Analyticity of the Pressure via Maximizing Measures

Rodrigo Bissacot - IME USP

Jointly with Ricardo Freire (IME-USP)
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Ergodic Optimization

and Related Fields

http://ergodicoptimization.ime.usp.br **December 9 - 13, 2013**

This international workshop will be held at Institute of Mathematics and Statistics (IME-USP) of the University of São Paulo, São Paulo (Brazil). It will focus on recent research on the field and also on providing a good immersion on the related topics for graduate students.

Participants

Jon Aaronson (Tel-Aviv University)
Alexandre Baraviera (UFRGS)
Jairo Bochi (PUC-RJ)
Gonzalo Contreras (CIMAT)
Edson de Faria (IME-USP)
Lorenzo Díaz (PUC-RJ)
Albert Fisher (IME-USP)
Eduardo Garibaldi (UNICAMP)
Diogo Gomes (IST-Lisboa) (*)
Godofredo Iommi (PUC Chile)
Oliver Jenkinson (Queen Mary, Univ. of London)
Tom Kempton (Utrecht University)
Zemer Kosloff (Tel-Aviv University)
Tamara Kucherenko (CUNY)
Renaud Leplaideur (Université de Brest)
Artur Lopes (UFRGS)
Jairo Mengue (UFRGS)
Tom Meyerovitch (Ben-Gurion Univ. of the Negev) (*)
Joana Mohr (UFRGS)
Krerley Oliveira (UFAL)
Vilton Pinheiro (UFBA)
Mark Pollicott (Warwick)
Anthony Quas (University of Victoria)
Rafael Rigão (UFRGS)
Isabel Rios (UFF)
Samuel Senti (UFRJ)
Daniel Smania (ICMC-USP)
Manuel Stadlbauer (UFBA)
Ali Tahzibi (ICMC-USP)
Fábio Tal (IME-USP)
Philippe Thieullen (Universitè Bordeaux 1)
Mariusz Urbanski (University of North Texas)
Paulo Varandas (UFBA)
Edson Vargas (IME-USP)
Christian Wolf (CUNY)

(*) = *pending confirmation*

Minicourses

Ergodic optimization, ergodic dominance, and thermodynamic formalism *by Oliver Jenkinson*

Ergodic Transport *by Jairo Mengue and Rafael Rigão*

Multiplicative ergodic theorems and applications *by Anthony Quas*

Scientific Committee

Albert Fisher (IME-USP) Artur Lopes (UFRGS) Ali Tahzibi (ICMC-USP) Fábio Tal (IME-USP)

Organizers

Rodrigo Bissacot (IME-USP) Ricardo Freire (IME-USP)

Supported by

Setting:

- X compact or non-compact (Polish Space) [our case].
  Ex: One-dimensional lattices: $S^{\mathbb{Z}}, S^{\mathbb{N}}$...
  where $S = \{-1, +1\}, \{1, 2, ..., k\}, \mathbb{N}, \mathbb{S}^1, \mathbb{R}$.
- The potential $f : X \to \mathbb{R}$ always continuous or more (Lipschitz, Hölder, summable variation, Locally Hölder...).
  Ex: $\Phi = (\Phi_C)_{C \subset\subset \mathbb{Z}}$ regular interaction and $f_\Phi = \sum_{C \ni 0} \frac{\Phi_C}{|C|}$.
- T is always continuous or more (expansive...). Ex: Shift map $\sigma : S^{\mathbb{N}} \to S^{\mathbb{N}}; \sigma(x) = \sigma(x_0, x_1, x_2, ...) = (x_1, x_2, x_3, ...)$.

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Ergodic Optimization

The main problem in *Ergodic Optimization* is to guarantee the existence and to describe the *maximizing measures* for the system, that is, to describe the set of probability measures satisfying:

$$\alpha(f) := \sup_{\mu \in \mathcal{M}_T} \int f \, d\mu$$

where M_T denotes the set of the T-invariant borel probability measures and f is a fixed potential $f : X \to \mathbb{R}$.

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Connection with Equilibrium Statistical Mechanics

Let $\sigma : \Sigma_{\mathbf{A}} \to \Sigma_{\mathbf{A}}$ the shift map where $\Sigma_{\mathbf{A}}$ is a topological Markov Shift.

Now you can think $\Sigma_{\mathbf{A}} = S^{\mathbb{N}}$ where $S = \{1, 2, ..., k\}$ or $S = \mathbb{N}$.

