Dynamics of condensation in inclusion processes

Stefan Grosskinsky

Warwick

in collaboration with Frank Redig, Kiamars Vafayi

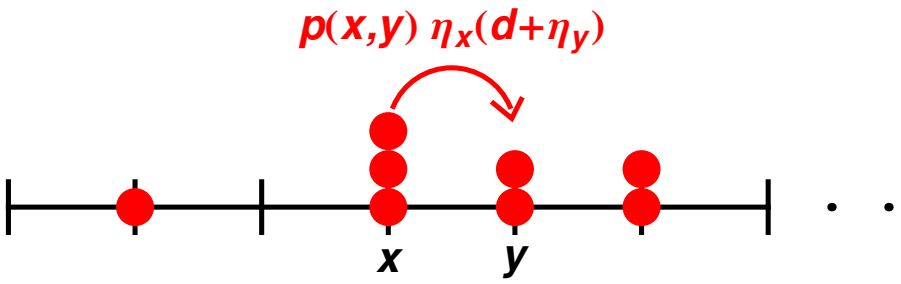
August 8, 2013

Inclusion process

Lattice: Λ of size L

State space: $X = \{0, 1, ..\}^{\Lambda}$

$$\boldsymbol{\eta} = (\eta_x)_{x \in \Lambda}$$



Jump rates: $p(x,y)\eta_x(d+\eta_y)$, $\quad d > 0$

$p(x,y) \geq 0$ irreducible on Λ

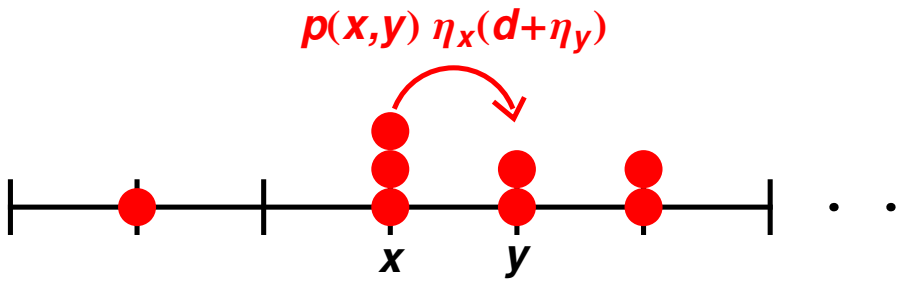
Generator: $\mathcal{L}f(\boldsymbol{\eta}) = \sum_{x,y \in \Lambda_L} p(x,y)\, \eta_x(d+\eta_y)\big(f(\boldsymbol{\eta}^{x,y}) - f(\boldsymbol{\eta})\big)$

Inclusion process

Lattice: Λ of size L

State space: $X = \{0, 1, ..\}^{\Lambda}$

$$\boldsymbol{\eta} = (\eta_x)_{x \in \Lambda}$$



Jump rates: $p(x,y)\eta_x(d+\eta_y)$, $\quad d > 0$

$p(x,y) \geq 0$ irreducible on Λ

Generator: $\mathcal{L}f(\boldsymbol{\eta}) = \sum_{x,y \in \Lambda_L} p(x,y)\, \eta_x(d+\eta_y)\big(f(\boldsymbol{\eta}^{x,y}) - f(\boldsymbol{\eta})\big)$

misanthrope model with rates $\quad c(n,m) = n(d+m)$

[Giardina, Kurchan, Redig, Vafayi (2009); G., Redig, Vafayi (2011)] [Cocozza-Thivent (1985)]

Inclusion process

Applications of IP

- 2 sites, N particles: rates $c(k, N-k) = d\,k + k(N-k)$
  $\rightarrow$ multi-species Moran model (related to Wright-Fisher)
- duality with Brownian energy/momentum process

  [Giardina, Kurchan, Redig, Vafayi (2009); Giardina, Redig, Vafayi (2010)]

