Theory of metric Lie groups and H-surfaces in homogeneous 3-manifolds.

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Based on joint work with Mira, Pérez, Ros and Tinaglia.

Definition

A **2**-dimensional submanifold with constant mean curvature $\mathbf{H} \geq 0$ in a Riemannian **3**-manifold is called an **H-surface**.

Definition

If the isometry group of a Riemannian manifold **Y** acts transitively, then **Y** is called **homogeneous**.

Definition

A Lie group with left invariant metric is called a **metric Lie group**.

Notation and Language

- $\mathbf{Y}$ = simply connected homogeneous $\mathbf{3}$-manifold.
- $\mathbf{X}$ = simply connected $\mathbf{3}$-dimensional Lie group with left invariant metric ($\mathbf{X}$ is a **metric Lie group**).
- $\mathbf{H}(\mathbf{Y}) = \mathbf{Inf}\{\max|\mathbf{H}_\mathbf{M}| : \mathbf{M} = \text{immersed closed surface in } \mathbf{Y}\}$, where $\max|\mathbf{H}_\mathbf{M}|$ denotes max of absolute mean curvature function $\mathbf{H}_\mathbf{M}$.
- The number $\mathbf{H}(\mathbf{Y})$ is called the **critical mean curvature** of $\mathbf{Y}$.
- $\mathrm{Ch}(\mathbf{Y}) = \mathbf{Inf}_{\mathbf{K}\subset\mathbf{Y}\text{ compact}} \dfrac{\mathbf{Area}(\partial\mathbf{K})}{\mathbf{Volume}(\mathbf{K})}$ = Cheeger constant of $\mathbf{Y}$.

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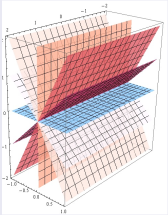
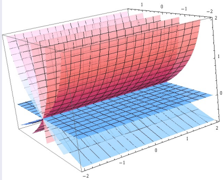
The unique **algebraic open book** decomposition of Nil_3, where all subgroups are minimal planes. Here the binding Γ is the x-axis.



Algebraic open book decomposition of Sol_3, where all subgroups are minimal and the only planar leaves are the (x, y) and (x, z)-planes. Here the binding Γ is the x-axis.



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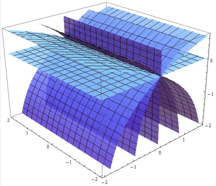
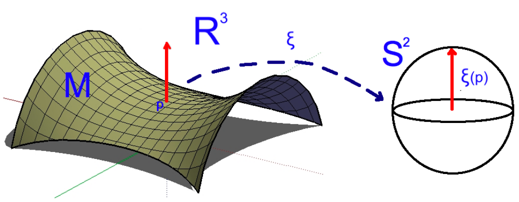


Definition (Left Invariant Gauss Map)

- Let $\mathbf{M} \subset \mathbf{X}$ be an oriented surface, $\mathbb{S}^2$ be the unit sphere in $\mathbf{T}_e\mathbf{X}$ and $N(\mathbf{p})$ be the unit normal to $\mathbf{M}$ at $\mathbf{p}$.
- The **left invariant Gauss map** of $\mathbf{M}$ is the mapping $\mathbf{G}\colon \mathbf{M} \to \mathbb{S}^2$, where $\mathbf{G}(\mathbf{p})$ is the left translation of $N(\mathbf{p})$ from $\mathbf{T}_\mathbf{p}\mathbf{X}$ to $\mathbf{T}_e\mathbf{X}$.



Here $\mathbf{M} \subset \mathbf{X}$ is an oriented surface with unit normal vector field $N = \xi$. Then: $\mathbf{G}\colon \mathbf{M} \to \mathbb{S}^2 \subset \mathbf{T}_e\mathbf{X}$ is defined by $\mathbf{G}(\mathbf{p}) = \mathbf{p}^{-1}N(\mathbf{p})$.

Lemma (Transversality Lemma, Meeks-Mira-Pérez-Ros)

Suppose that $\mathbf{S}$ is an immersed sphere in an $\mathbf{X}$ and $\mathbf{F}$ is a $\mathbf{2}$-dimensional subgroup. If the left invariant Gauss map $\mathbf{G}\colon \mathbf{S} \to \mathbb{S}^2$ is a diffeomorphism, then:

- The set of left cosets of $\mathbf{F}$ which intersect $\mathbf{S}$ can be parameterized by the interval $[0, 1]$, i.e., $\{g(t)\mathbf{F} \mid t \in [0, 1]\}$ are these cosets.
- Each of the cosets $g(0)\mathbf{F}$ and $g(1)\mathbf{F}$ intersects $\mathbf{S}$ in a single point.
- For $t \in (0, 1)$, $\mathbf{S} \cap g(t)\mathbf{F}$ is a connected immersed closed curve.

