Theory of metric Lie groups and H-surfaces in homogeneous 3-manifolds.

William H. Meeks III
University of Massachusetts at Amherst
Based on joint work with Mira, Pérez, Ros and Tinaglia.

Definition

A **2**-dimensional submanifold with constant mean curvature $\mathbf{H} \geq 0$ in a Riemannian **3**-manifold is called an **H-surface**.

Definition

If the isometry group of a Riemannian manifold **Y** acts transitively, then **Y** is called **homogeneous**.

Definition

A Lie group with left invariant metric is called a **metric Lie group**.

Notation and Language

- $\mathbf{Y}$ = simply connected homogeneous $\mathbf{3}$-manifold.
- $\mathbf{X}$ = simply connected $\mathbf{3}$-dimensional Lie group with left invariant metric ($\mathbf{X}$ is a **metric Lie group**).
- $\mathbf{H}(\mathbf{Y}) = \mathbf{Inf}\{\max|\mathbf{H_M}| : \mathbf{M} = \text{immersed closed surface in } \mathbf{Y}\}$, where $\max|\mathbf{H_M}|$ denotes max of absolute mean curvature function $\mathbf{H_M}$.
- The number $\mathbf{H}(\mathbf{Y})$ is called the **critical mean curvature** of $\mathbf{Y}$.
- $\mathrm{Ch}(\mathbf{Y}) = \mathbf{Inf}_{\mathbf{K}\subset\mathbf{Y} \text{ compact}} \dfrac{\mathbf{Area}(\partial\mathbf{K})}{\mathbf{Volume}(\mathbf{K})}$ = Cheeger constant of $\mathbf{Y}$.

Goals of Lecture 3

- Classification of algebraic open book decompositions.
- The left invariant Lie group Gauss map and the embeddedness problem for $\mathbf{H}$-spheres.
- The **Meeks-Tinaglia** curvature and radius estimates for $\mathbf{H}$-disks.
- Existence and uniqueness of $\mathbf{H}$-spheres in $\mathbf{X}$ **(Meeks-Mira-Pérez-Ros)**.
- Discuss various conjectures on embedded $\mathbf{H}$-surfaces in $\mathbf{Y}$.

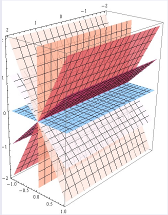
Definition

- Let $\mathbf{\Gamma}$ be a 1-parameter subgroup of $\mathbf{X}$.
- We will call a foliation $\mathcal{B} = \{L(\theta) \mid \theta \in [0, 2\pi)\}$ of $\mathbf{X} - \mathbf{\Gamma}$ an **algebraic open book decomposition of $\mathbf{X}$** if the set $H(\theta) = L(\theta) \cup \mathbf{\Gamma} \cup L(\pi + \theta)$ is a subgroup for each $\theta \in \mathbb{R}$.
- We call $L(\theta)$ a **leaf**, $\mathbf{\Gamma}$ the **binding** and $H(\theta)$ a **subgroup** of the algebraic open book decomposition $\mathcal{B}$.

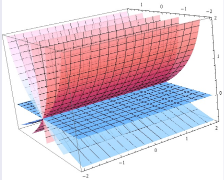
Example

- Let $\mathbf{X} = \mathbf{R}^{\mathbf{3}}$ and $\mathbf{\Gamma}$ be the $\mathbf{z}$-axis.
- Then the foliation $\mathcal{B}$ of $\mathbf{R}^{\mathbf{3}} - \mathbf{\Gamma}$ by half planes with boundaries $\mathbf{\Gamma}$ is an algebraic open book decomposition of $\mathbf{R}^{\mathbf{3}}$.

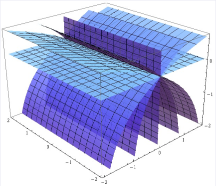
The unique **algebraic open book** decomposition of Nil_3, where all subgroups are minimal planes. Here the binding Γ is the x-axis.



Algebraic open book decomposition of Sol_3, where all subgroups are minimal and the only planar leaves are the (x, y) and (x, z)-planes. Here the binding Γ is the x-axis.



Algebraic open book decomposition of Sol_3, where all subgroups are minimal and the only planar leaves are the (x, y) and (y, z)-planes. Here the binding Γ is the y-axis.



Examples of algebraic open book decomposition of a semidirect product.