Inclusion process

Applications of IP

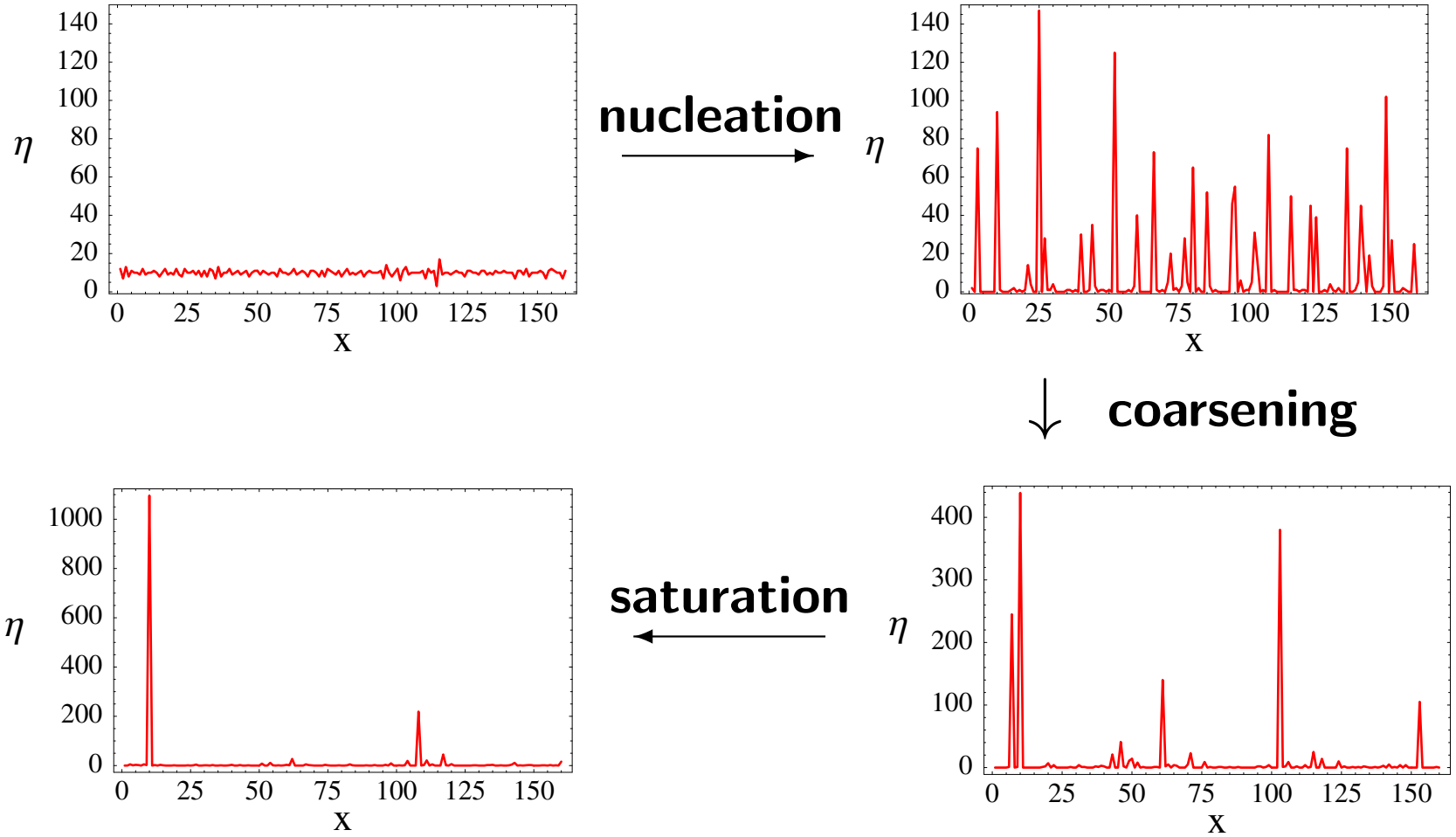
- 2 sites, N particles: rates $c(k, N-k) = d\,k + k(N-k)$
  $\rightarrow$ multi-species Moran model (related to Wright-Fisher)
- duality with Brownian energy/momentum process

  [Giardina, Kurchan, Redig, Vafayi (2009); Giardina, Redig, Vafayi (2010)]