Theorem (Meeks-Mira-Pérez-Ros)

Suppose that $\mathbf{X}$ has an algebraic open book decomposition. If $\mathbf{f}\colon \mathbf{S} \to \mathbf{X}$ is an immersed sphere in whose left invariant Gauss map $\mathbf{G}\colon \mathbf{S} \to \mathbb{S}^2$ is a diffeomorphism, then $\mathbf{S}$ is embedded.

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Proof when $\mathbf{X}$ has $\mathbf{2}$ algebraic open book decomposition.

- By the previous theorem, $\mathbf{X} = \mathbb{R}^2 \rtimes_{\mathbf{A}} \mathbb{R}$ for some diagonal matrix $\mathbf{A}$.

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Proof when X has 2 algebraic open book decomposition.

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- The new Gauss map in this new metric, which we denote the same way, is still a diffeomorphism.

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- The new Gauss map in this new metric, which we denote the same way, is still a diffeomorphism.
- Consider the algebraic open book decompositions $\mathcal{B}_1 = \{\mathbf{J}_1(\theta) \mid \theta \in \mathbb{S}^1\}$ and $\mathcal{B}_2 = \{\mathbf{J}_2(\theta) \mid \theta \in \mathbb{S}^1\}$ of $\mathbb{R}^2 \rtimes_{\mathbf{A}} \mathbb{R}$ with bindings $\mathbf{\Gamma}_1 = \mathbf{x}$-axis and $\mathbf{\Gamma}_2 = \mathbf{y}$-axis, respectively.

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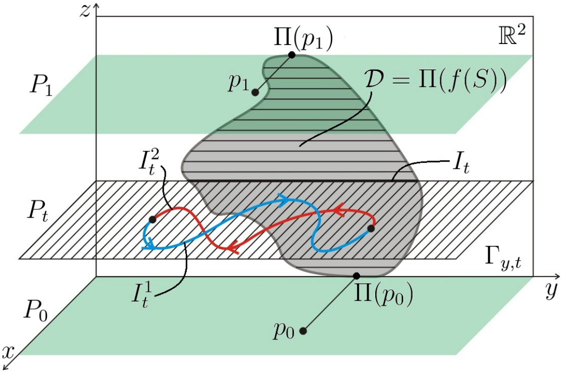
- The theorem follow easily from the next assertion, since we can express $\mathbf{S}$ as the union of $\mathbf{2}$ "disjoint" disks which are graphs over the $\mathbf{yz}$-plane; see the argument in the next slide.

Assertion

For $i = 1, 2$ and for each $a \in \mathbf{X}$, $\mathbf{f}^{-1}(a\mathbf{\Gamma}_i)$ contains at most $\mathbf{2}$ points.



- Suppose the left coset P_t intersects transversely in the immersed curve $\mathbf{f}(\mathbf{S}) \cap P_t$.
- For $i = 1, 2$ and for each $a \in \mathbf{X}$, $\mathbf{f}^{-1}(a\boldsymbol{\Gamma}_i)$ contains at most **2** points.
- For $i = 1$, the immersed curve $\mathbf{f}(\mathbf{S}) \cap P_t$ consists of two Jordan arcs $\mathbf{I}_t^1, \mathbf{I}_t^2$ with the same extrema.
- Hence, projection along $\boldsymbol{\Gamma}_1$ represents $\boldsymbol{\Sigma}$ as a bigraph over the **yz**-plane.
- Also, projection along $\boldsymbol{\Gamma}_2$ represents $\boldsymbol{\Sigma}$ as a bigraph over the **xz**-plane.
- So, $\boldsymbol{\Sigma}$ is embedded!!

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- After left translating $\mathbf{S}$ by a^{-1}, assume that $a\boldsymbol{\Gamma} = \boldsymbol{\Gamma}$.
- Since each subgroup of $\mathcal{B}_1$ intersects $\mathbf{S}$ in at least $\mathbf{4}$ points, the Transversality Lemma implies each leaf L_θ intersects $\mathbf{S}$ transversely.