- Let $\mathbf{G} = \mathbb{R}^2 \rtimes_{\mathbf{A}} \mathbb{R}$, where $\mathbf{A} = \begin{pmatrix} 1 & 0 \\ 0 & b \end{pmatrix}$.
- Note that when $b = 0$, $\mathbb{R}^2 \rtimes_{\mathbf{A}} \mathbb{R}$ is isometric to $\mathbb{H} \times \mathbb{R}$.
- Let E_1, E_2, E_3 be the usual basis for $\mathbf{L}(\mathbf{G})$ where:

$$[E_1, E_2] = 0, \quad [E_3, E_1] = E_1, \quad [E_3, E_2] = bE_2.$$

- Special normal $\mathbb{R}^2$-subgroup $\mathbf{J}_1(\infty) = \mathbf{J}_2(\infty) = \mathbb{R}^2 \rtimes_{\mathbf{A}} \{0\}$ corresponds to the distribution $\{aE_1 + cE_2 \mid a, c \in \mathbb{R}\}$.
- $\mathbb{H}$-**subgroups,** $b \neq 0$. $\mathbf{J}_1(\lambda) = \{(x, \lambda(e^z - 1), z) \mid x, z \in \mathbb{R}\}$ correspond to distributions $\mathbf{\Delta_d} = \{aE_1 + c(E_3 + \mathbf{d}E_2) \mid a, c \in \mathbb{R}\}$.
- $\mathbb{H}$-**subgroups,** $b \neq 0$. $\mathbf{J}_2(\lambda) = \{(\lambda(e^z - 1), y, z) \mid x, z \in \mathbb{R}\}$ correspond to distributions $\mathbf{\Delta_d} = \{aE_2 + c(E_3 + \mathbf{d}E_2) \mid a, c \in \mathbb{R}\}$.
- $\mathbb{H}$-**subgroups,** $b = 0$. $\mathbf{J}_1(\lambda) = \{(x, \lambda z, z) \mid x, z \in \mathbb{R}\}$ correspond to distributions $\mathbf{\Delta_d} = \{aE_1 + c(E_3 + \mathbf{d}E_2) \mid a, c \in \mathbb{R}\}$.
- $\mathbb{R}^2$-**subgroups,** $b = 0$. $\mathbf{J}_2(\lambda) = \{(\lambda(e^z - 1), y, z) \mid y, z \in \mathbb{R}\}$ correspond to distributions $\mathbf{\Delta_d} = \{aE_2 + c(E_3 + \mathbf{d}E_1) \mid a, c \in \mathbb{R}\}$.

Example

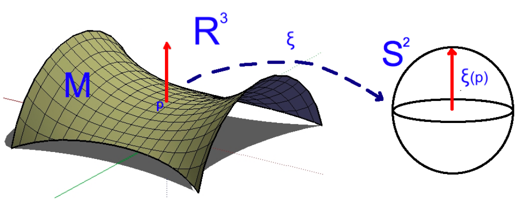
- The collection $\mathcal{B}_1 = \{\mathbf{J}_1(\theta) \mid \theta \in \mathbb{S}^1 = \mathbb{R} \cup \{\infty\}\}$ consists of the subgroups of an algebraic open book decomposition of $\mathbb{R}^2 \rtimes_{\mathbf{A}} \mathbb{R}$ with binding the **x**-axis.
- The collection $\mathcal{B}_2 = \{\mathbf{J}_2(\theta) \mid \theta \in \mathbb{S}^1 = \mathbb{R} \cup \{\infty\}\}$ consists of the subgroups of an algebraic open book decomposition of $\mathbb{R}^2 \rtimes_{\mathbf{A}} \mathbb{R}$ with binding the **y**-axis.

Theorem (Meeks-Mira-Pérez-Ros)

- A necessary and sufficient condition for a simply connected Lie group **G** to admit **2** different open book decompositions is that it can be expressed as a semidirect product $\mathbb{R}^2 \rtimes_{\mathbf{A}} \mathbb{R}$, where **A** is a diagonal matrix.
- These are the non-unimodular groups with invariant $\mathbf{D}(\mathbf{G}) < 1$, $\mathbb{H}^3$ and the unimodular group $\mathbf{Sol_3}$.
- $\mathbf{Nil_3}$ and $\mathbf{G} \neq \mathbb{H}^3$ with $\mathbf{D}(\mathbf{G}) = 1$ each admit **1** such algebraic open book decomposition.
- The algebraic open books in the previous **3** items together with those given by planes in $\mathbf{R^3}$ are the only ones.

Definition (Left Invariant Gauss Map)

- Let $\mathbf{M} \subset \mathbf{X}$ be an oriented surface, $\mathbb{S}^2$ be the unit sphere in $T_e\mathbf{X}$ and $N(\mathbf{p})$ be the unit normal to $\mathbf{M}$ at $\mathbf{p}$.
- The **left invariant Gauss map** of $\mathbf{M}$ is the mapping $\mathbf{G}\colon \mathbf{M} \to \mathbb{S}^2$, where $\mathbf{G}(\mathbf{p})$ is the left translation of $N(\mathbf{p})$ from $T_{\mathbf{p}}\mathbf{X}$ to $T_e\mathbf{X}$.



Here $\mathbf{M} \subset \mathbf{X}$ is an oriented surface with unit normal vector field $N = \xi$. Then: $\mathbf{G}\colon \mathbf{M} \to \mathbb{S}^2 \subset T_e\mathbf{X}$ is defined by $\mathbf{G}(\mathbf{p}) = \mathbf{p}^{-1} N(\mathbf{p})$.

