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CURRENT RESERVOIRS IN THE SIMPLE EXCLUSION PROCESS

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Abstract. We consider the symmetric simple exclusion process in the interval $[-N, N]$ with additional birth and death processes respectively on $(N-K, N]$, $K > 0$, and $[-N, -N+K)$. The exclusion is speeded up by a factor N^2, births and deaths by a factor N. Assuming propagation of chaos (a property proved in a companion paper, [3]) we prove convergence in the limit $N \to \infty$ to the linear heat equation with Dirichlet condition on the boundaries; the boundary conditions however are not known a priori, they are obtained by solving a non linear equation. The model simulates mass transport with current reservoirs at the boundaries and the Fourier law is proved to hold.

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1. Introduction

Stationary non equilibrium states are characterized by the presence of steady currents flowing through the system and a basic question in statistical mechanics is to understand their structure. Many papers have been devoted to the subject, we just quote a few of them where the issue is addressed in the context of stochastic interacting particle systems, [1], [2], [4], [5]. We want to produce currents by acting only at the boundaries [of the region where our particle system is confined], so that the bulk dynamics is left unchanged. There are two natural ways to proceed. Thinking of a one dimensional system in an interval (this paper is indeed about the one dimensional symmetric simple exclusion process, SSEP) we may produce a current by sending in particles from the right at some small rate (i.e. inversely proportional to the size of the region) and taking out at same rate particles from the left. Alternatively we may add and subtract at unit rate (i.e. much faster than previously) particles on the right trying to keep fixed a given density ρ_+ close to the right boundary; same is done on the left and if the two densities $\rho_\pm$ are different, say $\rho_+ > \rho_-$, then we have

a positive density gradient which by the Fourier law induces a negative current (inversely proportional to the size of the region).
While the second mechanism has been much studied in the context of the Fourier law and to investigate the invariant measures when stationary currents are present, as in the references quoted above, to our knowledge the first mechanism has not been examined so far, even though it may look the most natural and direct to produce a current and our purpose here is to start its analysis in the simplest possible context. For this reason we consider the $d = 1$ SSEP in an interval $\Lambda_N = [-N, N]$, N a positive integer (we are interested in the asymptotics as $N \to \infty$). The process takes place in $\{0,1\}^{\Lambda_N}$ (at most one particle per site) and time is speeded up by a factor N^2 (to match the length of the interval Λ_N). Thus, independently each particle tries to jump at rate $N^2/2$ to each one of its n.n. sites, the jump then takes place if and only if the chosen site is empty, see the next section for a formal definition; jumps outside Λ_N are suppressed. To induce a current we modify the process by sending in from the right and taking out from the left particles at rate $Nj/2$, $j > 0$ a fixed parameter independent of N (to compare with the statements in the beginning we should divide the rate by N^2 because times have been speeded up by a factor N^2). As we want the boundary processes localized at the boundaries we fix two intervals $I_\pm$ of length K at the boundaries and we send in particles only in I_+ and take out particles only from I_-. It may however happen that I_+ is already full or I_- empty, then our mechanisms abort, so that the current really flowing in the system will not be exactly what desired (but hopefully close if K is large). This seems unavoidable if we insist to localize at the boundaries the birth-death processes or to consider lattice gases. In this paper K may be arbitrarily large but fixed; we cannot allow K to grow with N and a system of positive and negative charges (for instance a zero range process with positive and negative particles) requires a different approach.
In this paper we derive the hydrodynamic equations for our model (in the limit as $N \to \infty$) under the hypothesis of "propagation of chaos", a property proved in a companion paper, [3]. The hydrodynamic equation is the linear heat equation in the "macroscopic" interval $(-1, 1)$ with Dirichlet boundary conditions at time 0 and at ± 1: the values at ± 1 are not a-priori given, they are obtained by solving a coupled system of two non linear integral equations. We also prove the validity of the Fourier law; in particular the currents which enter and exit from the system are at all times equal to the local density gradient at ± 1.
We hope to continue this research project by studying the fluctuations field in the hydrodynamic limit and then the structure of the invariant measure in the limit as $N \to \infty$.

2. Model and main results

Notation and definitions.
Λ_N is the interval in $\mathbb{Z}$ with endpoints $\pm N$, denoted by $\Lambda_N := [-N, N]$. We write $\epsilon \equiv 1/N$, fix an integer $K \geq 1$, write $I_+ \equiv [N - K + 1, N]$ and $I_- \equiv [-N, -N + K - 1]$. Particle configurations are elements η of $\{0,1\}^{\Lambda_N}$, $\eta(x) = 0, 1$ being the occupation number at $x \in \Lambda_N$.

We shall study the Markov process on $\{0,1\}^{\Lambda_N}$ with generator $L_\epsilon := \epsilon^{-2}\big(L_0 + \epsilon L_b\big)$, where $L_b = L_{b,+} + L_{b,-}$ and

$$L_0 f(\eta) := \frac{1}{2}\sum_{x=-N}^{N-1}[f(\eta^{(x,x+1)}) - f(\eta)], \quad L_{b,\pm} f(\eta) := \frac{j}{2}\sum_{x\in I_\pm} D_\pm\eta(x)[f(\eta^{(x)}) - f(\eta)\Big], \tag{2.1}$$

$\eta^{(x)}$ being the configuration obtained from η by changing the occupation number at x, $\eta^{(x,x+1)}$ by exchanging the occupation numbers at $x, x+1$; for any $u : \Lambda_N \to [0,1]$

$$\begin{aligned} D_+u(x) &= [1-u(x)]u(x+1)u(x+2)\dots u(N), \quad x\in I_+ \\ D_-u(x) &= u(x)[1-u(x-1)][1-u(x-2)]\dots[1-u(-N)], \quad x\in I_-. \end{aligned} \tag{2.2}$$

L_0 is the generator of the SSEP (and of the stirring process as well). $L_{b,+}$ and $L_{b,-}$ are generators of birth respectively death processes, the former is active in I_+ the latter in I_-. The parabolic nature of the stirring process suggests to scale time as the square of space, hence the factor ϵ^{-2} in the definition of L_ϵ. It readily follows from the structure of the generators that the expectations $\mathbb{E}_\epsilon[\eta(x,t)]$ satisfy the relations

$$\frac{d}{dt}\mathbb{E}_\epsilon[\eta(x,t)] = \frac{1}{2}\Delta_\epsilon\mathbb{E}_\epsilon[\eta(x,t)] + \epsilon^{-1}\frac{j}{2}\Big(\mathbf{1}_{x\in I_+}\mathbb{E}_\epsilon\big[D_+\eta(x,t)\big] - \mathbf{1}_{x\in I_-}\mathbb{E}_\epsilon\big[D_-\eta(x,t)\big]\Big) \tag{2.3}$$

where $\Delta_\epsilon = \epsilon^{-2}\Delta$, Δ the discrete Laplacian in Λ_N with reflecting boundary conditions:

$$\begin{aligned} \Delta u(x) &= u(x+1) + u(x-1) - 2u(x), \qquad |x| < \epsilon^{-1} \\ \Delta u(\pm N) &= u(\pm(N-1),t) - u(\pm N,t). \end{aligned} \tag{2.4}$$

Propagation of chaos.
Due to the last term, (2.3) is not a closed equation in $\mathbb{E}_\epsilon[\eta(x,t)]$, but since the stirring generator is the leading term in L_ϵ and the invariant measures for the stirring process on the line are product Bernoulli measures, it looks natural to conjecture "propagation of chaos", i.e. that the measures at time $t > 0$ are approximately product (as $\epsilon \to 0$). If the law at time $t > 0$ were a true product measure, then, instead of (2.3) the expectations $\mathbb{E}_\epsilon[\eta(x,t)]$ would satisfy the closed equation:

$$\frac{d}{dt}\rho_\epsilon(x,t) = \frac{1}{2}\Delta_\epsilon\rho_\epsilon(x,t) + \epsilon^{-1}\frac{j}{2}\Big(\mathbf{1}_{x\in I_+}D_+\rho_\epsilon(x,t) - \mathbf{1}_{x\in I_-}D_-\rho_\epsilon(x,t)\Big), \tag{2.5}$$

which will be referred to as "the discretized time evolution".
The Cauchy problem for (2.5) with $[0,1]$–valued initial datum $\rho_\epsilon(\cdot,0)$ has a unique global solution which also takes values in $[0,1]$. Indeed (2.5) is a first order system of ordinary differential equations with polynomial non linearity hence local existence and uniqueness. Global existence follows because the solution has values in $[0,1]$, which in turns is a consequence of the fact that $D_\pm u(x)$ vanishes when $u(x) = 1$, respectively $u(x) = 0$. A formal proof is given in Proposition 3.1.