For each β (inverse of the temperature) the *equilibrium measures* μ_β (for a potential βf) are the probability measures satisfying the *variational principle* for *pressure* $P(\beta f)$:

$$P(\beta f) = \sup_{m \in M_\sigma} \left\{ h(m) + \int \beta f dm ; \int \beta f dm > -\infty \right\} = h(\mu_\beta) + \int \beta f d\mu_\beta$$

where $h(m)$ is the Kolmogorov-Sinai entropy of the measure m.

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Connection with Equilibrium Statistical Mechanics

Under suitable hypothesis on $\Sigma_{\mathbf{A}}$ and f we can prove that there exist equilibrium measures μ_β for all positive β and any zero-temperature accumulation point of the family $(\mu_\beta)_{\beta>0}$ is a maximizing measures for the potential f.

Main ex-conjecture

Roughly:

Generically in the space of Lipschitz potentials with X compact and T with suitable properties the maximizing measure is unique and supported in an periodic orbit.

Ground States are Generically a Periodic Orbit. (Gonzalo Contreras) Abstract. We prove that for an expanding transformation the maximizing measures of a generic Lipschitz function are supported on a single periodic orbit. (arxiv)

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compact versus non-compact

In the compact setting ($X = \Sigma_{\mathbf{A}}$) is compact the maximizing measures always exist.

- When X is compact since the potential f is always assume continuous by compactness (of M_T) there exists a probability measure m in M_T such that $\alpha(f) := \sup_{\mu \in \mathcal{M}_T} \int f \, d\mu = \int f \, dm$

The problem in this context it is to describe the support of this measures.

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non-compact setting

There exist examples where the maximizing measures do not exist and the problem of the existence is nontrivial.

Ergodic optimization for noncompact dynamical systems.
- O. Jenkinson, R. D. Mauldin and M. Urbański - (DS-07')

Ergodic optimization for countable alphabet subshifts of finite type.
- O. Jenkinson, R. D. Mauldin and M. Urbański - (ETDS-06')

Zero Temperature limits of Gibbs-Equilibrium states for countable alphabet subshifts of finite type.
- O. Jenkinson, R. D. Mauldin and M. Urbański - (JSP-05')

These earlier papers assume that $\Sigma_{\mathbf{A}}$ is Finitely Primitive:

The matrix $\mathbf{A}$ is *finitely primitive*, when there exist a finite subset $\mathbb{F} \subseteq \mathbb{N}$ and an integer $K_0 \geq 0$ such that, for any pair of symbols $i, j \in \mathbb{N}$, one can find $\ell_1, \ell_2, \ldots, \ell_{K_0} \in \mathbb{F}$ satisfying

$$\mathbf{A}(i, \ell_1)\mathbf{A}(\ell_1, \ell_2) \cdots \mathbf{A}(\ell_{K_0}, j) = 1.$$

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$$i \quad \underbrace{\underline{i_1} \quad \underline{i_2} \quad \underline{i_3} \quad \ldots \quad \underline{i_{k_0-1}} \quad \underline{i_{k_0}}}_{k_0} \quad j$$

$i_1, i_2, \ldots, i_{k_0}$
elements of $\mathbb{F}$

They assume some regularity for f *summable variation* and also assume that f is *coercive*

$$\lim_{i\to\infty} \sup f|_{[i]} = -\infty\,,$$

$[i] := \{x \in \Sigma_{\mathbf{A}}(\mathbb{N}),\, \pi(x) = i\}$ is the cylinder beginning with i.

This condition is satisfied when we have for example:

$$\sum_{i\in\mathbb{N}} \exp(\sup f|_{[i]}) < \infty.$$

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When the matrix $\mathbf{A}$ is finitely primitive and f satisfies the last condition:

Theorem (O. Jenkinson, R. D. Mauldin and M. Urbański 05')

The family of Gibbs measures $(\mu_{\beta f})_{\beta \geq 1}$ has at list one weak accumulation point as $\beta \to \infty$. Any accumulation point μ is a maximizing measure for f, and $\lim_{\beta\to\infty} \int f d\mu_{\beta f} = \int f d\mu$.

Proof: Prohorov 's theorem and use that the measures $\mu_{\beta f}$ are Gibbs Measures.