The rank of the left invariant Gauss map; Meeks-Mira-Pérez-Ros

- If $\mathbf{H}$ is a $\mathbf{2}$-dimensional subgroup of $\mathbf{X}$, then for any $a \in \mathbf{H}$, $a\mathbf{H} = \mathbf{H}$, and so the left invariant Gauss map of $\mathbf{H}$ is constant.
- Let $\mathbf{\Sigma}$ be an $\mathbf{H}$-surface in $\mathbf{X}$.
- If the left invariant Gauss map of $\mathbf{\Sigma}$ has rank $\mathbf{0}$ at every point, then $\mathbf{\Sigma}$ is contained in some $\mathbf{2}$-dimensional subgroup of $\mathbf{X}$ (Lecture $\mathbf{4}$).
- If the left invariant Gauss map of $\mathbf{\Sigma}$ has rank $\mathbf{0}$ at some point p, then $\exists$ some $\mathbf{2}$-dimensional subgroup of $\mathbf{X}$ with the same normal vector as $\mathbf{\Sigma}$ at p (see Lecture $\mathbf{4}$).
- So, if $\mathbf{X}$ is isomorphic to $\mathbf{SU}(\mathbf{2})$, then the rank of the left invariant Gauss map of any $\mathbf{H}$-surface is everywhere at least $\mathbf{1}$.
- If the left invariant Gauss map of $\mathbf{\Sigma}$ has rank $\mathbf{1}$ everywhere, then $\mathbf{\Sigma}$ is invariant under the left action of a 1-parameter subgroup of $\mathbf{X}$.

Lemma (Transversality Lemma, Meeks-Mira-Pérez-Ros)

Suppose that $\mathbf{S}$ is an immersed sphere in an $\mathbf{X}$ and $\mathbf{F}$ is a $\mathbf{2}$-dimensional subgroup. If the left invariant Gauss map $\mathbf{G}\colon \mathbf{S} \to \mathbb{S}^2$ is a diffeomorphism, then:

- The set of left cosets of $\mathbf{F}$ which intersect $\mathbf{S}$ can be parameterized by the interval $[0,1]$, i.e., $\{g(t)\mathbf{F} \mid t \in [0,1]\}$ are these cosets.
- Each of the cosets $g(0)\mathbf{F}$ and $g(1)\mathbf{F}$ intersects $\mathbf{S}$ in a single point.
- For $t \in (0,1)$, $\mathbf{S} \cap g(t)\mathbf{F}$ is a connected immersed closed curve.

Proof.

- Note that the Gauss map of $\mathbf{F}$ has a constant value, and so, the Gauss maps of the left cosets of $\mathbf{F}$ have the same constant value.
- The set of left cosets $\{g\mathbf{F} \mid g \in \mathbf{X}\}$ is parameterized smoothly by $\mathbb{R}$.
- Let $\Pi\colon \mathbf{X} \to \mathbb{R} = \{g\mathbf{F} \mid g \in \mathbf{X}\}$ be the related smooth quotient map.
- Since the Gauss maps of the leaves of level sets for Π have the same constant unit normal and the Gauss map of $\mathbf{S}$ is a diffeomorphism, $\Pi|_{\mathbf{S}}$ can have at most $\mathbf{2}$ critical points, $\mathbf{1}$ maximum and $\mathbf{1}$ minimum.
- Elementary Morse theory implies all of the statements hold.

$\square$

Theorem (Meeks-Mira-Pérez-Ros)

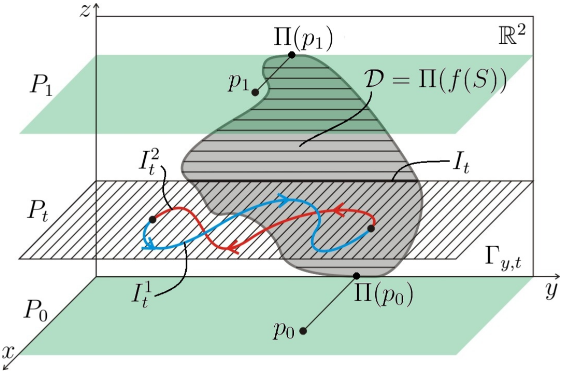
Suppose that $\mathbf{X}$ has an algebraic open book decomposition. If $\mathbf{f}\colon \mathbf{S} \to \mathbf{X}$ is an immersed sphere in whose left invariant Gauss map $\mathbf{G}\colon \mathbf{S} \to \mathbb{S}^2$ is a diffeomorphism, then $\mathbf{S}$ is embedded.

Proof when $\mathbf{X}$ has 2 algebraic open book decomposition.