Hydrodynamic limit.
The first result in this paper (proved in Section 5) shows that ρ_ϵ converges as $\epsilon \to 0$ to a limit function which then identifies the hydrodynamics of the system.

Theorem 2.1. *Suppose that $\rho_\epsilon(x,0)$, $x \in \Lambda_N$, with values in $[0,1]$, converges weakly as $\epsilon \to 0$ to $\rho(r,0) \in L^\infty([-1,1],[0,1])$ in the sense that*

$$\lim_{\epsilon\to 0} \epsilon \sum_{x\in\Lambda_N} \rho_\epsilon(x,0)\phi(\epsilon x) = \int_{[-1,1]} \rho(r,0)\phi(r)dr, \quad \text{for any } \phi \in L^\infty([-1,1],\mathbb{R}). \tag{2.6}$$

Then there is $\rho(r,t)$, $r \in [-1,1]$, $t > 0$ so that for any $t_1 > t_0 > 0$:

$$\lim_{\epsilon\to 0} \sup_{x\in\Lambda_N} \sup_{t_0\le t\le t_1} |\rho_\epsilon(x,t) - \rho(\epsilon x,t)| = 0. \tag{2.7}$$

The function $\rho(r,t)$ solves and is the unique solution of the integral equation

$$\rho(r,t) = \int_{[-1,1]} P_t(r,r')\rho(r',0)dr' + \frac{j}{2}\int_0^t \Big\{P_s(r,1)(1-\rho(1,t-s)^K) \\ -P_s(r,-1)(1-(1-\rho(-1,t-s))^K)\Big\}ds, \tag{2.8}$$

where $P_t(r,r')$ is the density kernel of the semigroup (also denoted as P_t) with generator $\Delta/2$, Δ the laplacian in $[-1,1]$ with reflecting, Neumann, boundary conditions, (see the Remarks below).

Remarks.
- The density kernel $P_t(r,r')$ can be expressed in terms of the Gaussian kernel

$$G_t(r,r') = \frac{e^{-(r-r')^2/(2t)}}{\sqrt{2\pi t}}, \quad r,r' \in \mathbb{R}, \tag{2.9}$$

as follows: if $\psi : \mathbb{R} \to [-1,1]$ denotes the usual reflection map, i.e. $\psi(x) = x$ for $x \in [-1,1]$, $\psi(x) = 2-x$ for $x \in [1,3]$, with ψ extended to the whole line as periodic of period 4, then

$$\begin{aligned} P_t(r,r') &= \sum_{r'':\psi(r'')=r'} G_t(r,r'') \quad \text{for } r' \neq \pm 1 \\ P_t(r,\pm 1) &= \sum_{r'':\psi(r'')=\pm 1} 2G_t(r,r''). \end{aligned} \tag{2.10}$$

- From the expressions above and (2.8) it follows that $\rho(\cdot,t)$ is "smooth" for any $t > 0$: we are calling "smooth" a function $f(r)$, $r \in [-1,1]$, if it is C^∞ in $(-1,1)$, continuous in $[-1,1]$ and if for each n exist the limits $\dfrac{d^n f(r)}{dr^n}$ as $r \to \pm 1$.

• Since ρ is smooth we can write (2.8) in differential form: it then becomes the heat equation with Dirichlet boundary conditions:

$$\begin{aligned}&\frac{\partial}{\partial t}\rho(r,t)=\frac{1}{2}\frac{\partial^2}{\partial r^2}\rho(r,t), \qquad r\in(-1,1), t>0\\ &\rho(r,0)=u_0(r), \quad \rho(\pm 1,t)=u_\pm(t)\end{aligned} \tag{2.11}$$

However the boundary conditions $u_\pm(t)$ are not a-priori known, they must be obtained by solving a nonlinear system of two integral equations:

$$\begin{aligned}&u_\pm(t)=\int_0^t\{p(s)f_\pm(u_\pm(t-s))-q(s)f_\mp(u_\mp(t-s))\}ds+w_{\pm,t}\\ &f_+(u)=\frac{j}{2}\Big(1-u^K\Big), \quad f_-(u)=\frac{j}{2}\Big(1-(1-u)^K\Big),\end{aligned} \tag{2.12}$$

where, writing $G(r)=G(0,r)$, the latter as in (2.9),

$$\begin{aligned}&p(t)=2\sum_{k\in\mathbb{Z}}G_t(4k), \quad q(t)=2\sum_{k\in\mathbb{Z}}G_t(4k+2)\\ &w_{+,t}=\sum_{k\in\mathbb{Z}}\int_{-1}^1 u_0(r')2G_t(1-r'+4k)dr', \quad w_{-,t}=\sum_{k\in\mathbb{Z}}\int_{-1}^1 u_0(r')2G_t(r'+1+4k)dr'\end{aligned} \tag{2.13}$$

• By a simple computation one can check that

$$\frac{\partial\rho(r,t)}{\partial r}|_{r=1}=j(1-\rho(1,t)^K), \quad \frac{\partial\rho(r,t)}{\partial r}|_{r=-1}=j(1-(1-\rho(-1,t))^K). \tag{2.14}$$

This remark will be important in the analysis of the Fourier law.

• To characterize the asymptotic behavior of the invariant measure as $N\to\infty$ it will be important to study the evolution starting from arbitrary initial configurations $\eta^{(N)}$. Since the functions $f^{(N)}(r):=\eta^{(N)}([\epsilon^{-1}r])$ are in a ball of $L^2([-1,1],\mathbb{R})$ they converge weakly by subsequences to an element of $L^2([-1,1],\mathbb{R})$ and we can then apply to any convergent subsequence Theorem 2.1.

• The identification of (2.8) and (2.11) as the hydrodynamic equation of the system is based on the assumption that ρ_ϵ gives an accurate description of the process. This is indeed correct because the "empirical averages" are close to the functions ρ_ϵ in the following sense. There is $\tau>0$ so that calling $J_M(x)=[x-M,x+M]\cap\Lambda_N$, M the integer part of N^a, $a\in(0,1)$, then for any $t_0>0$

$$\lim_{\delta\to 0}\lim_{\epsilon\to 0}\sup_{t_0\le t\le\tau\log\epsilon^{-1}}\sup_\eta\mathbb{P}_\epsilon\Big[\sup_{x\in\Lambda_N}|\frac{1}{|J_M(x)|}\sum_{y\in J_M(x)}\{\eta(y,t)-\rho_\epsilon(y,t)\}|\ge\delta\Big]=0 \tag{2.15}$$

(2.15) follows from Theorem 6.1 using the Chebishev inequality, Theorem 6.1 is proved in a companion paper, [3].