Theorem (I. D. Morris 07')

Assuming the conditions above we have that there exists a finite set $\mathcal{A} \subset \mathbb{N}$ such that

$$\alpha(f) = \max_{\substack{\mu\in\mathcal{M}_\sigma \\ supp\mu\subseteq\Sigma_{\mathbf{A}}(\mathcal{A})}} \int f \, d\mu.$$

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Theorem (_ and E. Garibaldi (Bull. Braz. Math. Soc. 2010))

Suppose $\Sigma_{\mathbf{A}}(\mathbb{N})$ *is shift with* **A** *finitely primitive. Let* $f : \Sigma_{\mathbf{A}}(\mathbb{N}) \to \mathbb{R}$ *be a bounded above, coercive and locally Hölder continuous potential satisfying* $\inf_{i \in \mathbb{F}} f|_{[i]} > -\infty$.
Then, there exists an integer $\mathcal{I} > I_{\mathbb{F}}$ *such that*

$$\alpha(f) = \max_{\substack{\mu \in \mathcal{M}_\sigma \\ \mathit{supp}\mu \subseteq \Sigma_{\mathbf{A}}(\mathcal{I})}} \int f \, d\mu.$$

In particular, maximizing measures do exist. Furthermore, there exists a compact σ*-invariant set* $\Omega \subseteq \Sigma_{\mathbf{A}}(\mathcal{I})$ *such that* $\mu \in \mathcal{M}_\sigma$ *is* f*-maximizing if, and only if,* μ *is supported in* Ω.

Key Fact:The invariant measures supported in periodic orbits are dense on the set of ergodic measures. (Parthasarathy- 1961)

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Connection with phase transitions

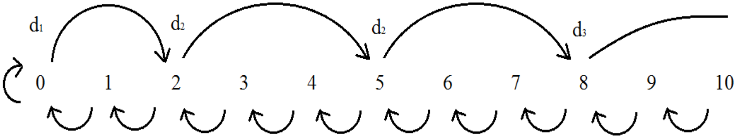
Beyond the Finite Primitive case

Renewal shifts

Example: Let $A = (a_{ij})_{i,j\in\mathbb{N}}$ be the transition matrix such that there exists an increasing sequence of naturals $(d_i)_{i\in\mathbb{N}}$ for which

$$a_{00} = a_{i+1,i} = a_{1,d_i} = 1,\ \forall\, i \in \mathbb{N}$$

and the others coefficients are zero.


d_1
d_2
d_2
d_3
0 1 2 3 4 5 6 7 8 9 10

Theorem (O. Sarig - CMP - 2001)

Let Σ be a Renewal Shift and f a locally Hölder potential such that $\sup f < \infty$. Then there exists a constant $\beta_c \in (0, \infty]$ such that

- *For $0 < t < \beta_c$ there exists an equilibrium probability measure μ_t corresponding to βf. For $t > \beta_c$ there is no equilibrium probability measures corresponding to tf;*
- *$P(\beta f)$ is real analytic on $(0, \beta_c)$ and linear on (β_c, ∞). At β_c, it is continuous but not analytic.*

Theorem (G. Iommi - 2007)

Let Σ be a Renewal Shift and f a locally Hölder potential such that $\sup f < \infty$. Then

- *For $\beta_c = \infty$, then there exists maximizing measures μ_β for βf.*
- *If $\beta_c < \infty$, then there are no maximizing measures for f*

Here, $\alpha(f) = \sup_{\mu \in \mathcal{M}_T} \int f \, d\mu$ is the slope linear part of the pressure $P(\beta f)$.

We say that $\mathbf{A}$ is *irreducible* when for any $(i, j) \in \mathbb{N}^2$ there exists a natural number $k(i, j)$ and a word $y_1 y_2 \ldots y_k$ such that $i y_1 y_2 \ldots y_k j$ is an allowable word: $\mathbf{A}(i, y_1) = 1$, $\mathbf{A}(y_i, y_{i+1}) = 1$ for $i = 1, \ldots, k-1$ and $\mathbf{A}(y_k, j) = 1$.

Naive idea:

Since the potential f decays to $-\infty$ when the symbols grow, we can restrict ourselves to periodic orbits whose symbols are all small.

Theorem (_ and R. Freire - To appear in ETDS)

Let σ be the shift on $\Sigma_{\mathbf{A}}(\mathbb{N})$ with $\mathbf{A}$ irreducible and let be $f : \Sigma_{\mathbf{A}}(\mathbb{N}) \to \mathbb{R}$ be a function with bounded variation and coercive. Then, there is a finite set $\mathcal{A} \subset \mathbb{N}$ such that $\mathbf{A}|_{\mathcal{A}\times\mathcal{A}}$ is irreducible and

$$\alpha(f) = \sup_{\mu \in \mathcal{M}_\sigma(\Sigma_{\mathbf{A}}(\mathcal{A}))} \int f \, d\mu \,.$$

Furthermore, if ν is a maximizing measure, then

$$supp\, \nu \subset \mathcal{M}_\sigma(\Sigma_{\mathbf{A}}(\mathcal{A}))\,.$$

Corollary

If you assume f coercive and locally Hölder potential then there is no phase transition in the Renewal Shift.