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- The normal variational vector field ∂_θ for $\mathcal{B}_1$ defined on $\mathbf{X} - \mathbf{\Gamma}$ induces a nonzero tangential vector field $\partial_\theta^{\mathbf{T}}$ on $\mathbf{S} - \mathbf{f}^{-1}(\mathbf{\Gamma})$.

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- By the Hopf index theorem, the Euler characteristic of $\mathbf{S}$ is $2n \geq 4$.
- This is a contradiction since the Euler characteristic of $\mathbf{S}$ is **2**. □

Theorem (Meeks-Mira-Pérez-Ros)

Let **X** be a **3**-dimensional metric Lie group. Then:

1. **X** admits and open book decomposition **iff** it is isomorphic to $\mathbf{R}^3$, $\mathbb{H}^3$, $\mathbf{Nil_3}$, $\mathbf{Sol_3}$, or to a non-unimodular Lie group with **D**-invariant $\mathbf{D} \leq 1$.
2. If **X** admits an algebraic open book decomposition and $\mathbf{f}\colon \mathbf{S} \to \mathbf{X}$ is an immersed sphere whose left invariant Gauss map of is a diffeomorphism, then **f** is injective.
3. The left invariant Gauss map of **H**-spheres in **X** are diffeomorphisms.
4. **H**-spheres in homogeneous **3**-manifolds are always Alexandrov embedded (bound submersed **3**-balls) except for the minimal spheres $\mathbb{S}^2(\kappa) \times \{t\}$ in $\mathbb{S}^2(\kappa) \times \mathbb{R}$.

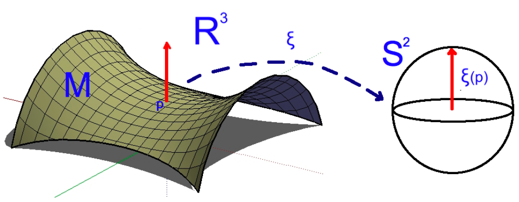
Remark

The fact that the Gauss map of an **H**-sphere in **X** is a diffeomorphism in item 4 in the above theorem depends on proving that **H**-spheres always have index **1**.

Conjecture (Alexandrov Embeddedness Conjecture, Meeks-Mira-Pérez-Ros)

- If $\mathbf{Y}$ be a simply connected, homogeneous $\mathbf{3}$-manifold not diffeomorphic to $\mathbb{S}^3$, then $\mathbf{H}$-spheres in $\mathbf{Y}$ are embedded.
- Furthermore, if $\mathbf{Y}$ is diffeomorphic to $\mathbf{R}^3$, then compact Alexandrov embedded $\mathbf{H}$-surfaces in $\mathbf{Y}$ are spheres.

Definition (Second Fundamental Form/Shape Operator)



Let **M** be an oriented surface in $\mathbf{R}^3$, let ξ be the unit vector field normal to **M**:

$$\mathbf{A} = -d\xi \colon T_\mathbf{p}\mathbf{M} \to T_{\xi(\mathbf{p})}\mathbf{S}^2 \simeq T_\mathbf{p}\mathbf{M}$$

is the **shape operator** of **M**.

Definition

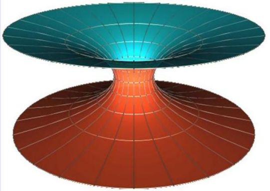
- The eigenvalues k_1, k_2 of $\mathbf{A}_p$ are the **principal curvatures** of $\mathbf{M}$ at p.
- $\mathbf{K}_G = \det(\mathbf{A}) = k_1 k_2$ is the **Gauss curvature**.
- $\mathbf{H} = \frac{1}{2}\mathbf{Trace}(\mathbf{A}) = \frac{k_1+k_2}{2}$ is the **mean curvature**.
- $\mathbf{K}_\mathbf{M} = k_1 k_2$ is the **Gaussian curvature**.
- $|\mathbf{A}| = \sqrt{k_1^2 + k_2^2}$ is the **norm of the shape operator.**

Gauss equation

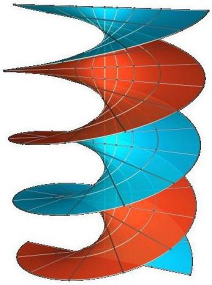
$$4\mathbf{H}^2 = |\mathbf{A}|^2 + 2\mathbf{K}_\mathbf{G} \quad (\mathbf{K}_\mathbf{G} = \text{Gaussian curvature})$$

Theorem

An **H**-surface **M** is a **minimal surface** $\iff$ **H** $\equiv 0$ $\iff$ **M** is a critical point for the area functional under compactly supported variations.