Fourier law.
In the "hydrodynamic limit literature" the limit function $\rho(r,t)$ is usually interpreted as the "density profile" at time t: this comes from attributing to each particle a mass ϵ so that $\mathbb{E}_\epsilon[\epsilon\eta(x,t)]$ is the average mass in the interval $[x-\frac{1}{2}, x+\frac{1}{2}]$ which in macroscopic units has length ϵ (as $[-N,N]$ in the macroscopic limit shrinks to $[-1,1]$). Thus $\mathbb{E}_\epsilon[\eta(x,t)]$ is the mass density, which in the limit converges to $\rho(r,t)$ (when $\epsilon x \to r$). Analogously, the expected current through a point is the average *signed* mass crossing that point per unit time. Let x be away from the boundaries in the sense that $|x| \le N-K$. Then it follows from (2.1) that the expected current through $x+\frac{1}{2}$ is

$$j^{(\epsilon)}(x,t) = \frac{\epsilon^{-2}}{2}\,\mathbb{E}_\epsilon\big[\epsilon\{\eta(x,t)-\eta(x+1,t)\}\big] = -\frac{1}{2}\mathbb{E}_\epsilon\big[\frac{\eta(x+1,t)-\eta(x,t)}{\epsilon}\big]. \tag{2.16}$$

By a similar argument the expected currents through N and $-N$ are:

$$j^{(\epsilon)}_\pm(t) = -\frac{\epsilon^{-1}j}{2}\sum_{y\in I_\pm}\mathbb{E}_\epsilon\Big[\epsilon D_\pm(\eta(y,t))\Big] = -\frac{j}{2}\sum_{y\in I_\pm}\mathbb{E}_\epsilon\Big[D_\pm(\eta(y,t))\Big]. \tag{2.17}$$

By (2.16) $j^{(\epsilon)}(x,t)$, $|x|\le N-K$, is equal to $-\frac{1}{2}$ times the discrete gradient of the density in agreement with the Fourier's law, which is then satisfied before the macroscopic limit $\epsilon\to 0$, (but not necessarily in the limit, as this requires that the limit of the derivative is the derivative of the limit). One would expect that also $j^{(\epsilon)}_\pm(t)$ are equal to $-\frac{1}{2}$ times the discrete gradient of the density, at least in the limit as $\epsilon\to 0$. This is settled in the next theorem where using the factorization properties proved in [3] we show that the limit of the current is both in the bulk and at the boundaries equal to $-\frac{1}{2}$ times the gradient of the density.

Theorem 2.2 (Validity of the Fourier law)**.** *Suppose that the process starts with a product measure μ^ϵ such that $\mu^\epsilon[\eta(x)=1]=\rho(\epsilon x,0)$ with $\rho(\cdot,0)\in C([-1,1],[0,1])$ with bounded derivative in $(-1,1)$. Let $\rho_\epsilon(x,t)$ be the solution of (2.5) starting from $\rho_\epsilon(x,0)=\rho(\epsilon x,0)$. Then, for any $t\ge 0$ and $r\in(-1,1)$, denoting by $[u]$ the integer part of u, we have*

$$\lim_{\epsilon\to 0} j^{(\epsilon)}([\epsilon^{-1}r],t) = -\frac{1}{2}\,\frac{d\rho_t(r)}{dr}. \tag{2.18}$$

Moreover for any $t>0$

$$\lim_{\epsilon\to 0} j^{(\epsilon)}_\pm(t) = -\frac{1}{2}\,\frac{d\rho_t(r)}{dr}\Big|_{r=\pm 1}. \tag{2.19}$$

Theorem 2.2 is proved in Section 6.

3. The discretized evolution

We begin the analysis of (2.5) by proving:

Proposition 3.1. *The Cauchy problem for* (2.5) *with initial datum* $\rho_\epsilon(x,0) \in [0,1]$ *has a unique global solution* $\rho_\epsilon(x,t)$*. Moreover* $\rho_\epsilon(x,t)$ *takes values* $[0,1]$*.*

Proof. Write

$$D_+^* u(x) = (1-u(x))\Big|u(x+1)u(x+2)\dots u(N)\Big|, \quad x \in I_+$$
$$D_-^* u(x) = u(x)\Big|(1-u(x-1))(1-u(x-2))\dots(1-u(-N))\Big|, \quad x \in I_-$$

If $0 \le u(x) \le 1$ then $D_\pm^* \equiv D_\pm$. A local existence and uniqueness theorem holds for the Cauchy problem (2.5) as well as for the problem with $D_\pm$ replaced by $D_\pm^*$ (as these are Lipschitz functions of u in the sup-norm topology). Denote the solution of the latter by $\rho_\epsilon^*(x,t)$, $t \le \tau$, $\tau > 0$, recalling that the initial datum $\rho_\epsilon(x,0)$ verifies $0 \le \rho_\epsilon(x,0) \le 1$ for any $x \in \Lambda_N$. We shall next prove that $0 \le \rho_\epsilon^*(x,t) \le 1$ for all x and $t \le \tau$. Define $u(s) = \max_{x\in\Lambda_N} \rho^*(x,s)$ and suppose by contradiction that there is $T \le \tau$ such that $u(T) > 1$. Then there is $t \le T$ so that (i) $u(t) > 1$ and (ii) $du(t)/dt > 0$ (because $u(0) \le 1$ and it cannot be that $du(s)/ds \le 0$ for almost all $s \le T$ such that $u(s) > 1$). Moreover there exists x such that (a) $\rho^*(x,t) = u(t)$ and (b) $du(t)/dt = d\rho^*(x,t)/dt$. All that leads to a contradiction because $d\rho^*(x,t)/dt = \frac{1}{2}\Delta_\epsilon \rho^*(x,t) + \frac{j}{2}(D_+^* - D_-^*)\rho^*(x,t) \le 0$. Indeed $\Delta_\epsilon \rho_\epsilon^*(x,t) \le 0$, because (x,t) maximizes $\rho_\epsilon^*(\cdot,t)$. $D_+^*\rho_\epsilon^*(x,t) = 0$ if $x \notin I_+$ and ≤ 0 in I_+, because $\rho_\epsilon^*(x,t) > 1$. $D_-^*\rho_\epsilon^* = 0$ if $x \notin I_-$ and ≥ 0 in I_-, because $\rho_\epsilon^*(x,t) \ge 0$. Thus $(D_+^* - D_-^*)\rho_\epsilon^*(x,t) \le 0$. Analogous arguments show that the solution cannot exit $[0,1]$ through 0, hence $\rho_\epsilon^*(x,t) \in [0,1]$. As a consequence $D_\pm^*\rho_\epsilon^* = D_\pm\rho_\epsilon^*$ and therefore ρ_ϵ^* solves (2.5) as well. By iteration, the previous argument extends to all times. □

We shall study (2.5) in its integral form:

$$\rho_\epsilon(x,t) = \sum_{y\in\Lambda_N} P_t^{(\epsilon)}(x,y)\rho_\epsilon(y,0) + \epsilon^{-1}\frac{j}{2}\int_0^t ds\Big(\sum_{y\in I_+} P_s^{(\epsilon)}(x,y)D_+\rho_\epsilon(y,t-s) - \sum_{y\in I_-} P_s^{(\epsilon)}(x,y)D_-\rho_\epsilon(y,t-s)\Big) \tag{3.1}$$

where $P_t^{(\epsilon)}$ is the semigroup with generator $\frac{1}{2}\Delta_\epsilon$, $P_t^{(\epsilon)}(x,y)$ its kernel:

$$P_t^{(\epsilon)} := e^{\frac{1}{2}\Delta_\epsilon t}, \qquad P_t^{(\epsilon)}(x,y) = P_t^{(\epsilon)}(y,x) \tag{3.2}$$

The analysis of (3.1) will exploit the nice regularity properties of $P_t^{(\epsilon)}(x,y)$ which are established in the next section.

4. Probability estimates for a random walk with reflections

In this section we shall consider a simple random walk on Λ_N which jumps with intensity $\epsilon^{-2}/2$ to each of its n.n. sites, the jumps outside Λ_N being suppressed. We denote by $P_t^{(\epsilon)}$

its law and call $Q_t^{(\epsilon)}$ the law of the corresponding unrestricted random walk on the whole $\mathbb{Z}$. In the sequel we shall prove (in many cases just recall) bounds and estimates on $P_t^{(\epsilon)}$ which will be used in the next sections to prove Theorems 2.1 and 2.2. We start by relating $P_t^{(\epsilon)}$ and $Q_t^{(\epsilon)}$, through a "reflection map" from $\mathbb{Z}$ to Λ_N which is a discrete analogue of the map ψ defined in the first remark after Theorem 2.1. Since the jump rate from $\pm N$ is $\epsilon^{-2}/2$ for $P_t^{(\epsilon)}$ and ϵ^{-2} for $Q_t^{(\epsilon)}$ to relate the two it just suffices to identify $N+1$ with N (as well as $-N-1$ with $-N$), then the jumps of $Q_t^{(\epsilon)}$ from N to $N+1$ and $-N$ to $-N-1$ are like suppressed. We thus define:

Definition. The "reflection map" $\psi_N : \mathbb{Z} \to \Lambda_N$ is:

- $|x| \le N$: $\psi_N(x) = x$
- $x < -N$: $\psi_N(x) = -\psi_N(-x)$
- $x > N$: $\psi_N(N+k) = N-(k-1)$, for $k = 1, \dots, 2N+1$,
  $\psi_N\big((N+2N+1)+k\big) = -N+(k-1)\big)$, for $k = 1, \dots, 2N+1$,
  $\psi_N\big((N+2(2N+1)+k\big) = N-(k-1)\big)$, for $k = 1, \dots, 2N+1$ and so on.