Condensation

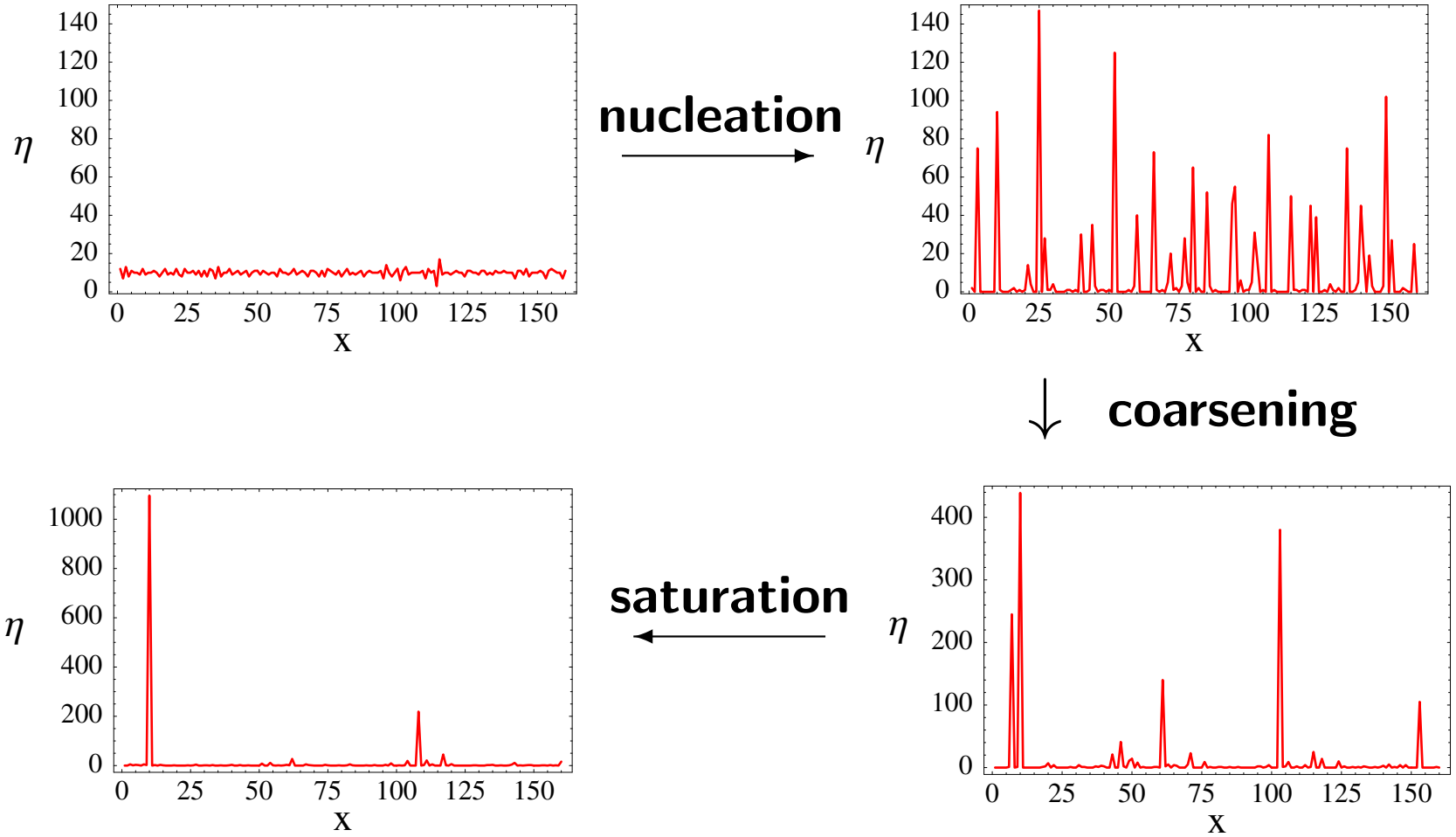
- spatial heterogeneity $p(x, y)$
  $\Rightarrow$ condensation on 'slow sites'
- effective attraction of particles
  $\Rightarrow$ condensation on a random site
  models with $c(n, m) \searrow$ in n and/or $\nearrow$ in m

Homogeneous condensation



[S. G., F. Redig, K. Vafayi, EJP 18, no. 66, 123 (2013)]

Homogeneous condensation



Dynamics of condensation

Λ fixed ; $N \to \infty$, $d_N \to 0$ such that $d_N \gg 1/N$

$$\mathcal{L}_N f(\boldsymbol{\eta}) = \sum_{x,y \in \Lambda_L} p(x,y)\, \eta_x (d_N + \eta_y) \big(f(\boldsymbol{\eta}^{x,y}) - f(\boldsymbol{\eta}) \big)$$

time scale $\theta_N := 1/d_N$

$$\mathbf{u}^N(t) := \big(\eta_x(\theta_N t)/N : x \in \Lambda \big)$$

process on the simplex $E = \big\{ \mathbf{u} \in [0,1]^\Lambda : \sum_{x \in \Lambda} u_x = 1 \big\}$

