A theorical basis for lower semi-continuous functions's conjugation

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Abstract:

In the theory of Fenchel Conjugation, we associate an extended real-valued function f of several real variables with a convex function defined as

$$f_F^*(a) = \sup_{x \in \mathbb{R}^n} \left\{ \langle p, x \rangle - f(x) \right\}$$

where $a \in \mathbb{R}^n$. It is called the Conjugate Function. This relationship has an important economic interpretation: let x be the level of production of some company, $f(x)$ be the cost of the level of production x and p be the price, then the conjugate of f can represent the gain of the company .

However, in some real problems, the price depends on the level of productions. For example, when there is a large amount of product available to the consumer's market, the price tends to decrease. Based on this philosophy, the Fenchel Conjugate should be modified so that this price variation was included in its modeling.

Moreau was one of the first to extend the theory of conjugation. He considered functions $c : K_1 \times K_2 \to \mathbb{R} \cup \{+\infty\}$ where K_1 and K_2 are arbitrary sets, and thus he defined the conjugate by

$$f^c(a) = \sup_{x \in K_2} \left\{ c(a, x) - f(x) \right\}$$

where $a \in K_1$. Cotrina, Karas, Ribeiro, Sosa and Yuan modified the conjugate function to try to solve the problem of modeling. In this one, c is the generalized inner product, in other words, $c(p, x) = \langle p(x), x \rangle$ where $p : \mathbb{R}^n \to \mathbb{R}^n$ is a continuous function .

Those authors had an interest in using this modified conjugated to build the duality scheme for lower semi-continuous programming. This one are problems of finding minimizers for lower semi-continuous functions whose existence of solution is ensured for some coerciviness conditions. However, this function can not have a good behavior as the case of nonconvex functions. Thus, it is hard to solve analytically or numerically and so this is a motivation to investigate an associated dual problem that could be easily solved.

The theory of Fenchel Conjugation was used to build the Convex Duality, but this duality has good characteristics for lower semicontinuous convex functions. The generalization of Cotrina, Karas, Ribeiro, Sosa and Yuan had the purpose to weaken those assumptions and so to build a scheme duality well-behaved for lower semicontinuous functions.

Furthermore, one has developed a study about the conjugate dual spaces which are subspaces of continuous functions whose modified conjugation is symmetric. This is important in theory and practice, because in many situations, it is necessary to calculate the conjugate only in those subspaces.

The term "generalization" given for conjugate in [2] caused discomfort among many researchers because they thought it was a particular case of Moreau Conjugate. Thus, Sosa investigated the relationship between the theory of Fenchel Conjugation with the Classical Separation Theorems. Those theorems ensure the existence of hyperplanes which separate convex and closed sets.

Thanks to Michael's Selection Theorem which separated specific closed subsets of $\mathbb{R}^n$ using continuous functions, Sosa was able to generalize the Separation Theorems extending hyperplanes for continuous functions and showed the modified conjugation in [2] comes from those theorems.

Finally, Cotrina, Raupp and Sosa developed the theory of duality for lower semi-continuous programming and investigated their properties.

The purpose of this work was to do a remake of [2], [3], [9] and relate them so to convince this proposed modified conjugate is actually a generalization of Fenchel Conjugate.

Key words: Separation Theorems; Conjugate Functions; Semi-continuous Programming.

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