Proposition 4.1. *With the above notation,*

$$P_t^{(\epsilon)}(x,z) = \sum_{y:\psi_N(y)=z} Q_t^{(\epsilon)}(x,y) \tag{4.1}$$

Proof. Let f be ψ_N measurable, i.e. $f(\psi_N(x)) = f(x)$, and g the function on Λ_N defined by setting $g(x) = f(x)$, $x \in \Lambda_N$. Calling L_Q and L_P the generators of $Q_t^{(\epsilon)}$ and $P_t^{(\epsilon)}$ we have

$$L_Q f(x) = L_P g(\psi_N(x))$$

so that $e^{L_Q t} f = e^{L_P t} g$, hence (4.1). □

By the local central limit theorem (see for instance [6]):

Theorem 4.2. *There exist positive constants $c_1, ..., c_5$ so that*

$$\begin{aligned} &|Q_t^{(\epsilon)}(x,y) - G_{\epsilon^{-2}t}(x,y)| \le \frac{c_1}{\sqrt{\epsilon^{-2}t}} G_{\epsilon^{-2}t}(x,y), \quad |x-y| \le (\epsilon^{-2}t)^{5/8}, \\ &Q_t^{(\epsilon)}(x,y) \le \min\Big\{c_2 e^{-c_3|x-y|^2/(\epsilon^{-2}t)}, c_4 e^{-|y-x|(\log|y-x|-c_5)}\Big\}, \quad |x-y| > (\epsilon^{-2}t)^{5/8}, \end{aligned} \tag{4.2}$$

G_t being the Gaussian kernel defined in (2.9).

The next corollary follows directly from Theorem 4.2 and Proposition 4.1.

Corollary 4.3. *For any $T > 0$ there exist c so that the following holds.*

- *For all ϵ, all $t \in (0, T]$ and all x, y in Λ_N,*
$$P_t^{(\epsilon)}(x, y) \le c\, G_{\epsilon^{-2}t}(x, y). \tag{4.3}$$
- *For all ϵ, all $t \in (0, T]$ and all $-N \le x \le N - 1$,*
$$\Big| P_t^{(\epsilon)}(x, y) - P_t^{(\epsilon)}(x + 1, y) \Big| \le \frac{c}{\sqrt{\epsilon^{-2}t}} G_{\epsilon^{-2}t}(x, y). \tag{4.4}$$
- *For all ϵ, all $t \in (0, T]$, $s > 0$ and all $x \in \Lambda_N$,*
$$\sum_{y \in \Lambda_N} \Big| P_{t+s}^{(\epsilon)}(x, y) - P_t^{(\epsilon)}(x, y) \Big| \le c\sqrt{\frac{s}{t}}. \tag{4.5}$$
$$\Big| P_{t+s}^{(\epsilon)}(x, y) - P_t^{(\epsilon)}(x, y) \Big| \le c\frac{\sqrt{\epsilon^{-2}s}}{\epsilon^{-2}t}. \tag{4.6}$$

Proof. (4.3) and (4.4) follow directly from (4.2). By (4.3) and (4.4) we can bound the left hand side of (4.5) by
$$\sum_{y \in \Lambda_N} \sum_{z \in \Lambda_N} P_s^{(\epsilon)}(x, z) \Big| P_t^{(\epsilon)}(z, y) - P_t^{(\epsilon)}(x, y) \Big| \le c \sum_{z \in \Lambda_N} G_{\epsilon^{-2}s}(x, z) \frac{|z - x|}{\sqrt{\epsilon^{-2}t}}$$
hence (4.5). (4.6) is obtained similarly, recalling that $G_t(x, y) \le ct^{-1/2}$. □

In the proof of Theorem 2.1 we shall use the following convergence results:

Lemma 4.4. *As in Theorem 2.1 suppose that $\rho_\epsilon(\cdot, 0)$ converges weakly to $\rho(\cdot, 0)$. Then, for any $t > 0$*
$$\lim_{\epsilon \to 0} \sup_{x \in \Lambda_N} \Big| \sum_{y \in \Lambda_N} P_t^{(\epsilon)}(x, y) \rho_\epsilon(y, 0) - \int_{[-1,1]} P_t(\epsilon x, r) \rho(r, 0) dr \Big| = 0. \tag{4.7}$$

Proof. By (4.4) the family of functions $f_\epsilon(r)$:
$$f_\epsilon(r) := \sum_{y \in \Lambda_N} P_t^{(\epsilon)}([\epsilon^{-1} r], y) \rho_\epsilon(y, 0), \quad r \in [-1, 1] \tag{4.8}$$
is uniformly Lipschitz, it will therefore suffice to prove pointwise convergence. We thus fix $r^* \in [-1, 1]$ and take $x = [\epsilon^{-1} r^*]$. By (4.1) and (4.2)
$$\begin{aligned} \sum_{y \in \Lambda_N} P_t^{(\epsilon)}(x, y) \rho_\epsilon(y, 0) &= \sum_{y \in \Lambda_N} \rho_\epsilon(y, 0) \sum_{z : \psi_N(z) = y} Q_t^{(\epsilon)}(x, z) \\ &= \sum_{y \in \Lambda_N} \rho_\epsilon(y, 0) \sum_{z : \psi_N(z) = y} G_{\epsilon^{-2}t}(x, z) + O\Big((\epsilon^{-2}t)^{-1/2} \Big) + O\Big(e^{-\epsilon^{-1}} \Big) \end{aligned}$$

Call Ψ_N the discrete analogue of the reflection map ψ of the Remarks after Theorem 2.1, i.e. $\Psi_N(x) = N\psi(x/N)$ for $x \in \Lambda_N$. It differs from ψ_N by shifts and we have:

$$\Big| \sum_{z:\psi_N(z)=y} G_{\epsilon^{-2}t}(x,z) - \sum_{z:\Psi_N(z)=y} G_{\epsilon^{-2}t}(x,z) \Big| \le \frac{c}{\sqrt{\epsilon^{-2}t}}$$

But (see (2.10))

$$\sum_{z:\Psi_N(z)=y} G_{\epsilon^{-2}t}(x,z) = \sum_{r':\psi(r')=\epsilon y} \epsilon G_t(\epsilon x, r') = \epsilon P_t(\epsilon x, \epsilon y)$$

and we conclude that

$$\Big| \sum_{y\in\Lambda_N} P_t^{(\epsilon)}(x,y)\rho_\epsilon(y,0) - \epsilon \sum_{y\in\Lambda_N} P_t(\epsilon x, \epsilon y)\rho_\epsilon(y,0) \Big| \le \frac{c}{\sqrt{\epsilon^{-2}t}}.$$

Let $x = [\epsilon^{-1}r^*]$, then by (2.6) with $\phi(r) := P_t(r^*, r)$ we have

$$\lim_{\epsilon\to 0} \sum_{y\in\Lambda_N} P_t^{(\epsilon)}([\epsilon^{-1}r^*], y)\rho_\epsilon(y,0) = \int_{[-1,1]} P_t(r^*, r)\rho(r,0)dr \tag{4.9}$$

which proves pointwise convergence and hence the lemma as argued at the beginning of the proof. □

Lemma 4.5. *Let $h_\epsilon(t)$ be a continuous function with values in $[0,1]$ which converges pointwise to $h(t)$. Then for any $t > 0$, $r \in [-1,1]$ and $y \in I_+$*

$$\lim_{\epsilon\to 0} \int_0^t \epsilon^{-1} P_s^{(\epsilon)}([\epsilon^{-1}r], \pm y) h_\epsilon(t-s)ds = \int_0^t P_s(r, \pm 1)h(t-s)ds. \tag{4.10}$$

Proof. Again, this follows easily from (4.1) and (4.2), after recalling also (2.10). Details are omitted. □