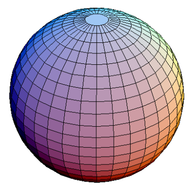
- Catenoid



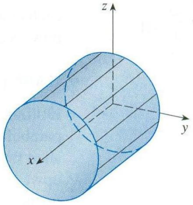
- Helicoid

Theorem

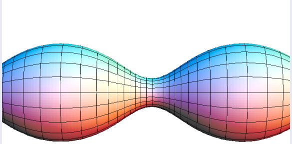
M is a **H-surface** $\Longleftrightarrow$ **M** is a critical point for the area functional under compactly supported variations **preserving the volume**.



- Sphere



- Cylinder



- Delaunay surfaces

Uniqueness, existence, index and embeddedness of H-spheres in $\mathbf{R}^3$.

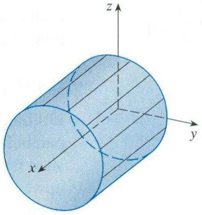
Let **M** be a closed (compact without boundary) **H**-surface in $\mathbf{R}^3$:

- **M** is **not** minimal. (coordinate functions cannot be harmonic)
- **M** has **genus 0** $\iff$ it is a round sphere (1951, **Hopf**).
- **M** is **embedded** $\iff$ it is a round sphere (1956, **Alexandrov**).
- **Index**(**M**) $= 1 \iff$ **M** is a round sphere (1984, **Barbosa-do Carmo**).

Question

Is the round sphere the only complete simply connected surface **embedded** in $\mathbb{R}^3$ with **non-zero** constant mean curvature?

NOT simply connected



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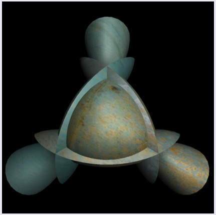

- Smyth surface

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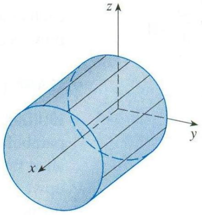

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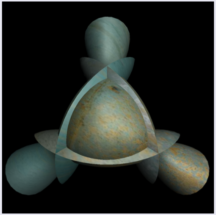

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Answer (Meeks-Tinaglia)

Yes!

New uniqueness results for CMC surfaces.

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Round spheres are the only complete simply connected surfaces **embedded** in $\mathbf{R}^3$ with non-zero constant mean curvature.

1997, **Meeks** for **properly embedded**.

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Corollary

Let $\mathbf{M}$ be a complete simply connected $\mathbf{H}$-surface **embedded** in $\mathbf{R}^3$. Then $\mathbf{M}$ is either

a plane, **a helicoid** **or** **a round sphere**.

(2008, **Colding-Minicozzi** and **Meeks-Rosenberg** for $\mathbf{H} = 0$**)**

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Theorem (Radius Estimate, Meeks-Tinaglia)

There exists a universal constant **R** such that if **M** is a **1**-disk, then **M** has **radius** less than **R**, i.e.,

$$\forall \mathbf{p} \in \mathbf{M}, \quad \mathrm{dist}_{\mathbf{M}}(\mathbf{p}, \partial \mathbf{M}) < \mathbf{R}.$$

Definition

A $\mathbf{1}$-disk is a compact disk **embedded** in $\mathbf{R}^3$ with constant mean curvature $\mathbf{1}$.

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In particular, if $\mathbf{M}$ is a complete simply connected $\mathbf{H} = \mathbf{1}$ surface embedded in $\mathbf{R}^3$, then:
Radius Estimate $\Longrightarrow$ $\mathbf{M}$ is compact $\Longrightarrow$ $\mathbf{M}$ is an embedded sphere.

Definition

A $\mathbf{1}$-disk is a compact disk **embedded** in $\mathbf{R}^3$ with constant mean curvature $\mathbf{1}$.