- By the previous theorem, $\mathbf{X} = \mathbb{R}^2 \rtimes_{\mathbf{A}} \mathbb{R}$ for some diagonal matrix $\mathbf{A}$.
- The new Gauss map in this new metric, which we denote the same way, is still a diffeomorphism.
- Consider the algebraic open book decompositions $\mathcal{B}_1 = \{\mathbf{J}_1(\theta) \mid \theta \in \mathbb{S}^1\}$ and $\mathcal{B}_2 = \{\mathbf{J}_2(\theta) \mid \theta \in \mathbb{S}^1\}$ of $\mathbb{R}^2 \rtimes_{\mathbf{A}} \mathbb{R}$ with bindings $\mathbf{\Gamma}_1 = \mathbf{x}$-axis and $\mathbf{\Gamma}_2 = \mathbf{y}$-axis, respectively.
- The theorem follow easily from the next assertion, since we can express $\mathbf{S}$ as the union of $\mathbf{2}$ "disjoint" disks which are graphs over the $\mathbf{yz}$-plane; see the argument in the next slide.

Assertion

For $i = 1, 2$ and for each $a \in \mathbf{X}$, $\mathbf{f}^{-1}(a\mathbf{\Gamma}_i)$ contains at most $\mathbf{2}$ points.



- Suppose the left coset P_t intersects transversely in the immersed curve $\mathbf{f}(\mathbf{S}) \cap P_t$.
- For $i = 1, 2$ and for each $a \in \mathbf{X}$, $\mathbf{f}^{-1}(a\mathbf{\Gamma}_i)$ contains at most **2** points.
- For $i = 1$, the immersed curve $\mathbf{f}(\mathbf{S}) \cap P_t$ consists of two Jordan arcs $\mathbf{I}_t^1, \mathbf{I}_t^2$ with the same extrema.
- Hence, projection along $\mathbf{\Gamma}_1$ represents $\mathbf{\Sigma}$ as a bigraph over the **yz**-plane.
- Also, projection along $\mathbf{\Gamma}_2$ represents $\mathbf{\Sigma}$ as a bigraph over the **xz**-plane.
- So, $\mathbf{\Sigma}$ is embedded!!

Assertion

For $i = 1, 2$ and for each $a \in \mathbf{X}$, $\mathbf{f}^{-1}(a\mathbf{\Gamma}_i)$ *contains at most* **2** *points.*

Proof.

- Suppose that the assertion fails, assume $i = 1$ and let $\mathbf{\Gamma} = \mathbf{\Gamma}_1$.
- Then for some $a \in \mathbf{X}$, $a\mathbf{\Gamma}$ intersects $\mathbf{S}$ transversely with $\mathbf{f}^{-1}(a\mathbf{\Gamma}) = \{p_1, \ldots, p_{2n}\} \subset \mathbf{S}$ for some $n \geq 2$.
- After left translating $\mathbf{S}$ by a^{-1}, assume that $a\mathbf{\Gamma} = \mathbf{\Gamma}$.
- Since each subgroup of $\mathcal{B}_1$ intersects $\mathbf{S}$ in at least **4** points, the Transversality Lemma implies each leaf L_θ intersects $\mathbf{S}$ transversely.
- The normal variational vector field ∂_θ for $\mathcal{B}_1$ defined on $\mathbf{X} - \mathbf{\Gamma}$ induces a nonzero tangential vector field $\partial_\theta^{\mathbf{T}}$ on $\mathbf{S} - \mathbf{f}^{-1}(\mathbf{\Gamma})$.
- Since $\mathbf{f}$ is transverse to the binding $\mathbf{\Gamma}$ of $\mathcal{B}_1$, the index of $\partial_\theta^{\mathbf{T}}$ is $+1$ at each of the points in $\{p_1, \ldots, p_{2n}\}$.
- By the Hopf index theorem, the Euler characteristic of $\mathbf{S}$ is $2n \geq 4$.
- This is a contradiction since the Euler characteristic of $\mathbf{S}$ is **2**. $\square$

Theorem (Meeks-Mira-Pérez-Ros)

Let $\mathbf{X}$ be a $\mathbf{3}$-dimensional metric Lie group. Then:

1. $\mathbf{X}$ admits and open book decomposition **iff** it is isomorphic to $\mathbf{R}^3$, $\mathbb{H}^3$, $\mathbf{Nil_3}$, $\mathbf{Sol_3}$, or to a non-unimodular Lie group with $\mathbf{D}$-invariant $\mathbf{D} \leq 1$.
2. If $\mathbf{X}$ admits an algebraic open book decomposition and $\mathbf{f}\colon \mathbf{S} \to \mathbf{X}$ is an immersed sphere whose left invariant Gauss map of is a diffeomorphism, then $\mathbf{f}$ is injective.
3. The left invariant Gauss map of $\mathbf{H}$-spheres in $\mathbf{X}$ are diffeomorphisms.
4. $\mathbf{H}$-spheres in homogeneous $\mathbf{3}$-manifolds are always Alexandrov embedded (bound submersed $\mathbf{3}$-balls) except for the minimal spheres $\mathbb{S}^2(\kappa) \times \{t\}$ in $\mathbb{S}^2(\kappa) \times \mathbb{R}$.

