ON THE DYNAMICS OF THE SCHRÖDINGER-KORTEWEG DE VRIES SYSTEM IN THE ENERGY SPACE

Nonlinear PDE's@IMPA

Xavier Carvajal and Adán J. Corcho
Instituto de Matemática/IM
Universidade Federal o Rio de Janeiro/UFRJ

Abstract

In this work we study the interactions between long and short waves that arise in various physical situations (see [2, 4, 7, 9]). Specifically, we study interactions modeled by the Initial Value Problem (IVP) for the Schrödinger-Korteweg de Vries system, that is,

$$(0.1)\qquad \begin{cases} i\phi_\tau + \phi_{\xi\xi} = \alpha\phi\eta + \beta|\phi|^2\phi, \quad \tau,\ \xi \in \mathbb{R},\\ \eta_\tau + \eta_{\xi\xi\xi} + \eta\eta_\xi = \gamma(|\phi|^2)_\xi,\\ \phi(\xi,0) = \phi_0(\xi), \quad \eta(\xi,0) = \eta_0(\xi), \end{cases}$$

where the short wave $\phi = \phi(\xi,\tau)$ is a complex valued function, the long wave $\eta = \eta(\xi,\tau)$ is a real valued function and α, β and γ are real constants with $\alpha\gamma \neq 0$.

Under the transformations

$$\phi(\xi,\tau) = \sqrt{\tfrac{8}{|\alpha\gamma|}}u(2\xi,4\tau) = \sqrt{\tfrac{8}{|\alpha\gamma|}}u(x,t) \quad \text{and} \quad \eta(\xi,\tau) = \tfrac{4}{\alpha}v(2\xi,4\tau) = \tfrac{4}{\alpha}v(x,t)$$

the system (0.1) becomes into the coupled equations:

$$(0.2)\qquad \begin{cases} iu_t + u_{xx} = uv + p|u|^2u, \qquad t,\ x \in \mathbb{R},\\ v_t + 2v_{xxx} + 3q(v^2)_x = \epsilon(|u|^2)_x,\\ u(x,0) = u_0(x), \quad v(x,0) = v_0(x), \end{cases}$$

with $p = \frac{2\beta}{\alpha\gamma}$, $q = \frac{1}{3\alpha}$ and $\epsilon = sgn(\alpha\gamma)$. We will refer to the cases $\epsilon = -1$ and $\epsilon = 1$ as *focusing* and *defocusing* case, respectively. Moreover, the following quantities:

$$(0.3)\qquad E_1(t) = \int_{-\infty}^{+\infty} |u|^2 dx = E_1(0) \quad \textit{(\textbf{Mass})},$$

$$(0.4)\qquad E_2(t) = \int_{-\infty}^{+\infty} \Big\{ v^2 + 2\epsilon\,\mathrm{Im}(u\bar{u}_x)\Big\} dx = E_2(0) \quad \textit{(\textbf{Moment})},$$

$$(0.5)\qquad E_3(t) = \int_{-\infty}^{+\infty} \Big\{ |u_x|^2 + \epsilon|v_x|^2 + v|u|^2 + \epsilon p|u|^4 - \epsilon q v^3\Big\} dx = E_3(0) \quad \textit{(\textbf{Energy})}$$

are invariants by the flux of the system.

Many works concerning local and global well-posedness in the Hadamard sense, for the IVP (0.2) have been developed by several authors for initial data (u_0, v_0) in the classical Sobolev spaces $H^s \times H^\kappa$. For example, we refer the articles [1, 3, 5, 6, 8, 10, 11].

Focusing case. The global results established in [8, 11] were obtained in the *defocusing* situation ($\epsilon = 1$). Here we focus our attention on the *focusing case*($\epsilon = -1$), where it is not clear the existence of an a-priori estimate in the energy space $H^1 \times H^1$ since the terms $\displaystyle\int |u_x|^2dx$ and $\displaystyle\int |v_x|^2dx$ appear with oppositive signals in the energy conservation

law E_3. Specifically, we study the hyperbolic character of the energy E_3 in the focusing regime and we obtain the following *viriel type identity*:

Proposition 0.1 (*Viriel Identity*). *Let $(u(\cdot,t),v(\cdot,t))$ a solution in $C\big([0,T_0];H^{1+}\times H^{2+}\big)$ of the IVP (0.2) provided by the local theory developed in* [11] *and assume in addition that the initial data $(u_0,v_0)\in L^2(x^2dx)\times L^2(x^2dx)$. Then, for all $t\in[0,T_0]$, we have that*

$$\frac{d}{dt}\int x^2|u|^2dx = 4\mathcal{I}m\int x\bar{u}u_x dx, \tag{0.6}$$

$$f''(t) = 8\int |u_x|^2dx - 12\int |v_x|^2dx + 4\int |u|^2v + 8q\int v^3dx + 2p\int |u|^4dx, \tag{0.7}$$

where

$$f(t) := \int x^2|u|^2dx + 2\int_0^t\int xv^2dxdt'.$$

As a consequence of the Proposition 0.1 we obtain the following viriel type estimates for radially symmetric solutions:

Theorem 0.2. *Assume that $(u(\cdot,t),v(\cdot,t))$ is a radial solution in $C\big([0,T_{max});H^{1+}\times H^{2+}\big)$ of the IVP (0.2) provided by the local theory developed in* [11] *with $(u_0,v_0)\in L^2(x^2dx)\times L^2(x^2dx)$. Then for any $8<\delta<12$ there exists a positive function G_δ, only depending on $\|u_0\|_{L^2}$ and $|E_2(0)|$ such that*

$$\frac{d^2}{dt^2}\int x^2|u(x,t)|^2dx \leq \delta E_3(0) + G_\delta\big(\|u_0\|_{L^2},|E_2(0)|\big). \tag{0.8}$$

Then, using the Theorem 0.2 we give some sufficient conditions on the initial data for the existence of blow-up solutions.

Defocusing case. In this situation we also obtain a viriel identity for radially symmetric solutions, but in this regime we have global well-posedness in the energy space $H^1\times H^1$ for any data. Combine these facts we conclude the existence of radial initial data such that the corresponding solution does not preserve the original symmetry (breakdown of the symmetry).

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