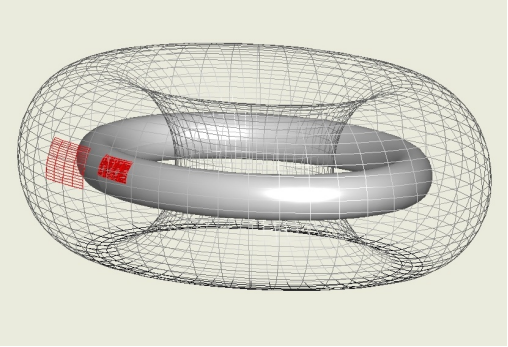

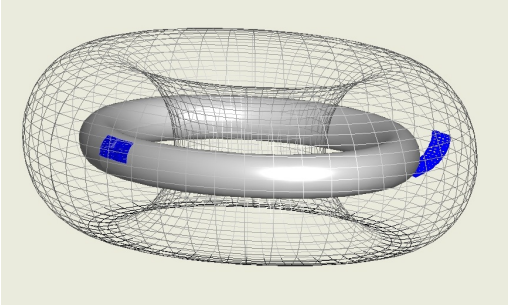

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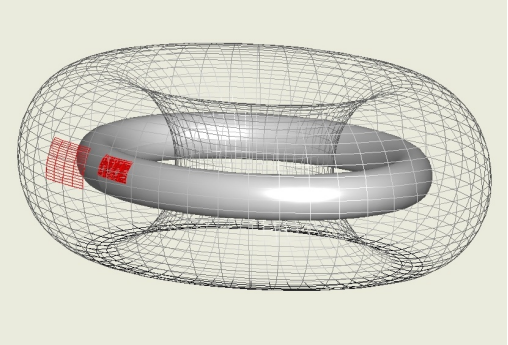
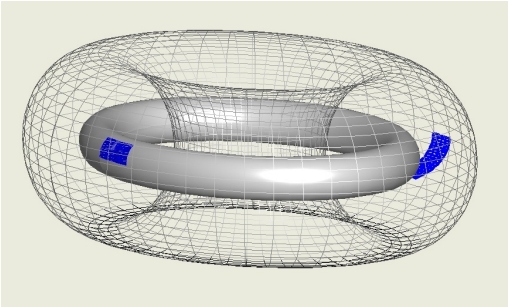
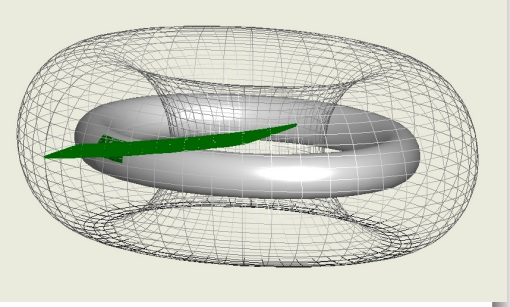
3-manifolds







stronger conjecture

stronger non-dynamically coherent conjecture

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intermediate conjecture

intermediate conjecture

f volume preserving $\Rightarrow$ f dynamically coherent

evidence

hammerlind-potrie13

f partially hyperbolic & non-dynamically coherent on a 3-manifold with solvable fundamental group, then

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examples of ph dynamics

known ph dynamics in dimension 3

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- perturbations of time-one maps of Anosov flows

examples of ph dynamics

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- perturbations of time-one maps of Anosov flows
- certain skew-products

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new example

- non-dynamically coherent example

question

question

are there more examples?

conjecture pujals

classification conjecture (pujals01)

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If $f : M^3 \to M^3$ is partially hyperbolic and dynamically coherent, then f is

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conjecture

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invariant tori in ph dynamics

invariant tori in PH dynamics

T invariant torus tangent to

invariant tori in ph dynamics

invariant tori in PH dynamics

T invariant torus tangent to

- $E^s \oplus E^u$

invariant tori in ph dynamics

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$\Rightarrow$ T Anosov torus

conjectures

conjectures

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conjectures

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non-dyn. coh. conjecture

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why stronger conjectures?

hertz-hertz-ures11

M irreducible contains an Anosov torus,

why stronger conjectures?

hertz-hertz-ures11

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hertz-hertz-ures11

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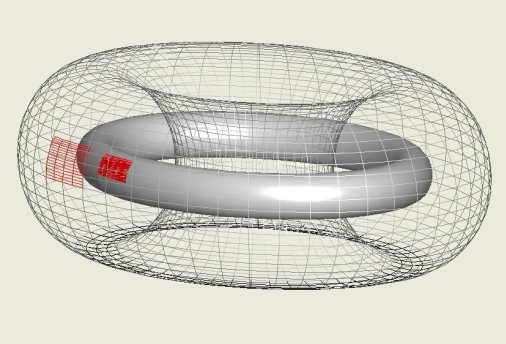
3-manifolds