5. Proof of the hydrodynamic limit

In this section we shall prove Theorem 2.1. We start by proving equicontinuity which is a direct consequence of the estimates of the previous section:

Proposition 5.1. *For any $T > 0$ there is a constant c so that for any solution $\rho_\epsilon(x,t)$ of (2.5) with $\rho_\epsilon(\cdot,0) \in [0,1]$ the following holds. For any $x \in [-N, N-1]$, any $t \in (0,T]$ and any $\epsilon > 0$*

$$|\rho_\epsilon(x,t) - \rho_\epsilon(x+1,t)| \le \min\Big\{1, c\Big(\epsilon \log_+(\epsilon^{-2}t) + \frac{1}{\sqrt{\epsilon^{-2}t}}\Big)\Big\} \tag{5.1}$$

where $\log_+ u = \max\{\log u, 1\}$. *For any* $0 < s < t$, $x \in \Lambda_N$ *and* $\epsilon > 0$*:*

$$|\rho_\epsilon(x,t) - \rho_\epsilon(x,t+s)| \le \min\Big\{1, c\Big(\sqrt{\frac{s}{t}} + \sqrt{s}\log(\frac{t}{s})\Big)\Big\} \tag{5.2}$$

Proof. By (4.4)

$$|\sum_y \big(P_t^{(\epsilon)}(x,y) - P_t^{(\epsilon)}(x+1,y)\big)\rho_\epsilon(y,0)| \le \frac{c}{\sqrt{\epsilon^{-2}t}}$$

and for any $y \in I_+ \cup I_-$,

$$|\int_0^t \epsilon^{-1}|P_s^{(\epsilon)}(x,y) - P_s^{(\epsilon)}(x+1,y)|ds \le \epsilon + \int_{\epsilon^2}^t \frac{c\epsilon^{-1}}{\epsilon^{-2}s}ds$$

hence (5.1). By (4.5)

$$|\sum_y \big(P_{t+s}^{(\epsilon)}(x,y) - P_t^{(\epsilon)}(x,y)\big)\rho_\epsilon(y,0)| \le c\sqrt{\frac{s}{t}}$$

By (4.6) for any $y \in I_+ \cup I_-$ and denoting by $f(t) := D_\pm \rho_\epsilon(y,t)$, $y \in I_\pm$,

$$\begin{aligned}
&\epsilon^{-1}\Big|\int_0^{t+s} P_{s'}^{(\epsilon)}(x,y)f(t+s-s')ds' - \int_0^t P_{s'}^{(\epsilon)}(x,y)f(t-s')ds'\Big| \\
&\quad \le \epsilon^{-1}\Big(\int_0^{2s} P_{s'}^{(\epsilon)}(x,y)ds' + \int_0^s P_{s'}^{(\epsilon)}(x,y)ds' + \int_s^t |P_{s'+s}^{(\epsilon)}(x,y) - P_{s'}^{(\epsilon)}(x,y)|ds'\Big) \\
&\quad \le c\Big(\sqrt{s} + \sqrt{s}\log(\frac{t}{s})\Big)
\end{aligned} \tag{5.3}$$

□

We now turn to the proof of Theorem 2.1, we fix $T > 0$ and study the evolution in the finite time interval $[0,T]$. Since we only have that $\rho_\epsilon(\cdot,0) \to \rho(\cdot,0)$ weakly, it is convenient to introduce a regularized equation. We denote by $\rho_\epsilon(x,t|u,s)$, $t \ge s \ge 0$, the solution of (2.5) for $t \ge s$ with u the initial datum at time s, $u = u(x)$, $x \in \Lambda_N, u(x) \in [0,1]$. With such notation we then set for any $\delta \in (0,T)$

$$\rho_\epsilon^{(\delta)}(x,t) = \begin{cases} \sum_{y\in\Lambda_N} P_t^{(\epsilon)}(x,y)\rho_\epsilon(y,0) & 0 \le t \le \delta \\ \rho_\epsilon\big(x,t|\rho_\epsilon^{(\delta)}(\cdot,\delta),\delta\big) & t \in (\delta,T]. \end{cases} \tag{5.4}$$

By Proposition 5.1 the family of functions $(r,t) \to \rho_\epsilon^{(\delta)}([\epsilon^{-1}r],t)$, $r \in [-1,1], t \in [\delta,T]$ is equicontinuous and bounded, hence it converges in sup norm by subsequences to a limit function $u^{(\delta)}(r,t)$. By Lemma 4.4

$$u^{(\delta)}(r,\delta) = \int_{[-1,1]} P_\delta(r,r')\rho(r',0)dr'. \tag{5.5}$$

Moreover for any integer $0 \le m \le K$

$$\lim_{\epsilon\to 0}\sup_{\delta\le t\le T}\Big|D_+\rho_\epsilon^{(\delta)}(N-m,t)-\big(1-u^{(\delta)}(1,t)\big)u^{(\delta)}(1,t)^m\}\Big|=0$$
$$\lim_{\epsilon\to 0}\sup_{\delta\le t\le T}\Big|D_-\rho_\epsilon^{(\delta)}(-N+m,t)-u^{(\delta)}(-1,t)\big(1-u^{(\delta)}(-1,t)\big)^m\}\Big|=0. \tag{5.6}$$

By Lemma 4.4 and Lemma 4.5 it then follows that

$$u^{(\delta)}(r,t)=u(r,t|u^{(\delta)}(\cdot,\delta),\delta), \tag{5.7}$$

where the latter is the solution of the limit equation in the time interval $[\delta,T]$ with initial datum at time δ equal to $u^{(\delta)}(\cdot,\delta)$. Uniqueness can be easily proved, it follows also from (5.9) below where we prove that the solution depends continuously on the initial datum. By uniqueness it then follows that $\rho_\epsilon^{(\delta)}$ converges in sup norm to $u^{(\delta)}$ as $\epsilon\to 0$ and not only by subsequences.
We shall next study the dependence on δ and define for $t\in[\delta,T]$

$$h_\epsilon^{(\delta)}(t)=\sup_{x\in\Lambda_N}|\rho_\epsilon^{(\delta)}(x,t)-\rho_\epsilon(x,t)|,\quad h^{(\delta)}(t):=\sup_{|r|\le 1}|u^{(\delta)}(r,t)-u(r,t)| \tag{5.8}$$

We are going to prove that there is c_T so that for all ϵ and δ positive

$$h_\epsilon^{(\delta)}(t)+h^{(\delta)}(t)\le c_T\sqrt{\delta} \tag{5.9}$$

(which in particular implies uniqueness of the solution of (2.8)). It follows from (5.9) that

$$\limsup_{\epsilon\to 0}\sup_{t\in[\delta,T]}\sup_{x\in\Lambda_N}\big|\rho_\epsilon(x,t)-u(\epsilon x,t)\big|\le c\sqrt{\delta} \tag{5.10}$$

which then proves Theorem 2.1.

Proof of (5.9). From (4.3) it follows that $h_\epsilon^{(\delta)}(\delta)\le c\sqrt{\delta}$, then using again (4.3),

$$h_\epsilon^{(\delta)}(t)\le c\sqrt{\delta}+C\int_\delta^t\frac{1}{\sqrt{s}}h_\epsilon^{(\delta)}(t-s)ds. \tag{5.11}$$

By iteration,

$$h_\epsilon^{(\delta)}(t)\le c\sqrt{\delta}\Big(1+\sum_{n=1}^\infty C^n a_n(t-\delta)\Big)$$
$$a_n(t):=\int_0^t\frac{1}{\sqrt{s_1}}ds_1\int_0^{t-s_1}\frac{1}{\sqrt{s_2}}ds_2\dots\int_0^{t-s_1\cdots-s_{n-1}}\frac{1}{\sqrt{s_n}}ds_n$$

By (5.12) below we then have $h_\epsilon^{(\delta)}(t)\le c'\sqrt{\delta}\Big(1+e^{\pi C^2T}\Big)$. Same argument applies to $h^{(\delta)}(t)$ and (5.9) is proved.

