Renormalisation theory and stochastic PDEs

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In this course, we will show how it is possible to deal with stochastic PDEs for which the mere question "What does it mean to be a solution?" is a very non-trivial one. Examples of SPDEs of such a kind include the stochastic Allen-Cahn equation (this is also the stochastic quantisation equation for the Φ^4_3 Euclidean quantum field theory)

$$\partial_t u = \Delta u + u - u^3 + \xi \;, \tag{AC}$$

where the spatial variable is 3-dimensional and ξ denotes space-time white noise, the KPZ equation

$$\partial_t h = \partial_x^2 h + (\partial_x h)^2 + \xi \;, \tag{KPZ}$$

where the spatial variable is 1-dimensional and ξ denotes again space-time white noise, as well as the continuous parabolic Anderson model

$$\partial_t u = \Delta u + u\,\eta \;, \tag{PAM}$$

where the spatial variable is either 2 or 3-dimensional and this time η denotes noise that is white in space but constant in time. In all of these equations, there is some product that is ill-posed. In the case of (AC), one does not expect to obtain function-valued solutions so that it is unclear what the meaning of u^3 is. Similarly, one does not expect solutions to the KPZ equation to be differentiable, so that $(\partial_x h)^2$ does not have any classical meaning. Finally, in the case of (PAM), the product $u\,\eta$ involves a distribution and a function that is sufficiently irregular for it not to make any classical sense.

We will introduce a new theory of "regularity structures" that allows to represent a large class of very irregular functions (or distributions!) by a kind of local Taylor expansion, but where the usual polynomials have been replaced by functions (or distributions) that are much better adapted to the problem at hand. The classes of functions / distributions exhibited by this theory are sufficiently large to allow to make sense of what it actually means to solve any of the above equations. It achieves this by encoding the divergencies arising in these problems in a systematic way, which allows to add suitable diverging counterterms, finally leading to solutions in a canonical way. This encoding is performed through a "renormalisation group" acting on the corresponding space of equations which is very similar in spirit to the constructions arising in the context of quantum field theory.

The aim of the course is to present the main results from the theory of regularity structures in a self-contained way and to show how each class of equations is canonically associated to a renormalisation group. We will then show in detail how to apply the theory to one of the examples mentioned above.