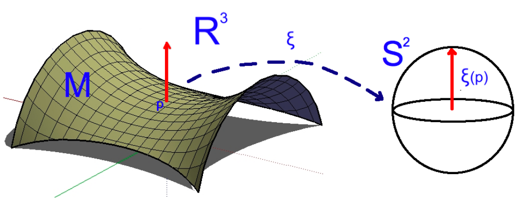
Remark

The fact that the Gauss map of an $\mathbf{H}$-sphere in $\mathbf{X}$ is a diffeomorphism in item 4 in the above theorem depends on proving that $\mathbf{H}$-spheres always have index $\mathbf{1}$.

Conjecture (Alexandrov Embeddedness Conjecture, Meeks-Mira-Pérez-Ros)

- If $\mathbf{Y}$ be a simply connected, homogeneous $\mathbf{3}$-manifold not diffeomorphic to $\mathbb{S}^3$, then $\mathbf{H}$-spheres in $\mathbf{Y}$ are embedded.
- Furthermore, if $\mathbf{Y}$ is diffeomorphic to $\mathbf{R}^3$, then compact Alexandrov embedded $\mathbf{H}$-surfaces in $\mathbf{Y}$ are spheres.

Definition (Second Fundamental Form/Shape Operator)



Let $\mathbf{M}$ be an oriented surface in $\mathbf{R}^3$, let ξ be the unit vector field normal to $\mathbf{M}$:

$$\mathbf{A} = -d\xi\colon\, T_{\mathbf{p}}\mathbf{M} \to T_{\xi(\mathbf{p})}\mathbf{S}^2 \simeq T_{\mathbf{p}}\mathbf{M}$$

is the **shape operator** of $\mathbf{M}$.

Definition

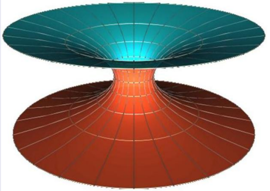
- The eigenvalues k_1, k_2 of $\mathbf{A}_p$ are the **principal curvatures** of $\mathbf{M}$ at p.
- $\mathbf{K}_G = \det(\mathbf{A}) = k_1 k_2$ is the **Gauss curvature**.
- $\mathbf{H} = \frac{1}{2}\mathbf{Trace}(\mathbf{A}) = \frac{k_1+k_2}{2}$ is the **mean curvature**.
- $\mathbf{K_M} = k_1 k_2$ is the **Gaussian curvature**.
- $|\mathbf{A}| = \sqrt{k_1^2 + k_2^2}$ is the **norm of the shape operator.**

Gauss equation

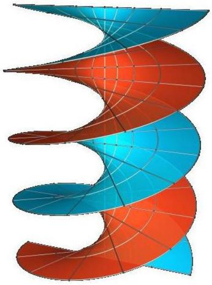
$$4\mathbf{H}^2 = |\mathbf{A}|^2 + 2\mathbf{K_G} \quad (\mathbf{K_G} = \text{Gaussian curvature})$$

Theorem

An **H**-surface **M** is a **minimal surface** $\iff$ **H** $\equiv 0$ $\iff$ **M** is a critical point for the area functional under compactly supported variations.



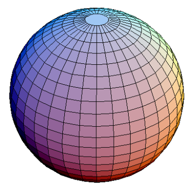
- Catenoid



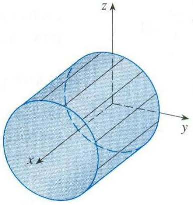
- Helicoid

Theorem

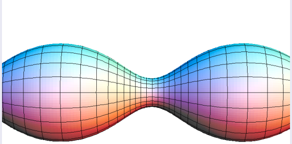
M is a **H-surface** $\iff$ **M** is a critical point for the area functional under compactly supported variations **preserving the volume**.



- Sphere



- Cylinder



- Delaunay surfaces

Uniqueness, existence, index and embeddedness of **H**-spheres in $\mathbb{R}^3$.

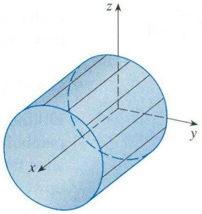
Let **M** be a closed (compact without boundary) **H**-surface in $\mathbb{R}^3$:

- **M** is **not** minimal. (coordinate functions cannot be harmonic)
- **M** has **genus 0** $\iff$ it is a round sphere (1951, **Hopf**).
- **M** is **embedded** $\iff$ it is a round sphere (1956, **Alexandrov**).
- **Index**(**M**) $= 1 \iff$ **M** is a round sphere (1984, **Barbosa-do Carmo**).

Question

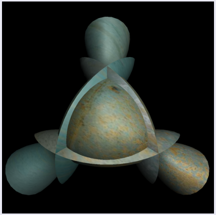
Is the round sphere the only complete simply connected surface **embedded** in $\mathbf{R}^3$ with **non-zero** constant mean curvature?

NOT simply connected



- Cylinder

NOT embedded



- Smyth surface

Answer (Meeks-Tinaglia)

Yes!

Theorem (Meeks-Tinaglia)

Round spheres are the only complete simply connected surfaces **embedded** in $\mathbf{R}^3$ with non-zero constant mean curvature.