Lemma 5.2. *With $a_n(t)$ as above,*

$$a_n(t) \le (\pi t)^{\frac{n}{2}} e^{-\frac{n}{2}[\log(\frac{n}{2})-1]} \tag{5.12}$$

Proof. We have

$$a_n(t) = \int_{[0,t]^n} \mathbf{1}_{s_1+\cdots+s_n \le t} \prod_{i=1}^n \frac{1}{\sqrt{s_i}}\ ds_1 \dots ds_n. \tag{5.13}$$

We change variables by setting $s_i = t_i\, t$ and get

$$a_n(t) = (\sqrt{t})^n \int_{[0,1]^n} \mathbf{1}_{t_1+\cdots+t_n \le 1} \prod_{i=1}^n \frac{1}{\sqrt{t_i}}\ dt_1 \dots dt_n. \tag{5.14}$$

Multiplying and dividing by $\exp\{-\alpha(t_1+\cdots+t_n)\}$ we have

$$a_n(t) \le (\sqrt{t})^n e^{\alpha} \int_{[0,1]^n} \prod_{i=1}^n \frac{e^{-\alpha t_i}}{\sqrt{t_i}}\, dt_1 \dots dt_n \le (\sqrt{t})^n e^{\alpha} \Big[\int_0^1 \frac{e^{-\alpha s}}{\sqrt{s}}\, ds \Big]^n \le (\sqrt{t})^n e^{\alpha} \big(\frac{\sqrt{\pi}}{\sqrt{\alpha}}\big)^n \tag{5.15}$$

By choosing $\alpha = \dfrac{n}{2}$ we get (5.12). □

6. Proof of the Fourier law

In this section we shall prove Theorem 2.2. The proof relies on Theorem 6.1 below which is proved in [3]. Writing $\Lambda_N^{n,\neq}$, $n \ge 1$, for the set of all sequences $\underline{x} = (x_1, .., x_n)$ in Λ_N^n with distinct entries, we first define the v-functions

$$v^\epsilon(\underline{x}, t|\mu^\epsilon) := \mathbb{E}_\epsilon \Big[\prod_{i=1}^n \{\eta(x_i,t) - \rho_\epsilon(x_i,t)\} \Big], \quad \underline{x} \in \Lambda_N^{n,\neq},\ n \ge 1 \tag{6.1}$$

where the process starts with a product measure μ^ϵ, in particular a single configuration, and $\rho_\epsilon(x,t)$ is the solution of (2.5) with initial datum $\rho_\epsilon(x,0) = \mu^\epsilon[\eta(x,0)=1]$.

Theorem 6.1. *There exist $\tau > 0$ and $c^* > 0$ so that the following holds. For any $\beta^* > 0$ and for any positive integer n there is a constant $c_n < \infty$ so that for any $\epsilon > 0$, any initial product measure μ^ϵ*

$$\sup_{\underline{x} \in \Lambda_N^{n,\neq}} |v^\epsilon(\underline{x}, t|\mu^\epsilon)| \le \begin{cases} c_n(\epsilon^{-2}t)^{-c^* n}, & t \le \epsilon^{\beta^*} \\ c_n \epsilon^{(2-\beta^*)c^* n} & \epsilon^{\beta^*} \le t \le \tau \log \epsilon^{-1} \end{cases} \tag{6.2}$$

Proof of (2.19). Recalling (2.17) and applying Theorem 6.1 we have:

$$\lim_{\epsilon\to 0} j_+^{(\epsilon)}(t) = -\frac{j}{2}(1-\rho(1,t)^K), \ \ \lim_{\epsilon\to 0} j_-^{(\epsilon)}(t) = -\frac{j}{2}(1-(1-\rho(-1,t))^K) \tag{6.3}$$

and (2.19) follows from (2.14).

Proof of (2.18). We can express $\rho(\cdot,t)$ using the Green function for (2.11) and get

$$\rho(r,t)=\tilde P_t\rho(r,0)+\int_0^t\{\mathbb P_{s,r,1}(ds)\rho(1,t-s)+\mathbb P_{s,r,-1}(ds)\rho(-1,t-s)\},\tag{6.4}$$

where $\tilde P_t\rho(r,0)=\mathbb E_r\big(\rho(B(t),0)\mathbf 1_{\tau>t}\big)$ with $B(t)$ the standard Brownian motion starting from r, and τ the hitting time of $\{-1,1\}$; $\mathbb P_{s,r,\pm1}(ds)$ are the corresponding hitting time distributions. Since $\rho'(r,t):=\partial\rho(r,t)/\partial r$ satisfies the heat equation we can write similarly to (6.4)

$$\rho'(r,t)=\tilde P_t\rho'(r,0)+\int_0^t\{\mathbb P_{s,r,1}(ds)\rho'(1,t-s)+\mathbb P_{s,r,-1}(ds)\rho'(-1,t-s)\}\tag{6.5}$$

with $\rho'(\pm1,t-s)$ explicitly given in (2.14). The idea then is to write $j^{(\epsilon)}(x,t)$ (which is defined in (2.16)) in a similar way. We are going to prove that

$$j^{(\epsilon)}(x,t)=-\frac{\bar\phi_\epsilon(x+1,t)}{2}-\frac12\int_0^t\Big\{\Theta_+^{(\epsilon)}(t-s)\mathbb P_{x,\epsilon;+}(ds)+\Theta_-^{(\epsilon)}(t-s)\mathbb P_{x,\epsilon;-}(ds)\Big\}\tag{6.6}$$

with

$$\begin{aligned}\Theta_+^{(\epsilon)}(t)&=\epsilon^{-1}\Big(E_\epsilon[\eta(N-K,t)]-E_\epsilon[\eta(N-K-1,t)]\Big)\\ \Theta_-^{(\epsilon)}(t)&=\epsilon^{-1}\Big(E_\epsilon[\eta(-N+K+1,t)]-E_\epsilon[\eta(-N+K,t)]\Big)\end{aligned}\tag{6.7}$$

and

$$\bar\phi_\epsilon(x+1,t)=\epsilon^{-1}\sum_y\Big(P_{x+1}(y(t)=y,\tau>t)-P_x(y(t)=y,\tau'>t)\Big)\rho_\epsilon(y,0),\tag{6.8}$$

where we have used the following notation: $y(t)$ is a random walk with transition kernel $P^{(\epsilon)}(x,y)$ on Λ_N, P_x its law when $y(0)=x$, τ its first hitting time of $\{-N+K+1,N-K\}$ and τ' the hitting time of $\{-N+K,N-K-1\}$.
To prove (6.6) we use (2.3), observing that when $x\notin I_-\cup I_+$, the second term on the r.h.s. of (2.3) vanishes, so that

$$\frac{d}{dt}\mathbb E_\epsilon[\eta(x,t)]=\frac12\Delta_\epsilon\mathbb E_\epsilon[\eta(x,t)],\quad |x|\le N-K.$$

This allows to express $g_\epsilon(x,t):=\mathbb E_\epsilon[\eta(x,t)]=P_x[g_\epsilon(y(\bar\tau\wedge t),t-\bar\tau\wedge t)]$ where here P_x refers to the expectation with respect to a random walk $y(\cdot)$ that starts at x (time running backward for $y(\cdot)$) and $\bar\tau$ is the first time it reaches $\{-N+K,N-K\}$, which corresponds to the (time variable) boundary condition. Doing the same for each of the terms in $E_\epsilon[\eta(x,t)]-E_\epsilon[\eta(x+1,t)]$ we arrive to (6.6).

We need to compare (6.5) and (6.6) recalling (2.14). By the weak convergence of the random walk to the Brownian motion $\tilde\phi(x+1,t)$, $x=[\epsilon^{-1}r]$, converges to $\tilde P_t\rho_0'(r)$ and $\mathbb P_{x,\epsilon;\pm}(ds)$

converges weakly to $\mathbb{P}_{r,;\pm}(ds)$ ($x = [\epsilon^{-1}r]$). Therefore, recalling (2.14), (6.5) and (6.6), the proof of (2.18) will follow from: for any $t > 0$

$$\lim_{\epsilon\to 0}\Theta_+^{(\epsilon)}(t) = j(1-\rho(1,t)^K),\ \ \lim_{\epsilon\to 0}\Theta_-^{(\epsilon)}(t) = j(1-(1-\rho(-1,t))^K) \tag{6.9}$$

which will be proved in the remaining part of this section. As the analysis of $\Theta_\pm^{(\epsilon)}(t)$ are similar we shall only prove (6.9) for $\Theta_+^{(\epsilon)}(t)$.