Theorem (Radius Estimate, Meeks-Tinaglia)

There exists a universal constant $\mathbf{R}$ such that if $\mathbf{M}$ is a $\mathbf{1}$-disk, then $\mathbf{M}$ has **radius** less than $\mathbf{R}$, i.e.,

$$\forall \mathbf{p} \in \mathbf{M}, \quad \mathrm{dist}_{\mathbf{M}}(\mathbf{p}, \partial \mathbf{M}) < \mathbf{R}.$$

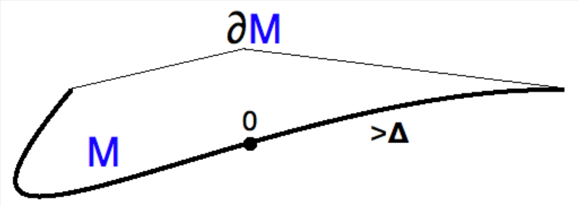
In particular, if $\mathbf{M}$ is a complete simply connected $\mathbf{H} = \mathbf{1}$ surface embedded in $\mathbf{R}^3$, then:
Radius Estimate $\implies$ $\mathbf{M}$ is compact $\implies$ $\mathbf{M}$ is an embedded sphere.
Hopf or Alexandrov Theorems $\implies$ $\mathbf{M}$ is a **round sphere.**

The Radius Estimate is a **trivial** consequence of the following Intrinsic Curvature Estimate.

Theorem (Intrinsic Curvature Estimate, Meeks-Tinaglia)

- Let $\mathbf{Y}$ be a homogenously regular $\mathbf{3}$-manifold with absolute sectional curvature less than $\mathbf{K} \geq 0$.
- Given $\mathbf{\Delta} > 0$ there exist a universal constant $\mathbf{C} = \mathbf{C}(\mathbf{\Delta}, \mathbf{K})$ such that:
- If $\mathbf{M}$ is an $\mathbf{H}$-disk in $\mathbf{Y}$ with $\mathbf{0} \in \mathbf{M}$, $\mathrm{dist}_{\mathbf{M}}(\mathbf{0}, \partial \mathbf{M}) > \mathbf{\Delta}$ and $\mathbf{H} \geq 1$, then

$$|\mathbf{A_M}|(\mathbf{0}) \leq \mathbf{C}.$$



Theorem (Radius/Curvature Estimate, Meeks-Tinaglia)

- Suppose that $\mathbf{X} = \mathbb{H}^3$.
- There exists a universal constant $\mathbf{R}$ such that if $\mathbf{M}$ is an $\mathbf{H}$-disk and $\mathbf{H} > \mathbf{H}(\mathbf{X})$, then $\mathbf{M}$ has **radius** less than $\mathbf{R}$, i.e.,

$$\forall \mathbf{p} \in \mathbf{M}, \quad \mathrm{dist}_{\mathbf{M}}(\mathbf{p}, \partial \mathbf{M}) < \mathbf{R}.$$

- Complete embedded $\mathbf{H}$-surfaces of finite topology in $\mathbf{X}$ with $\mathbf{H} > 0$ have bounded second fundamental forms and are properly embedded when $\mathbf{H} > 1$.

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Conjecture (Radius/Curvature Estimates, Meeks-Tinaglia)

- There exists a universal constant $\mathbf{R}$ such that if $\mathbf{M}$ is an embedded $\mathbf{H}$-disk with $\mathbf{H} > \mathbf{H}(\mathbf{Y})$, then $\mathbf{M}$ has **radius** less than $\mathbf{R}$, i.e.,

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- Complete embedded $\mathbf{H}$-surfaces in $\mathbf{Y}$ with $\mathbf{H} > 0$ have bounded second fundamental forms.
- Every complete embedded $\mathbf{H}$-surface of finite topology in $\mathbf{Y}$ is properly embedded, whenever $\mathbf{H} \geq \mathbf{H}(\mathbf{Y})$.

Theorem (Classification Theorem for $\mathbf{H}$-spheres, Meeks-Mira-Pérez-Ros)

Suppose $\mathbf{X}$ is a simply connected $\mathbf{3}$-dimensional metric Lie group.

- $\mathbf{X}$ is diffeomorphic to $\mathbf{R^3} \implies$ the moduli space of $\mathbf{H}$-spheres in $\mathbf{X}$ is parameterized by the mean curvature values $\mathbf{H}$ in $(\mathbf{H}(\mathbf{X}), \infty)$.
- $\mathbf{X}$ is diffeomorphic to $\mathbb{S}^3 \implies$ the moduli space of $\mathbf{H}$-spheres in $\mathbf{X}$ is parameterized by the mean curvature values $\mathbf{H}$ in $[0, \infty)$.
- $\mathbf{X}$ diffeomorphic to $\mathbb{S}^3 \implies$ the areas of all $\mathbf{H}$-spheres form a half-open interval $(0, \mathbf{A}(\mathbf{X})]$.
- $\mathbf{H}$-spheres in $\mathbf{X}$ are **Alexandrov embedded** with **index 1**, **nullity 3**.