1997, **Meeks** for **properly embedded**.

Corollary

Let $\mathbf{M}$ be a complete simply connected $\mathbf{H}$-surface **embedded** in $\mathbf{R}^3$. Then $\mathbf{M}$ is either

a plane, **a helicoid** **or** **a round sphere**.

(2008, **Colding-Minicozzi** and **Meeks-Rosenberg** for $\mathbf{H} = 0$**)**

Definition

A $\mathbf{1}$-disk is a compact disk **embedded** in $\mathbf{R}^3$ with constant mean curvature $\mathbf{1}$.

Theorem (Radius Estimate, Meeks-Tinaglia)

There exists a universal constant $\mathbf{R}$ such that if $\mathbf{M}$ is a $\mathbf{1}$-disk, then $\mathbf{M}$ has **radius** less than $\mathbf{R}$, i.e.,

$$\forall \mathbf{p} \in \mathbf{M}, \quad \mathrm{dist}_{\mathbf{M}}(\mathbf{p}, \partial \mathbf{M}) < \mathbf{R}.$$

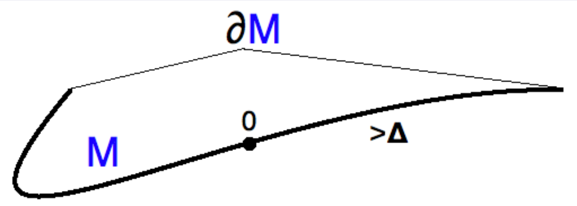
In particular, if $\mathbf{M}$ is a complete simply connected $\mathbf{H} = \mathbf{1}$ surface embedded in $\mathbf{R}^3$, then:
Radius Estimate $\Longrightarrow$ $\mathbf{M}$ is compact $\Longrightarrow$ $\mathbf{M}$ is an embedded sphere.
Hopf or Alexandrov Theorems $\Longrightarrow$ $\mathbf{M}$ is a **round sphere.**

The Radius Estimate is a **trivial** consequence of the following Intrinsic Curvature Estimate.

Theorem (Intrinsic Curvature Estimate, Meeks-Tinaglia)

- Let $\mathbf{Y}$ be a homogenously regular $\mathbf{3}$-manifold with absolute sectional curvature less than $\mathbf{K} \geq 0$.
- Given $\mathbf{\Delta} > 0$ there exist a universal constant $\mathbf{C} = \mathbf{C}(\mathbf{\Delta}, \mathbf{K})$ such that:
- If $\mathbf{M}$ is an $\mathbf{H}$-disk in $\mathbf{Y}$ with $\mathbf{0} \in \mathbf{M}$, $\mathrm{dist}_{\mathbf{M}}(\mathbf{0}, \partial \mathbf{M}) > \mathbf{\Delta}$ and $\mathbf{H} \geq 1$, then

$$|\mathbf{A_M}|(\mathbf{0}) \leq \mathbf{C}.$$



Theorem (Radius/Curvature Estimate, Meeks-Tinaglia)

- Suppose that $\mathbf{X} = \mathbb{H}^3$.
- There exists a universal constant $\mathbf{R}$ such that if $\mathbf{M}$ is an $\mathbf{H}$-disk and $\mathbf{H} > \mathbf{H}(\mathbf{X})$, then $\mathbf{M}$ has **radius** less than $\mathbf{R}$, i.e.,

$$\forall \mathbf{p} \in \mathbf{M}, \quad \mathrm{dist}_{\mathbf{M}}(\mathbf{p}, \partial \mathbf{M}) < \mathbf{R}.$$

- Complete embedded $\mathbf{H}$-surfaces in $\mathbf{X}$ with $\mathbf{H} > 0$ have bounded second fundamental forms and are properly embedded when $\mathbf{H} > 1$.

In particular, if $\mathbf{M}$ is a complete simply connected $\mathbf{H} > \mathbf{1}$ surface embedded in $\mathbb{H}^3$, then:
Radius Estimate $\implies$ $\mathbf{M}$ is compact $\implies$ $\mathbf{M}$ is an embedded sphere.
Hopf or Alexandrov Theorems $\implies$ $\mathbf{M}$ is a **round geodesic sphere.**

Conjecture (Radius/Curvature Estimates, Meeks-Tinaglia)

- There exists a universal constant $\mathbf{R}$ such that if $\mathbf{M}$ is an embedded $\mathbf{H}$-disk with $\mathbf{H} > \mathbf{H}(\mathbf{Y})$, then $\mathbf{M}$ has **radius** less than $\mathbf{R}$, i.e.,

$$\forall \mathbf{p} \in \mathbf{M}, \quad \mathrm{dist}_{\mathbf{M}}(\mathbf{p}, \partial \mathbf{M}) < \mathbf{R}.$$

- Complete embedded $\mathbf{H}$-surfaces in $\mathbf{Y}$ with $\mathbf{H} > 0$ have bounded second fundamental forms.
- Every complete embedded $\mathbf{H}$-surface of finite topology in $\mathbf{Y}$ is properly embedded, whenever $\mathbf{H} \geq \mathbf{H}(\mathbf{Y})$.