Dynamics of condensation

Λ fixed ; $N \to \infty$, $d_N \to 0$ such that $d_N \gg 1/N$

$$\mathcal{L}_N f(\boldsymbol{\eta}) = \sum_{x,y \in \Lambda_L} p(x,y)\, \eta_x (d_N + \eta_y) \big(f(\boldsymbol{\eta}^{x,y}) - f(\boldsymbol{\eta}) \big)$$

time scale $\theta_N := 1/d_N$

$$\mathbf{u}^N(t) := \big(\eta_x(\theta_N t)/N : x \in \Lambda \big)$$

process on the simplex $E = \big\{ \mathbf{u} \in [0,1]^\Lambda : \sum_{x \in \Lambda} u_x = 1 \big\}$

Taylor expansion ($p(x,y)$ symmetric)

$$\theta_N \mathcal{L}_N f(\mathbf{u}) = \frac{1}{2} \sum_{x,y \in \Lambda} p(x,y)(u_x - u_y)(\partial_{u_y} - \partial_{u_x}) f(\mathbf{u})$$

$$+ \frac{1}{2} \sum_{x,y \in \Lambda} p(x,y) u_x u_y \theta_N (\partial_{u_x} - \partial_{u_y})^2 f(\mathbf{u}) + O(\theta_N/N) =$$

$$= L\, f(\mathbf{u}) + \theta_N\, L'\, f(\mathbf{u}) + O(\theta_N/N)$$

two-scale system with drift and fast Wright-Fisher diffusion

Dynamics of condensation

WF-diffusion has absorbing set

$$\mathcal{A} := \left\{ \mathbf{u} \in E \,:\, p(x,y)\, u_x u_y = 0 \text{ for all } x, y \in \Lambda \right\} .$$

corner points $\mathcal{C} := \left\{ \mathbf{e}_x : x \in \Lambda \right\} \subset \mathcal{A}$

Dynamics of condensation

WF-diffusion has absorbing set

$$\mathcal{A} := \left\{ \mathbf{u} \in E \,:\, p(x,y)\, u_x u_y = 0 \text{ for all } x, y \in \Lambda \right\} .$$

corner points $\mathcal{C} := \left\{ \mathbf{e}_x : x \in \Lambda \right\} \subset \mathcal{A}$

Theorem 1

Assume $\mathbf{u}^N(0) \xrightarrow{d} \mathbf{u}^0 \in \mathcal{C}$. Then $\big(\mathbf{u}^N(t) : t \geq 0\big)$ converges weakly on path space to $\big(\mathbf{u}(t) : t \geq 0\big)$ on $\mathcal{C}$ with $\mathbf{u}(0) = \mathbf{u}^0$ and generator

$$Af(\mathbf{e}_x) = \sum_{y \in \Lambda} p(x,y) \big(f(\mathbf{e}_y) - f(\mathbf{e}_x) \big) .$$

Dynamics of condensation

WF-diffusion has absorbing set

$$\mathcal{A} := \left\{ \mathbf{u} \in E \,:\, p(x,y)\, u_x u_y = 0 \text{ for all } x, y \in \Lambda \right\} .$$

corner points $\mathcal{C} := \left\{ \mathbf{e}_x : x \in \Lambda \right\} \subset \mathcal{A}$