Theorem (Classification Theorem for H-spheres, Meeks-Mira-Pérez-Ros)

Suppose X is a simply connected 3-dimensional metric Lie group.

- X is diffeomorphic to $\mathbf{R}^3 \implies$ the moduli space of H-spheres in X is parameterized by the mean curvature values H in $(H(X), \infty)$.
- X is diffeomorphic to $\mathbb{S}^3 \implies$ the moduli space of H-spheres in X is parameterized by the mean curvature values H in $[0, \infty)$.
- X diffeomorphic to $\mathbb{S}^3 \implies$ the areas of all H-spheres form a half-open interval $(0, A(X)]$.
- H-spheres in X are **Alexandrov embedded** with **index 1**, **nullity 3**.

- It follows that an index-1 H-sphere S inherits all of the symmetries of X fixing a certain unique point in X called its **center of mass**. So, S is rotational when X has a 4 or 6-dimensional isometry group.

Theorem (Classification Theorem for H-spheres, Meeks-Mira-Pérez-Ros)

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- Our approach of some steps in the proof of this theorem appear in the proof by **Daniel-Mira** and **Meeks** on the uniqueness of **H**-spheres in the metric Lie group $\mathbf{Sol_3}$ with its most symmetric left invariant metric.

Steps of the proof of the Classification Theorem for **H**-spheres.

Throughout **Σ** denotes a fixed **H_0**-sphere in **X** of **index 1**.

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- **Step 2:** The left invariant Gauss map $\mathbf{\Gamma}\colon \mathbf{\Sigma} \to \mathbb{S}^2 \subset \mathbf{T}_e(\mathbf{X})$ is a diffeomorphism: **Nodal Domain Argument.**

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- **Step 3:** Curvature estimates for $\boldsymbol{\Sigma}$ (given any fixed upper bound $\mathbf{H}_1$ of $\mathbf{H}_0$): **Rescaling Argument** **or** **Use that Gauss map is a diffeo**.

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  - **(A)** If $\mathbf{X}$ is isomorphic to $\mathbf{SU}(\mathbf{2})$, areas of spheres in $\mathcal{M}(\mathbf{X})$ are uniformly bounded.

Steps of the proof of the Classification Theorem for $\mathbf{H}$-spheres.

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- **Step 1:** The moduli space $\mathcal{M}(\mathbf{X})$ of non-congruent index-$\mathbf{1}$ $\mathbf{H}$-spheres in $\mathbf{X}$ is an analytic $\mathbf{1}$-manifold locally parameterized by its mean curvature values: **Implicit Function Theorem**.
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  - **(A)** If $\mathbf{X}$ is isomorphic to $\mathbf{SU}(2)$, areas of spheres in $\mathcal{M}(\mathbf{X})$ are uniformly bounded.
  - **(B)** If $\mathbf{X}$ is **not** isomorphic to $\mathbf{SU}(2)$, then for any $\boldsymbol{\Delta} > 0$ the areas of spheres in $\mathcal{M}(\mathbf{X})$ with $\mathbf{H}_0 \in [\mathbf{H}(\mathbf{X}) + \boldsymbol{\Delta}, \infty)$ are uniformly bounded.

- **Step 5:** Each component of $\mathcal{M}(\mathbf{X})$ is an interval parameterized by the mean curvature values in a subinterval $\mathbf{I_X} \subset [0, \infty)$. $\mathbf{I_X} = [0, \infty)$ if $\mathbf{X}$ is isomorphic to $\mathbf{SU(2)}$ and otherwise $\mathbf{I_X} = (\mathbf{H}(\mathbf{X}), \infty)$.

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- **Step 6:** On any $\mathbf{H}_0$-sphere $\mathbf{M}$ different from a left translation of $\boldsymbol{\Sigma}$, $\exists$ a **NON-ZERO** complex valued quadratic differential $\omega_{\boldsymbol{\Sigma}}(\mathbf{M})$ with isolated negative index zeroes.

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- **Conclusion:**
  - The space of non-congruent $\mathbf{H}$-spheres in $\mathbf{X}$ equals $\mathcal{M}(\mathbf{X})$ which is an interval parameterized by the mean curvature values in $[0, \infty)$ if $\mathbf{X}$ is isomorphic to $\mathbf{SU}(2)$ and otherwise, in the interval $(\mathbf{H}(\mathbf{X}), \infty)$.