Theorem (Classification Theorem for H-spheres, Meeks-Mira-Pérez-Ros)

Suppose $\mathbf{X}$ is a simply connected $\mathbf{3}$-dimensional metric Lie group.

- $\mathbf{X}$ is diffeomorphic to $\mathbf{R}^3 \implies$ the moduli space of $\mathbf{H}$-spheres in $\mathbf{X}$ is parameterized by the mean curvature values $\mathbf{H}$ in $(\mathbf{H}(\mathbf{X}), \infty)$.
- $\mathbf{X}$ is diffeomorphic to $\mathbb{S}^3 \implies$ the moduli space of $\mathbf{H}$-spheres in $\mathbf{X}$ is parameterized by the mean curvature values $\mathbf{H}$ in $[0, \infty)$.
- $\mathbf{X}$ diffeomorphic to $\mathbb{S}^3 \implies$ the areas of all $\mathbf{H}$-spheres form a half-open interval $(0, \mathbf{A}(\mathbf{X})]$.
- $\mathbf{H}$-spheres in $\mathbf{X}$ are **Alexandrov embedded** with **index 1**, **nullity 3**.

- The proof of the above theorem will be the focus of Lecture **4**.
- We place it here in order to discuss the following related conjectures.

Conjecture (Index $\mathbf{1}$ Conjecture, Meeks-Mira-Pérez-Ros)

When $\mathbf{X}$ is diffeomorphic to $\mathbf{R}^{\mathbf{3}}$, then every compact $\mathbf{H}$-surface in $\mathbf{X}$ with **index** $\mathbf{1}$ is a sphere. In particular, solutions to the isoperimetric problem in such an $\mathbf{X}$ are spheres.

Conjecture (Isoperimetric Domains Conjecture, Meeks-Mira-Pérez-Ros)

If $\mathbf{X}$ is diffeomorphic to $\mathbb{R}^3$, then:

1. Isoperimetric surfaces (boundaries of isoperimetric domains) in $\mathbf{X}$ are spheres.
2. For each fixed volume $\mathbf{V}_0$, solutions to the isoperimetric problem in $\mathbf{X}$ for volume $\mathbf{V}_0$ are unique up to left translations in $\mathbf{X}$.

Conjecture (Topological Existence Conjecture, Meeks)

- If $\mathbf{X} \approx \mathbf{R}^3$ and $\mathbf{H} < \mathbf{H}(\mathbf{X})$, then $\mathbf{X}$ admits properly embedded $\mathbf{H}$-surfaces of every possible topology.
- If $\mathbf{X} \approx \mathbf{R}^3$ and $\mathbf{H} > \mathbf{H}(\mathbf{X})$, then $\mathbf{X}$ admits properly embedded $\mathbf{H}$-surfaces of every possible topology, except for finite genus with $\mathbf{1}$ end or finite positive genus with $\mathbf{2}$ ends which it never admits.

Conjecture (Topological Uniqueness Conjecture, Meeks)

If $\mathbf{M}_1$, $\mathbf{M}_2$ are two diffeomorphic, connected and properly embedded $\mathbf{H}$-surfaces of finite topology in $\mathbf{X}$ with $\mathbf{H} = \mathbf{H}(\mathbf{X})$, then there exists a diffeomorphism $\mathbf{f} \colon \mathbf{X} \to \mathbf{X}$ with $\mathbf{f}(\mathbf{M}_1) = \mathbf{M}_2$.

Meeks-Frohman

If $\mathbf{M}_1$, $\mathbf{M}_2$ are two diffeomorphic, connected and properly embedded minimal surfaces in $\mathbf{Y} = \mathbb{S}(\kappa) \times \mathbb{R}$, then there exists a diffeomorphism $\mathbf{f} \colon \mathbf{Y} \to \mathbf{Y}$ with $\mathbf{f}(\mathbf{M}_1) = \mathbf{M}_2$.

Theorem (Meeks-Pérez)

- Suppose that $\mathbf{Y}$ is a $\mathbf{3}$-dimensional metric Lie group, not necessarily simply connected.
- The compact, orientable strongly stable $\mathbf{H}$-surfaces in $\mathbf{Y}$ are precisely the left cosets of compact $\mathbf{2}$-dimensional subgroups of $\mathbf{Y}$ and furthermore, all such subgroups are tori which are normal subgroups of $\mathbf{Y}$.
- In particular, the existence of such strongly stable compact $\mathbf{H}$-surfaces in $\mathbf{Y}$ implies that the fundamental group of $\mathbf{Y}$ contains a subgroup isomorphic to $\mathbb{Z} \times \mathbb{Z}$.