Recalling (2.3) we can write:

$$\Theta_+^{(\epsilon)}(t) = \phi_\epsilon(N-K,t) + \sum_{y\in I_+}\Gamma_{\epsilon,t,y} - \sum_{y\in I_-}\Gamma_{\epsilon,t,y} \tag{6.10}$$

where

$$\phi_\epsilon(x,t) := \epsilon^{-1}\sum_{y\in\Lambda_N}\Big(P_t^{(\epsilon)}(x,y) - P_t^{(\epsilon)}(x-1,y)\Big)\rho_\epsilon(y,0), \qquad x\in\Lambda_N, t>0 \tag{6.11}$$

and for $y\in I_\pm$, respectively,

$$\Gamma_{\epsilon,t,y} := \epsilon^{-2}\int_0^t ds\left(P_s^{(\epsilon)}(N-K,y) - P_s^{(\epsilon)}(N-K-1,y)\right)\mathbb{E}_\epsilon\left(\frac{j}{2}(D_\pm\eta(\cdot,t-s))(y)\right). \tag{6.12}$$

As the analysis of (6.10) will involve several steps, we give first an outline.

- We shall first prove that $\phi_\epsilon(N-K,t)$ vanishes as $\epsilon\to 0$ (this will be simple).
- We will then show that also $\Gamma_{\epsilon,t,y}$ with $y\in I_-$ vanishes as $\epsilon\to 0$. This is less simple and involves couplings of random walks.
- The analysis in the previous step is then used to prove that

$$\lim_{\epsilon\to 0}|\Gamma_{\epsilon,t,y} - \Gamma^*_{\epsilon,t,y}| = 0, \quad \text{for all } y\in I_+, \text{ where:} \tag{6.13}$$

$$\Gamma^*_{\epsilon,t,y} := \epsilon^{-2}\int_0^t ds\left(P_s^{(\epsilon)}(N-K,y) - P_s^{(\epsilon)}(N-K-1,y)\right)\frac{j}{2}(D_+\rho_\epsilon(\cdot,t-s))(y).$$

- It is then proved that there exist numbers $a(h), h=0,\dots,K-1$, so that

$$\lim_{\epsilon\to 0}\sum_{y\in I_+}\Gamma^*_{\epsilon,t,y} = \frac{j}{2}\sum_{h=0}^{K-1}a(h)\Big(1-\rho(1,t)\Big)\rho(1,t)^h. \tag{6.14}$$

- The final step consists in recognizing that the right hand side of (6.14) is indeed equal to $j(1-\rho(1,t)^K)$.

By (4.1)

$$\phi_\epsilon(x,t) = \epsilon^{-1}\sum_{z\in\mathbb{Z}}Q_t^{(\epsilon)}(x,z)\left(\rho_\epsilon(\psi_N(z),0) - \rho_\epsilon(\psi_N(z-1),0)\right). \tag{6.15}$$

Recalling that $\rho_\epsilon(y,0)=\rho_0(\epsilon y)$ and that ρ_0', the derivative of ρ_0, is by assumption bounded, we then have for any $r\in[-1,1]$

$$\lim_{\epsilon\to 0}\phi_\epsilon([\epsilon^{-1}r],t)=\int_{\mathbb{R}}G_t(r,r')(-1)^{S(r')}\rho_0'(\psi(r'))dr'=:\phi(r,t), \tag{6.16}$$

where $S(r')=1$ if r' in $[-1,1]$, $\pm[3,5]$,... and $=-1$ in the complement. By symmetry $\phi(\pm 1,t)=0$ so that

$$\lim_{\epsilon\to 0}\phi_\epsilon(x,t)=0,\qquad x=N-K. \tag{6.17}$$

By rescaling the time we rewrite $\Gamma_{\epsilon,t,y}$ as

$$\Gamma_{\epsilon,t,y}:=\int_0^{\epsilon^{-2}t}ds\left(P^{(\epsilon)}_{\epsilon^2 s}(N-K,y)-P^{(\epsilon)}_{\epsilon^2 s}(N-K-1,y)\right)\mathbb{E}_\epsilon\Big[\frac{j}{2}(D_\pm\eta(\cdot,t-\epsilon^2 s))(y)\Big] \tag{6.18}$$

and recall that $P^{(\epsilon)}_{\epsilon^2 s}(x,y)$, which in this proof we denote by $p^{(N)}_s(x,y)$, is the transition probability of a reflected random walk in Λ_N with jump intensity $1/2$ for each pair of n.n. in Λ_N. In the sequel we shall also consider the transition probabilities $p_t(x,y)$ of the random walk on $\mathbb{Z}_+=\{0,1,2,\dots\}$ with jump intensity $1/2$ among nearest neighbors.

Lemma 6.2. *There exists a constant c so that for any $h=0,..,K-1$ and any t*

$$\big|p_t(K,h)-p_t(K+1,h)\big|\le\frac{c}{1+t^{3/2}}. \tag{6.19}$$

The integrals below are well defined:

$$\int_0^\infty\{p_t(K,h)-p_t(K+1,h))\}dt=:a(h). \tag{6.20}$$

Proof. The second statement follows at once from the first, which we now prove with a coupling argument. We write

$$p_t(K,h)-p_t(K+1,h)=\mathcal{E}\left[\mathbf{1}_{y_1(t)=h}-\mathbf{1}_{y_2(t)=h}\right],$$

where $\mathcal{E}$ is the expectation in a process which couples two simple random walks on $\mathbb{Z}_+$, denoted by $y_1(s)$ and $y_2(s)$, $s\in[0,t]$, with $y_1(0)=K$, $y_2(0)=K+1$. The coupling (whose law will be denoted by $\mathcal{P}$) is defined as follows: $y_2(s)$ moves as the random walk on $\mathbb{Z}_+$ (i.e. with transition probability $p_s(x,y)$) for all $s\in[0,t]$. Let $t_1=t/3$: in the time interval $[0,t_1]$, $y_1(s)$ copies exactly the jumps of $y_2(s)$ for all $s<\min\{\tau,t_1\}$, where τ is the first time when y_2 jumps to 0. If $\tau\le t_1$ then we set $y_1(s)=y_2(s)$ for all $s\in[\tau,t]$. When $\tau>t_1$, we let y_1 move independently of y_2 in $[t_1,\tau^*]$, where τ^* is the first time when $y_1=y_2$, and for $s>\tau^*$ we set $y_1(s)=y_2(s)$. $\mathcal{P}$ is evidently a coupling and we have:

$$p_t(K,h)-p_t(K+1,h)=\mathcal{E}\left[(\mathbf{1}_{y_1(t)=h}-\mathbf{1}_{y_2(t)=h})\mathbf{1}_{\tau^*>t}\right]$$

Letting $t_2 = 2t/3$ and

$$g(z_1, z_2) := \mathcal{E}\left[\mathbf{1}_{y_1(t)=h} + \mathbf{1}_{y_2(t)=h} \,\middle|\, y_1(t_2) = z_1, y_2(t_2) = z_2\right]$$
$$h(z_1, z_2) := \mathcal{P}\left[\tau^* > t_2; \,\middle|\, y_1(t_1) = z_1, y_2(t_1) = z_2\right],$$

the l.h.s. of (6.19) is bounded by:

$$\mathcal{E}\Big[g(y_1(t_2), y_2(t_2))h(y_1(t_1), y_2(t_1))\mathbf{1}_{\tau>t_1}\Big]$$

and (6.19) follows after recalling that $y_1(t_1) - y_2(t_1) = 1$ if $\tau > t_1$. □

Lemma 6.3. *There is a constant c so that for any $h = 0, .., K-1$, any N and any t*

$$\left|p_t^{(N)}(N-K, N-h) - p_t^{(N)}(N-K-1, N-h)\right| \le \frac{c}{1+t^{3/2}}. \tag{6.21}$$

Moreover for any $t \le N$ and $\bar{c}$ suitable positive constant,

$$|p_t^{(N)}(x, N-h) - p_t(N-x, h)| \le ce^{-\bar{c}N}, \quad x = N-K, N-K-1. \tag{6.22}$$