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Steps of the proof continued.

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  - Each $\mathbf{H}$-sphere in $\mathbf{X}$ has **index** $\mathbf{1}$ and **nullity** $\mathbf{3}$.
  - Each $\mathbf{H}$-sphere in $\mathbf{X}$ is the boundary of an immersed $\mathbf{3}$-ball $\mathbf{F}\colon \mathbf{B} \to \mathbf{X}$ (Alexandrov embedded).

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  - If $\mathbf{X}$ is isomorphic to $\mathbf{SU}(\mathbf{2})$, then the areas of $\mathbf{H}$-spheres in $\mathbf{X}$ form a half-open interval $(0, \mathbf{A}(\mathbf{X})]$. $\square$

Conjecture (Index 1 Conjecture, Meeks-Mira-Pérez-Ros)

When $\mathbf{X}$ is diffeomorphic to $\mathbf{R}^{\mathbf{3}}$, then every compact $\mathbf{H}$-surface in $\mathbf{X}$ with **index** $\mathbf{1}$ is a sphere. In particular, solutions to the isoperimetric problem in such an $\mathbf{X}$ are spheres.

Conjecture (Isoperimetric Domains Conjecture, Meeks-Mira-Pérez-Ros)

If $\mathbf{X}$ is diffeomorphic to $\mathbb{R}^3$, then:

1. Isoperimetric surfaces (boundaries of isoperimetric domains) in $\mathbf{X}$ are spheres.
2. For each fixed volume $\mathbf{V}_0$, solutions to the isoperimetric problem in $\mathbf{X}$ for volume $\mathbf{V}_0$ are unique up to left translations in $\mathbf{X}$.

Conjecture (Topological Existence Conjecture, Meeks)

- If $\mathbf{X} \approx \mathbf{R}^3$ and $\mathbf{H} < \mathbf{H}(\mathbf{X})$, then $\mathbf{X}$ admits properly embedded $\mathbf{H}$-surfaces of every possible topology.
- If $\mathbf{X} \approx \mathbf{R}^3$ and $\mathbf{H} > \mathbf{H}(\mathbf{X})$, then $\mathbf{X}$ admits properly embedded $\mathbf{H}$-surfaces of every possible topology, except for finite genus with $\mathbf{1}$ end or finite positive genus with $\mathbf{2}$ ends which it never admits.

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Conjecture (Topological Uniqueness Conjecture, Meeks)

If $\mathbf{M}_1$, $\mathbf{M}_2$ are two diffeomorphic, connected and properly embedded $\mathbf{H}$-surfaces of finite topology in $\mathbf{X}$ with $\mathbf{H} = \mathbf{H}(\mathbf{X})$, then there exists a diffeomorphism $\mathbf{f}\colon \mathbf{X} \to \mathbf{X}$ with $\mathbf{f}(\mathbf{M}_1) = \mathbf{M}_2$.

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Meeks-Frohman

If $\mathbf{M}_1$, $\mathbf{M}_2$ are two diffeomorphic, connected and properly embedded minimal surfaces in $\mathbf{Y} = \mathbb{S}(\kappa) \times \mathbb{R}$, then there exists a diffeomorphism $\mathbf{f} \colon \mathbf{Y} \to \mathbf{Y}$ with $\mathbf{f}(\mathbf{M}_1) = \mathbf{M}_2$.

Theorem (Unknottedness Theorem for Minimal Surfaces in $\mathbf{SU(2)}$)

Let $\mathbf{X}$ be isomorphic to $\mathbf{SU}(\mathbf{2})$ and let $\mathbf{M}_1$, $\mathbf{M}_2$ be two compact, diffeomorphic embedded minimal surfaces in $\mathbf{X}$. Then there exists a diffeomorphism $\mathbf{f}\colon \mathbf{X} \to \mathbf{X}$ with $\mathbf{f}(\mathbf{M}_1) = \mathbf{M}_2$.

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Proof.

- By Waldhausen, it suffices to prove that the closure of each complement of $\mathbf{X} - \mathbf{M}_i$ is a handlebody.

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Proof.