Theorem 1

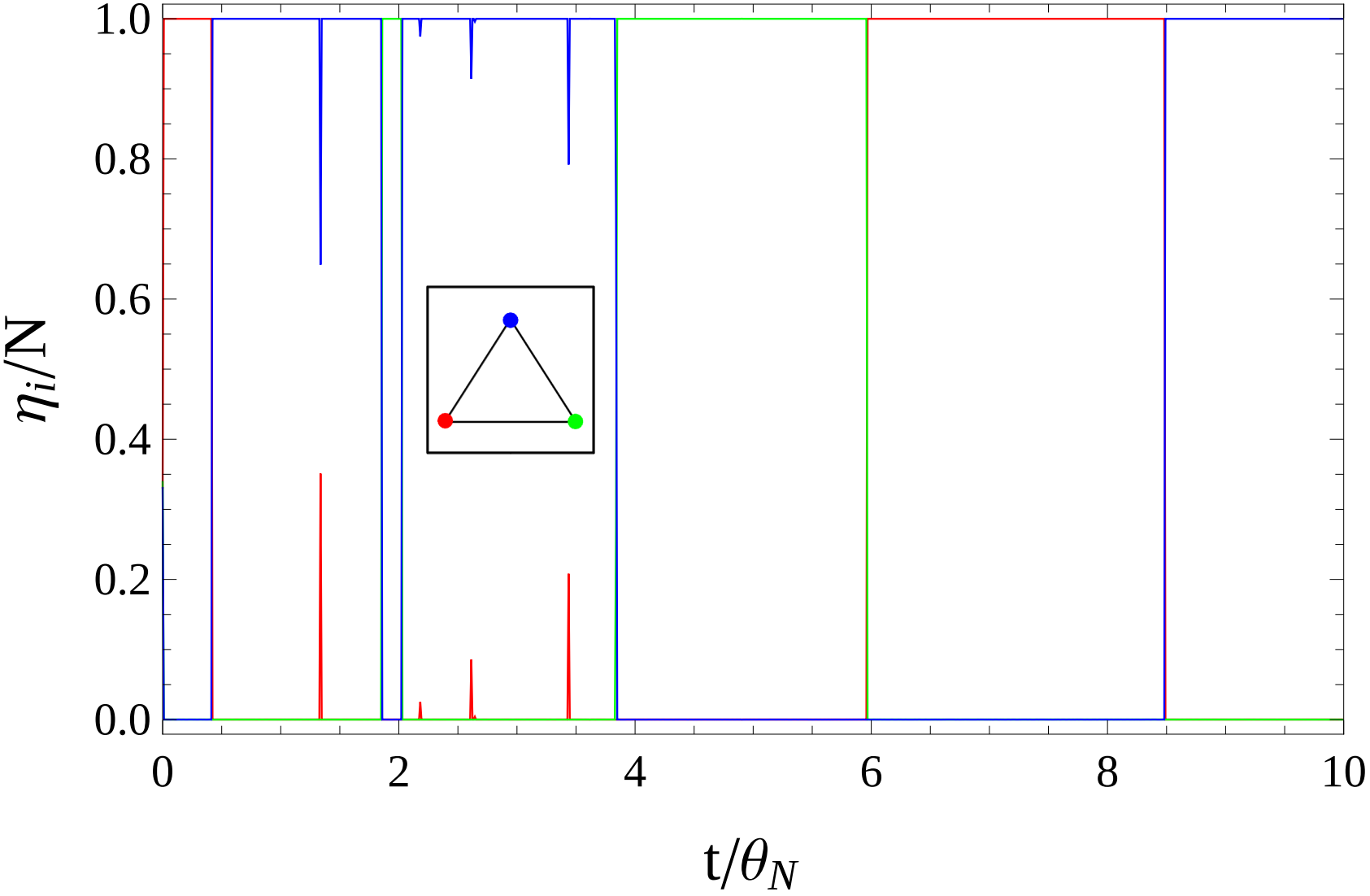
Assume $\mathbf{u}^N(0) \xrightarrow{d} \mathbf{u}^0 \in \mathcal{C}$. Then $\big(\mathbf{u}^N(t) : t \geq 0\big)$ converges weakly on path space to $\big(\mathbf{u}(t) : t \geq 0\big)$ on $\mathcal{C}$ with $\mathbf{u}(0) = \mathbf{u}^0$ and generator

$$Af(\mathbf{e}_x) = \sum_{y \in \Lambda} p(x,y) \big(f(\mathbf{e}_y) - f(\mathbf{e}_x) \big) .$$

If $p(x,y) > 0$ for all $x, y \in \Lambda$ the same holds (with $t > 0$) for general initial conditions $\mathbf{u}^N(0) \xrightarrow{d} \mathbf{u}^0 \in E$ with $\mathbb{P}(\mathbf{u}(0) = e_x) = u_x^0$.