Corollary (Unknottedness Theorem for Minimal Surfaces in SU(2))

Let $\mathbf{X}$ be isomorphic to $\mathbf{SU}(\mathbf{2})$ and let $\mathbf{M}_1$, $\mathbf{M}_2$ be two compact, diffeomorphic embedded minimal surfaces in $\mathbf{X}$. Then there exists a diffeomorphism $\mathbf{f}\colon \mathbf{X} \to \mathbf{X}$ with $\mathbf{f}(\mathbf{M}_1) = \mathbf{M}_2$.

Proof.

- By Waldhausen, it suffices to prove that the closure of each complement of $\mathbf{X} - \mathbf{M}_i$ is a handlebody.
- Otherwise by **Meeks-Simon-Yau**, $\mathbf{X}$ admits an embedded, compact, strongly stable minimal surface, which contradicts the above theorem. $\square$

Theorem (Positive Injectivity Radius of Classical Minimal Surfaces, Meeks-Pérez)

- A complete embedded minimal surface $\mathbf{M}$ of finite topology in a homogeneous $\mathbf{3}$-manifold $\mathbf{Y}$ has positive injectivity radius.
- More generally, a complete embedded $\mathbf{H}$-surface $\mathbf{M}$ of finite topology in a homogeneous $\mathbf{3}$-manifold $\mathbf{Y}$ has positive injectivity radius, when $\mathbf{H} < \mathbf{H}(\mathbf{Y})$.

Remark

- **Meeks-Tinaglia** proved that a complete embedded $\mathbf{H}$-surface of finite topology in a homogeneously regular $\mathbf{3}$-manifold of non-positive curvature has bounded second fundamental form when $\mathbf{H} > 0$, and hence, it positive injectivity radius.
- Their result in the non-positive sectional curvature setting was motivated by the earlier proof by **Meeks-Rosenberg** that a complete embedded minimal surface of finite topology in a homogeneously regular $\mathbf{3}$-manifold of non-positive curvature has positive injectivity radius.

Conjecture (Positive Injectivity Radius/Bounded Curvature Conjecture, Meeks-Pérez-Tinaglia)

- A complete embedded $\mathbf{H}$-surface of finite topology in $\mathbf{X}$ with $\mathbf{H} \geq \mathbf{H}(\mathbf{X})$ has positive injectivity radius and bounded curvature.
- A complete embedded $\mathbf{H}$-surface of finite genus surface in $\mathbf{X}$ with $\mathbf{H} = \mathbf{H}(\mathbf{X})$ has positive injectivity radius and bounded curvature.

Remark

Curvature estimates of **Meeks-Tinaglia** for embedded ($\mathbf{H} > 0$)-disks imply that any complete embedded ($\mathbf{H} > 0$)-surface of positive injectivity radius in a homogeneously regular $\mathbf{3}$-manifold has bounded second fundamental form.

Conjecture (Bounded Curvature Conjecture, Meeks-Tinaglia)

- The norm of the second fundamental of any complete embedded finite topology ($\mathbf{H} > 0$)-surface in $\mathbf{X}$ is bounded.
- More generally, the same result holds for the case for any complete $\mathbf{3}$-manifold locally isometric to $\mathbf{X}$.

Conjecture (Calabi-Yau Properness Problem, Meeks-Pérez)

- A complete, connected, embedded $\mathbf{H}$-surface of positive injectivity radius is proper in $\mathbf{X}$ whenever $\mathbf{H} \geq \mathbf{H}(\mathbf{X})$.
- In particular, by a result of **Meeks-Pérez**, complete, embedded minimal surfaces of finite topology in metric Lie groups isomorphic to $\mathbf{R}^3$, $\mathbf{Nil}_3$, $\widetilde{\mathbf{E}}(2)$, $\mathbf{Sol}_3$ would always be proper.

Conjecture (Stability Conjecture for $\mathrm{SU}(2)$, Meeks-Pérez-Ros)

If **X** is diffeomorphic to $\mathbb{S}^3$, then **X** contains no strongly stable complete **H**-surfaces.

Meeks-Pérez-Ros

- Conjecture is true if **X** is a Berger sphere with non-negative scalar curvature.
- Also true if **X** is **SU**(**2**) endowed with a left invariant metric of positive scalar curvature.
- If **Y** is a **3**-sphere with a Riemannian metric (not necessarily a left invariant metric) such that it admits no strongly stable complete minimal surfaces, then for each integer $g \in \mathbb{N} \cup \{0\}$, the space of compact embedded minimal surfaces of genus g in **Y** is compact.

Conjecture (Calabi-Yau Properness Problem, Meeks-Pérez)

- A complete, connected, embedded $\mathbf{H}$-surface of positive injectivity radius is proper in $\mathbf{X}$ whenever $\mathbf{H} \geq \mathbf{H}(\mathbf{X})$.
- In particular, by a result of **Meeks-Pérez**, complete, embedded minimal surfaces of finite topology in metric Lie groups isomorphic to $\mathbf{R}^3$, $\mathbf{Nil_3}$, $\widetilde{\mathbf{E}}(2)$, $\mathbf{Sol_3}$ are proper.