- By Waldhausen, it suffices to prove that the closure of each complement of $\mathbf{X} - \mathbf{M}_i$ is a handlebody.
- Otherwise by **Meeks-Simon-Yau**, $\mathbf{X}$ admits an embedded, compact, strongly stable minimal surface, which contradicts the above theorem. $\square$

Theorem (Positive Injectivity Radius of Classical Minimal Surfaces, Meeks-Pérez)

- A complete embedded minimal surface $\mathbf{M}$ of finite topology in a homogeneous $\mathbf{3}$-manifold $\mathbf{Y}$ has positive injectivity radius.
- More generally, a complete embedded $\mathbf{H}$-surface $\mathbf{M}$ of finite topology in a homogeneous $\mathbf{3}$-manifold $\mathbf{Y}$ has positive injectivity radius, when $\mathbf{H} < \mathbf{H}(\mathbf{Y})$.

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Remark

- **Meeks-Tinaglia** proved that a complete embedded $\mathbf{H}$-surface of finite topology in a homogeneously regular $\mathbf{3}$-manifold of non-positive curvature has bounded second fundamental form when $\mathbf{H} > 0$, and hence, it positive injectivity radius.
- Their result in the non-positive sectional curvature setting was motivated by the earlier proof by **Meeks-Rosenberg** that a complete embedded minimal surface of finite topology in a homogeneously regular $\mathbf{3}$-manifold of non-positive curvature has positive injectivity radius.

Conjecture (Positive Injectivity Radius/Bounded Curvature Conjecture, Meeks-Pérez-Tinaglia)

- A complete embedded $\mathbf{H}$-surface of finite topology in $\mathbf{X}$ with $\mathbf{H} \geq \mathbf{H}(\mathbf{X})$ has positive injectivity radius and bounded curvature.
- A complete embedded $\mathbf{H}$-surface of finite genus surface in $\mathbf{X}$ with $\mathbf{H} = \mathbf{H}(\mathbf{X})$ has positive injectivity radius and bounded curvature.

Conjecture (Positive Injectivity Radius/Bounded Curvature Conjecture, Meeks-Pérez-Tinaglia)

- A complete embedded $\mathbf{H}$-surface of finite topology in $\mathbf{X}$ with $\mathbf{H} \geq \mathbf{H}(\mathbf{X})$ has positive injectivity radius and bounded curvature.
- A complete embedded $\mathbf{H}$-surface of finite genus surface in $\mathbf{X}$ with $\mathbf{H} = \mathbf{H}(\mathbf{X})$ has positive injectivity radius and bounded curvature.

Remark

Curvature estimates of **Meeks-Tinaglia** for embedded ($\mathbf{H} > 0$)-disks imply that any complete embedded ($\mathbf{H} > 0$)-surface of positive injectivity radius in a homogeneously regular $\mathbf{3}$-manifold has bounded second fundamental form.

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Conjecture (Bounded Curvature Conjecture, Meeks-Tinaglia)

- The norm of the second fundamental of any complete embedded finite topology ($\mathbf{H} > 0$)-surface in $\mathbf{X}$ is bounded.
- More generally, the same result holds for the case for any complete $\mathbf{3}$-manifold locally isometric to $\mathbf{X}$.

Conjecture (Calabi-Yau Properness Problem, Meeks-Pérez)

- A complete, connected, embedded $\mathbf{H}$-surface of positive injectivity radius is proper in $\mathbf{X}$ whenever $\mathbf{H} \geq \mathbf{H}(\mathbf{X})$.
- In particular, by a result of **Meeks-Pérez**, complete, embedded minimal surfaces of finite topology in metric Lie groups isomorphic to $\mathbf{R}^3$, $\mathbf{Nil}_3$, $\widetilde{\mathbf{E}}(\mathbf{2})$, $\mathbf{Sol}_3$ would always be proper.

Conjecture (Stability Conjecture for $\mathrm{SU}(2)$, Meeks-Pérez-Ros)

If **X** is diffeomorphic to $\mathbb{S}^3$, then **X** contains no strongly stable complete **H**-surfaces.

Meeks-Pérez-Ros

- Conjecture is true if **X** is a Berger sphere with non-negative scalar curvature.
- Also true if **X** is **SU**(**2**) endowed with a left invariant metric of positive scalar curvature.
- If **Y** is a **3**-sphere with a Riemannian metric (not necessarily a left invariant metric) such that it admits no strongly stable complete minimal surfaces, then for each integer $g \in \mathbb{N} \cup \{0\}$, the space of compact embedded minimal surfaces of genus g in **Y** is compact.