Illustration

3-site ring $N = 10000,\ d_N = 0.001$



Dynamics of condensation

Theorem 2

Let $p(x,y) \in \{0,1\}$, $\mathbf{u}^N(0) \xrightarrow{d} \mathbf{u}^0 \in E$ and write

$$\hat{p}(x,y) = \big(1 - p(x,y)\big) \sum_{z \in \Lambda} p(x,z)p(z,y) \geq 0 \ .$$

Dynamics of condensation

Theorem 2

Let $p(x,y) \in \{0,1\}$, $\mathbf{u}^N(0) \xrightarrow{d} \mathbf{u}^0 \in E$ and write

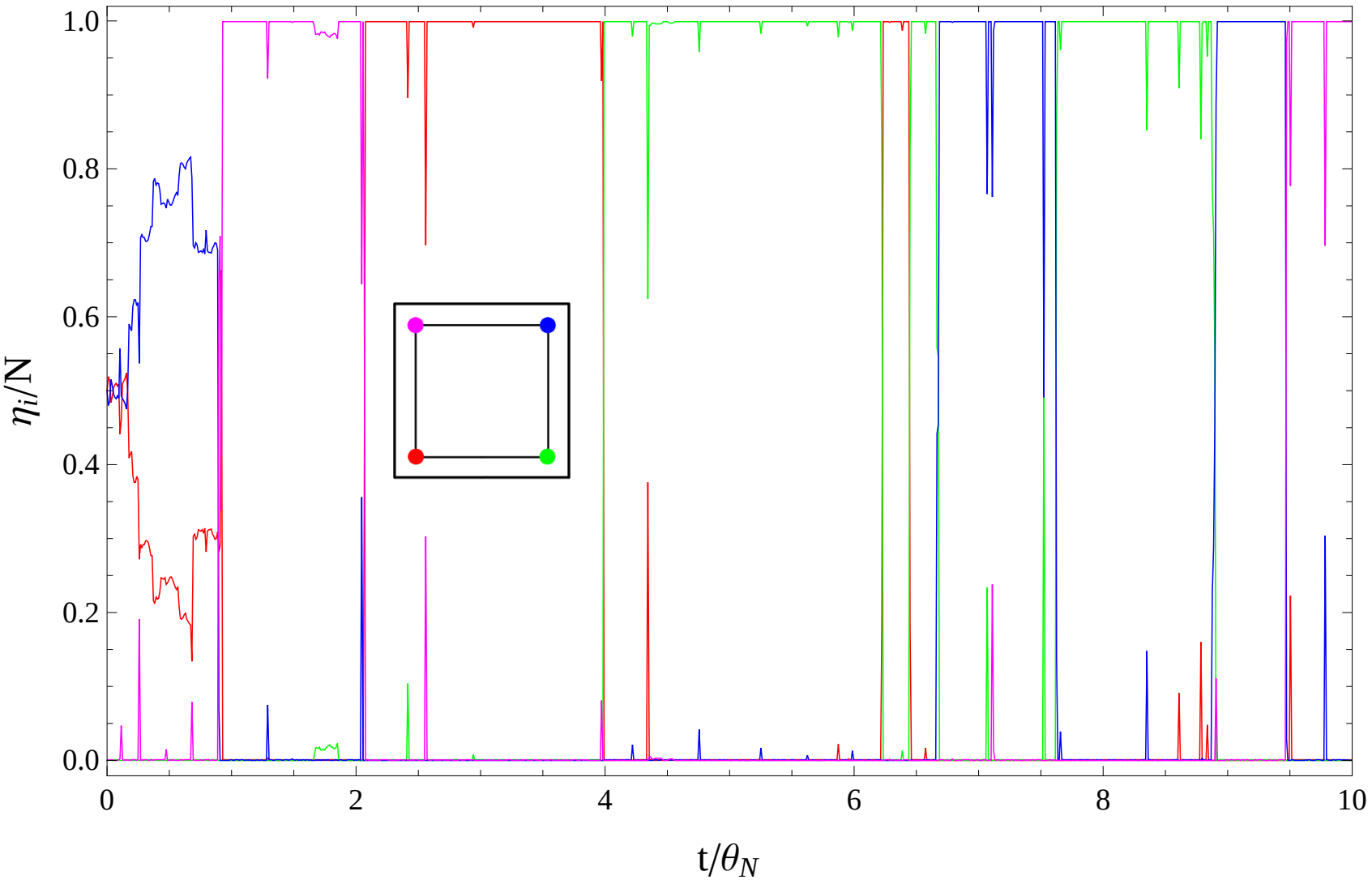
$$\hat{p}(x,y) = \big(1 - p(x,y)\big) \sum_{z \in \Lambda} p(x,z)p(z,y) \geq 0 \ .$$