Proof. The same argument used in the proof of Lemma 6.2 proves (6.21). Details are omitted. As for (6.22), just notice that $p_t(N-x, h)$ is the probability for the random walk on $\{y \in \mathbb{Z}\colon y \le N\}$ reflected at N and starting at x at time 0 to be at $N-h$ at time t, while $p_t^{(N)}(x, N-h)$ refers to the walk that is also reflected at $-N$. Letting the two walks move together before reaching $-N$, the difference on the l.h.s is bounded from above by the probability of reaching $-N$ by time N, and the estimate follows at once by very simple exponential bound on the Poisson clock process (or still using (4.2)). □

Lemma 6.4. *For any $y \in I_-$*

$$\lim_{\epsilon\to 0} \Gamma^-_{\epsilon,t,y} = 0, \quad y \in I_- \tag{6.23}$$

Proof. By (6.18)

$$|\Gamma_{\epsilon,t,y}| \le c\int_0^{\epsilon^{-2}t} |p_s^{(N)}(N-K, y) - p_s^{(N)}(N-K-1, y)|ds. \tag{6.24}$$

We bound the probability difference by coupling the two random walks as in the beginning of the proof of Lemma 6.2, namely the random walk $y_1(s)$ starting at $N-K$ copies the jumps of $y_2(s)$, the one starting at $N-K-1$. Calling τ_N the first hitting time of N by y_2, the two random walks become identical after τ_N. Let τ_- be the first hitting time of $-N+K$ by y_2. Thus the contribution to (6.24) comes from the event $\tau_- < \tau_N$. Calling $\mathcal{P}$ the law of the above coupling, $\mathcal{E}$ its expectation and $\mathcal{F}_{\tau_-}$ the canonical σ-algebra, we have

$$|\Gamma_{\epsilon,t,y}| \le c\int_0^{\epsilon^{-2}t} \mathcal{E}\left[\mathbf{1}_{\tau_-<s}\mathbf{1}_{\tau_N>\tau_-}|\mathcal{E}\left[\mathbf{1}_{y_1(s)=y} - \mathbf{1}_{y_2(s)=y}|\mathcal{F}_{\tau_-}\right]|\right] ds. \tag{6.25}$$

Since $\tau_N > \tau_-$, $y_1(\tau_-) = -N+K+1$ and $y_2(\tau_-) = -N+K$, the above conditional expectation can be bounded using (6.21) (changing x to $-x$). Thus

$$|\Gamma_{\epsilon,t,y}| \le c \int_0^{\epsilon^{-2}t} \mathcal{E}\left[\mathbf{1}_{\tau_-<s}\mathbf{1}_{\tau_N>\tau_-}\frac{c}{1+(s-\tau_-)^{3/2}}\right] ds \tag{6.26}$$

The r.h.s. of (6.26) involves only the random walk y_2, and due to the initial conditions we are considering (K is fixed), $\mathcal{P}(\tau_- < \tau_N) \le \tilde{c}\epsilon$ for a positive constant $\tilde{c}$. Calling $m(dt)$ the law of τ_- conditioned to $\tau_- < \tau_N$, we may write for $y \in I_-$

$$\begin{aligned}|\Gamma_{\epsilon,t,y}| &\le c'\epsilon \int_0^{\epsilon^{-2}t} ds \int_{(0,s]} m(du)\frac{1}{1+(s-v)^{3/2}}\\ &\le c'\epsilon \int_{(0,\epsilon^{-2}t]} m(du)\int_u^{\epsilon^{-2}t}\frac{1}{1+(s-u)^{3/2}}ds \le c''\epsilon.\end{aligned}$$

proving the lemma □

Proof of (6.13)**.** We split the integral in (6.18) into $s \le \epsilon^{-c^*}$ and $s > \epsilon^{-c^*}$, where c^* is as in Theorem 6.1 (assuming without any loss that $c^* < 2$). For the second one we use (6.21) to see that

$$\left|\int_{\epsilon^{-c^*}}^{\epsilon^{-2}t} ds\left(p_s^{(N)}(N-K,y) - p_s^{(N)}(N-K-1,y)\right)\mathbb{E}_\epsilon\left((D_\pm\eta(\cdot,t-\epsilon^2 s))(y)\right)\right| \le C\epsilon^{c^*/2}$$

Same estimates hold for $\Gamma^*_{\epsilon,t,y}$ so that using Theorem 6.1 we get

$$|\Gamma_{\epsilon,t,y} - \Gamma^*_{\epsilon,t,y}| \le c\epsilon^{(2-\beta^*)c^*}\int_0^{\epsilon^{-c^*}} |p_s^{(N)}(N-K,y) - p_s^{(N)}(N-K-1,y)|ds + 2C\epsilon^{c^*/2}$$

hence (6.13). □

Proof of (6.14)**.** Again by Lemma 6.3 we have

$$\left|\Gamma^*_{\epsilon,t,y} - \int_0^{\epsilon^{-1}}\left(p_s^{(N)}(N-K,y) - p_s^{(N)}(N-K-1,y)\right)\frac{j}{2}(D_+\rho_\epsilon(\cdot,t-\epsilon^2 s))(y)ds\right| \le c\sqrt{\epsilon}$$

We replace $(D_+\rho_\epsilon(\cdot,t-\epsilon^2 s))(y)$ by $(D_+\rho_\epsilon(\cdot,t))(y)$, the error being bounded by $c\sqrt{\frac{\epsilon}{t}}$, by (5.2). Using again Lemma 6.3 we obtain

$$\left|\Gamma^*_{\epsilon,t,y} - \frac{j}{2}(D_+\rho_\epsilon(\cdot,t))(y)\int_0^\infty\left(p_s^{(N)}(N-K,y) - p_s^{(N)}(N-K-1,y)\right)ds\right| \le c'\left(\sqrt{\frac{\epsilon}{t}} + \sqrt{\epsilon}\right)$$

By (2.7),

$$\lim_{\epsilon\to 0}(D_+\rho_\epsilon(\cdot,t))(y) = (1-\rho(1,t))\rho(1,t)^{N-y} \qquad y \in I_+$$

which, by (6.20), proves (6.14). □

We are left with the final step, namely to recognize that the right hand side of (6.14) is equal to $1-\rho(1,t)^K$. We use conservation of mass, namely from (2.5) it follows that

$$2\epsilon\left(\sum_{x=N-K}^{N}\rho_\epsilon(x,t+\tau)-\sum_{x=N-K}^{N}\rho_\epsilon(x,t)\right)=\int_t^{t+\tau}\left(-\frac{1}{2}J_+^{(\epsilon)}(s)+\sum_{y\in I_+}\frac{j}{2}D_+\rho_\epsilon(y,s)\right)ds \tag{6.27}$$

with $J_+^{(\epsilon)}(s)$ the analogue of $\Theta_+^{(\epsilon)}(s)$, namely

$$J_+^{(\epsilon)}(t):=\epsilon^{-1}\left(\rho_\epsilon(N-K,t)-\rho_\epsilon(N-K-1,t)\right).$$

Then, analogously to (6.10)

$$J_+^{(\epsilon)}(t)=\phi_\epsilon(N-K,t)+\sum_{y\in I_+}\Gamma^*_{\epsilon,t,y}-\sum_{y\in I_-}\Gamma^*_{\epsilon,t,y}.$$

The same arguments used for $\Gamma_{\epsilon,t,y}$, for $y\in I_-$ shows that $\lim_{\epsilon\to 0}\sum_{y\in I_-}\Gamma^*_{\epsilon,t,y}=0$, so that using (6.14) we get from (6.27) in the limit $\epsilon\to 0$

$$0=\int_t^{t+\tau}\frac{j}{2}\sum_{h=0}^{K-1}\left(-\frac{1}{2}a(h)(1-\rho(1,s))\rho(1,s)^h+(1-\rho(1,s))\rho(1,s)^h ds\right)ds \tag{6.28}$$

which by the continuity in t gives for any $t>0$

$$\frac{1}{2}\sum_{h=0}^{K-1}a(h)(1-\rho(1,t))\rho(1,t)^h=\sum_{h=0}^{K-1}(1-\rho(1,t))\rho(1,t)^h=1-\rho(1,t)^K. \tag{6.29}$$

□

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