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hertz-hertz-ures11

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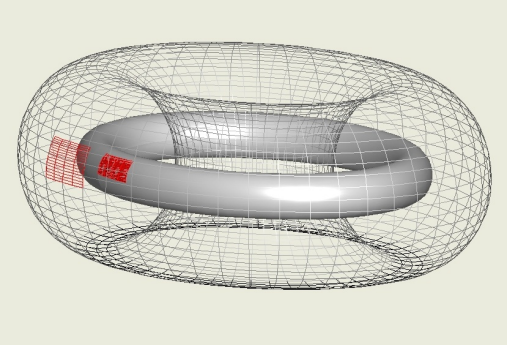
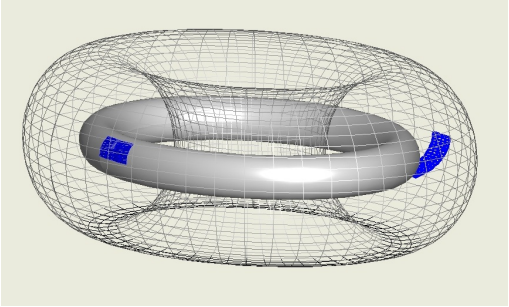
3-manifolds





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hertz-hertz-ures11

M irreducible contains an Anosov torus, then M is either

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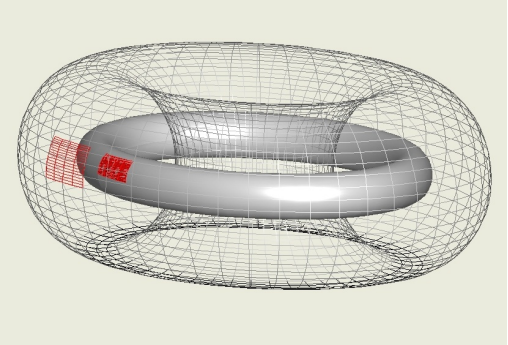
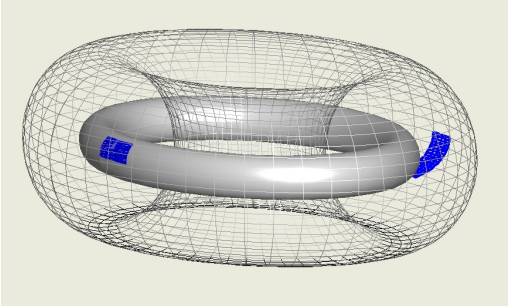
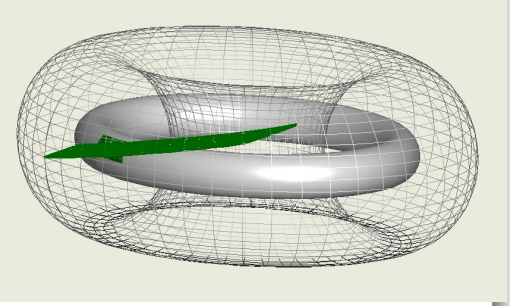
3. a mapping torus of a hyperbolic automorphism of $\mathbb{T}^2$

3-manifolds







remark

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$f : M^3 \to M^3$ partially hyperbolic $\Rightarrow$ M irreducible

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why

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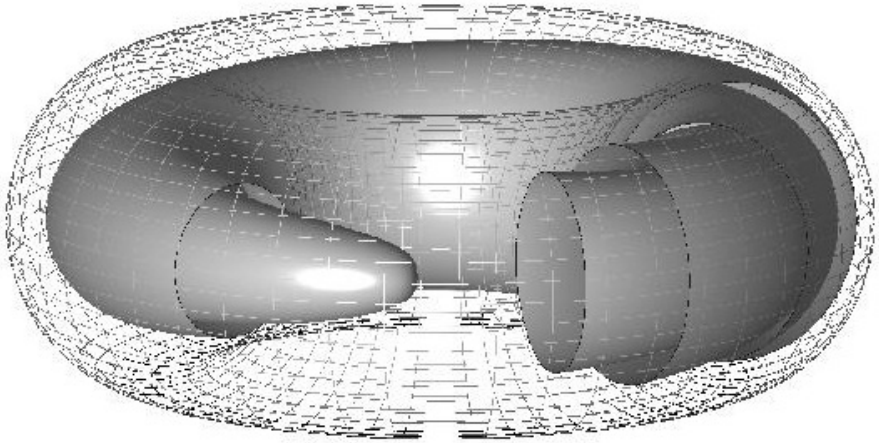


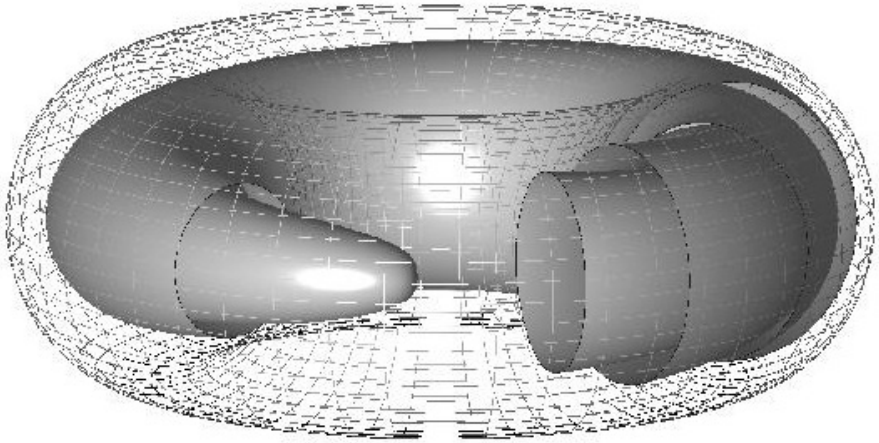
- Rosenberg68

remark

remark

$f : M^3 \to M^3$ partially hyperbolic $\Rightarrow$ M irreducible

why



- Rosenberg68
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