Then $\big(\mathbf{u}^N(t) : t > 0\big)$ converges weakly on path space to $\big(\mathbf{u}(t) : t > 0\big)$ on $\mathcal{A}$ with initial condition $\mathbf{u}(0) \sim \nu_{\mathbf{u}^0}$ and generator

$$Af(\mathbf{u}) = \sum_{x,y \in \Lambda} \frac{1}{2} \hat{p}(x,y)\, u_x u_y \left(\partial_{u_x} - \partial_{u_y}\right)^2 f(\mathbf{u})$$
$$+ \sum_{y \in \Lambda} \delta_{u_y,0} \Big(\sum_{x \in \Lambda} p(x,y) u_x \Big) \Big(f\Big(\mathbf{u} + \sum_{x \in \Lambda} p(x,y) u_x (\mathbf{e}_y - \mathbf{e}_x)\Big) - f(\mathbf{u}) \Big)$$

Illustration

4-site ring $N = 1000,\ d_N = 0.01$



Method of proof

convergence of the semigroups $e^{t\theta_N \mathcal{L}_N}$ and $e^{(L+\theta_N L')t}$

Central lemma. For all $t > 0$

$$p(x,y) \sup_{\mathbf{u}\in E} \mathbb{E}_{\mathbf{u}}\left[u_x^N(t)u_y^N(t)\right] \to 0 \quad \text{as } N \to \infty \ ,$$

$$p(x,y) \limsup_{N\to\infty} \theta_N \sup_{\mathbf{u}\in E} \mathbb{E}_{\mathbf{u}}\left[u_x^N(t)u_y^N(t)\right] \leq C \ .$$

from **Gronwall**-type estimate due to two-scale structure

Method of proof

convergence of the semigroups $e^{t\theta_N \mathcal{L}_N}$ and $e^{(L+\theta_N L')t}$

Central lemma. For all $t > 0$

$$p(x,y) \sup_{\mathbf{u}\in E} \mathbb{E}_{\mathbf{u}}\big[u_x^N(t)u_y^N(t)\big] \to 0 \quad \text{as } N \to \infty \ ,$$

$$p(x,y) \limsup_{N\to\infty} \theta_N \sup_{\mathbf{u}\in E} \mathbb{E}_{\mathbf{u}}\big[u_x^N(t)u_y^N(t)\big] \le C \ .$$

from **Gronwall**-type estimate due to two-scale structure

- **tightness** of $(\mathbf{u}^N(t) : t > 0)$ with Lemma
  for $t = 0$ use right-continuity of paths
- for Theorem 1, characterize through **martingale problem** on $\mathcal{C}$

$$M_x(t) := u_x(t) - u_x(0) - \sum_{y\in\Lambda} \int_0^t p(x,y)\big(u_y(s) - u_x(s)\big)\, ds$$

Method of proof

for Theorem 2 (general initial condition)

- **harmonic projection** $Pf(x) := \int_{\mathcal{A}} f(a)\nu_x(da)$ [Kurtz (1973)]

  $P : C(E, \mathbb{R}) \to \mathcal{H}(E, \mathbb{R})$, $L'(Pf) = 0$ with BC $f(a)$

Method of proof

for Theorem 2 (general initial condition)

- **harmonic projection** $Pf(x) := \int_{\mathcal{A}} f(a)\nu_x(da)$ [Kurtz (1973)]

  $P : C(E,\mathbb{R}) \to \mathcal{H}(E,\mathbb{R})$, $L'(Pf) = 0$ with BC $f(a)$

- convergence $e^{(L+\theta_N L')t} \to S(t)$ with $S(0) = P$

  semigroup on $\mathcal{H}(E,\mathbb{R})$ with **generator** $Af := (PL)f$

  process on $C(\mathcal{A},\mathbb{R})$ by uniqueness of harmonic functions

- computation $PL = \lim_{h\searrow 0} P\left(\frac{e^{hL}-I}{h}\right)$

  use **martingales** $u_x(t)$, $u_x(t)u_y(t)$ if $p(x,y) = 0$

Conclusion

- stationary results, large deviations
- condensation due to vanishing diffusion rate
- dynamics of condensation on finite lattices

Work in progress.

generalize condition on θ_N, include asymmetry,

dynamics of correlation functions in thermodynamic limit,

more general rates (such as $\gamma > 2$, ECM)